On the Involute B-scrolls In the Euclidean $3-space\ IE^3$

The Involute B-scrolls

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Introduction

Some of the earliest research results about plane curves were motivated by the desire to build more accurate clocks. Practical designs were based on the motion of a pendulum, requiring careful study of motion due to gravity first carried out by Galileo, Descartes, and Mersenne. The culmination of these studies was the work of Christian Huygens(1629-1695) in his 1673 treatise. He is also known for his work in optics. Some of the ideas introduced in Huygens's classic work,[6] such as the involute and evolute of a curve, are part of our current geometric language.The idea of a string involute is due to C. Huygens,he discovered involutes while trying to build a more accurate clock[1].

Introduction

The involute of a given curve is a well-known concept in Euclidean-3 space IE^3. We can say that ;evolute and evolvent is a method of derivying a new curve based on a given curve. The evolvent is often called the involute of the curve. Involvents play a part in the construction of gears[7]. Evolute is the locus of the centers of tangent circles of the given planar curve.

Introduction

It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. $B-scrolls$ are the special ruled surfaces. $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves [2] .
In this study we will define and work on involute curves and involute B-scroll of any curve in Euclidean-3 space IE^3.

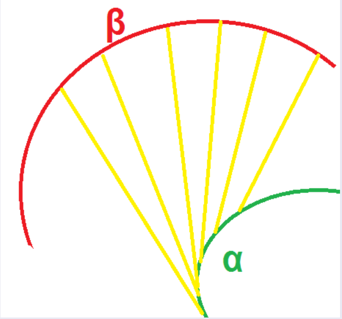
Preliminaries

Let α and β be the curves in Euclidean 3-space .The tangent lines to a curve $\alpha(s)$ generate a surface called the tores of α. If the curve $\beta(s)$ which lies on the tores intersect the tangent lines orthogonally is called an involute of $\alpha(s)$. If a curve $\beta(s)$ is an involute of $\alpha(s)$, then by definition $\alpha(s)$ is an evolute of $\beta(s)$. Hence given $\beta(s)$, its evolutes are the curves whose tangent lines intersect $\beta(s)$ orthogonally. If $\beta(s)$ is a point on an involute β, then $\beta(s) - \alpha(s)$ is propotional to $V_1(s)$. Thus the involute $\beta(s)$ will have a representation of the form
$\beta(s) = \alpha(s) + \lambda(s) V_1(s)$ [5].

Preliminaries

Theorem

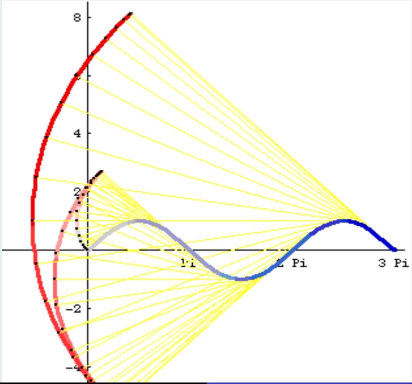
In the Euclidean $3-$ *space* E^3, $\beta \subset IE^3$, *if the curve* $\beta(s)$ *is the involute of* $\alpha(s)$ *with tangent vector* $V_1(s)$ [5] , *then we have that* $\beta(s) = \alpha(s) + (c-s)\,V_1(s)\,; \forall s \in I, c = constant$



Preliminaries

Example

As an example for the involute curve, the red curve is the involute of the blue sine curve



Preliminaries

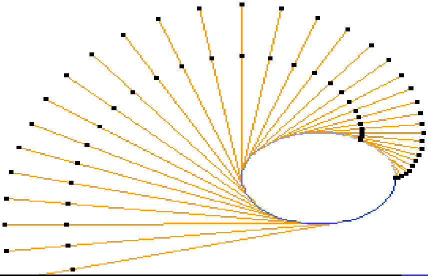
Thus there exist an infinite number of involutes for each constant c. If curve α is the evolute of curve β, then curve β is the involute of curve α. The opposite is true locally. When the tangent line of a curve $\alpha(s) = \alpha$ is given by $\beta = \alpha + \lambda V_1, -\infty < \lambda < +\infty$,then

$$\left\| \frac{d\beta(s)}{d\lambda} \right\| = \| V_1 \| = 1.$$

That is , λ is a natural parameter. Also since $\beta = \alpha$ for $\lambda = 0$, it follows that $|\lambda|$ is the distance between the point β on the tangent line and the point α on $\alpha(s)$ [5] .

Preliminaries

All involutes of a given curve are parallel to each other. This property also makes it easy to see that evolute of a curve is the envelope of its normals. If we calculate the distance between the congruent points of two involutes $\beta_1(s) = \alpha(s) + (c_1 - s)\, V_1(s)$ and $\beta_2(s) = \alpha(s) + (c_2 - s)\, V_1(s)$ we have remains constant for all s and equal to $|c_1 - c_2|, \forall s \in I$ [5]



Preliminaries

In the Euclidean $3-space\ IE^3$, α, $\beta \subset IE^3$, if the curve $\beta(s)$ is the involute of $\alpha(s)$, then for $\forall s \in I$, $c = constant$

$$d(\alpha(s), \beta(s)) = |c - s|$$

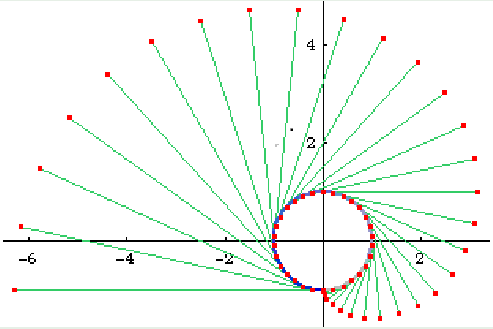
is the distance between the arclengthed curves $\alpha(s)$ and $\beta(s)$.

Preliminaries

Example

Along the circle $\alpha(t) = (a\cos t, a\sin t)$, $a > 0$, we have the involute curve of $\alpha(t)$ is

$$\beta(t) = (a\cos t - (c - at)\sin t, a\sin t + (c - at)\cos t)$$



Preliminaries

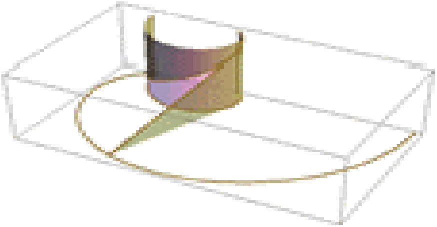
Example

Let us consider the circular helix $\alpha(t) = (a\cos t, a\sin t, bt)$, $a > 0$ is a , then the involute curve of the curve $\alpha(t)$ is

$$\begin{aligned} \beta(t) &= (a[(\cos t + t\sin t) - \gamma\sin t], a[(\sin t - t\cos t) + \gamma\cos t], \gamma b \\ \gamma &= c\left(a^2+b^2\right)^{\frac{-1}{2}} \ and \ t = s\left(a^2+b^2\right)^{\frac{-1}{2}} \end{aligned}$$

Frenet apparatus and Frenet formulas of the Involute curve

As shown in the following Figure, that the involute is a planar curve, whose plane is $z = \gamma b$



Frenet apparatus and Frenet formulas of the Involute curve

The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curves. The following result shows that we can write the Frenet apparatus of the involute curve based on the its evolute curve. And also we can introduce the Frenet Formulas of the involute curve based on the Frenet apparatus of its evolute curve.

Frenet apparatus and Frenet formulas of the Involute curve

Theorem

In the Euclidean $3-$ *space* IE^3, $\alpha, \beta \subset IE^3$, $\alpha(s)$ *and* $\beta(s^*)$ *are the arclengthed curves with the arcparametres s and* s^**,respectively. Let* V_1, V_2, V_3 *and* V_1^*, V_2^*, V_3^* *be Frenet vectors belong to the the curves* $\alpha(s)$ *and the involute* $\beta(s^*)$*, respectively. If the curve* $\beta(s^*)$ *is the involute of the curve* $\alpha(s)$ *then we have the equation*[3]

$$\langle V_1, V_1^* \rangle = 0.$$

Frenet apparatus and Frenet formulas of the Involute curve

Theorem

In the Euclidean $3-space\ IE^3$, $\alpha, \beta \subset IE^3$, $\alpha(s)$ *and* $\beta(s^*)$ *are the arclengthed curves with the arcparametres* s *and* s^**,respectively. Let the first and second curvatures of the curve* $\alpha(s)$ *and* $\beta(s^*)$ *be* k_1, k_2 *and* k_1^*, k_2^* *respectively. The Frenet vector fields* V_1, V_2, V_3 *and* V_1^*, V_2^*, V_3^* *which belong to the curve* α *and* β*, respectively. If* $\beta(s^*)$ *is the involute of the curve* $\alpha(s)$*.We have the following equations*[3]

$$\begin{aligned} V_1^* &= V_2; \lambda k_1 > 0, \lambda = (c-s) \\ V_2^* &= (-k_1 V_1 + k_2 V_3)(\lambda k_1 k_1^*)^{-1} \\ V_3^* &= (k_2 V_1 + k_1 V_3)(\lambda k_1 k_1^*)^{-1}. \end{aligned}$$

Frenet apparatus and Frenet formulas of the Involute curve

The ratio of the arc parametres of these curves is $\frac{ds}{ds^*} = \frac{1}{\lambda k_1}$.The first curvature of involute β is

$$k_1^* = \sqrt{\frac{k_1^2 + k_2^2}{\lambda^2 k_1^2}}, \lambda = (c - s), k_1 \neq 0.$$

If we use this result at the equation V_2^* and V_3^*,we have

$$\begin{aligned} V_1^* &= V_2; \lambda k_1 > 0 \\ V_2^* &= (-k_1 V_1 + k_2 V_3)\left(k_1^2 + k_2^2\right)^{\frac{-1}{2}} \\ V_3^* &= (k_2 V_1 + k_1 V_3)\left(k_1^2 + k_2^2\right)^{\frac{-1}{2}} \end{aligned}$$

Frenet apparatus and Frenet formulas of the Involute curve

Corollary

If the second curvature k_2 of the curve $\alpha(s)$ is equal to zero, that is $\alpha(s)$ is a planar curve, then

$$k_1^* = \frac{1}{\lambda}, \lambda > 0$$

Corollary

If the second curvature k_2 of the curve $\alpha(s)$ is constant but not equal to zero, then $\dot{k}_2 = 0$. Hence we have that

$$(k_1^*)^2 = \frac{k_1^2 + k_2^2}{\lambda^2 k_1^2}, \ \ k_1 \neq 0$$

Frenet apparatus and Frenet formulas of the Involute curve

Theorem

In the Euclidean $3-$ *space* IE^3, *let* $\alpha(s)$ *and* $\beta(s^*)$ *be the arclengthed curves with the arcparametres* s *and* s^*, *respectively. Let* $\beta(s^*)$ *be the involute of the curve* $\alpha(s)$. *The Frenet vector fields* V_1, V_2, V_3 *and* V_1^*, V_2^*, V_3^* *of* α *and* β, *respectively. Let the first and second curvatures of the curve* $\alpha(s)$ *be* k_1 *and* k_2,*respectively. We have the differential of the Frenet vector fields* $V_1^*(s), V_2^*(s), V_3^*(s)$ *as*

$$\begin{bmatrix} \dot{V}_1^* \\ \dot{V}_2^* \\ \dot{V}_3^* \end{bmatrix} = \begin{bmatrix} \frac{-1}{\lambda} & 0 & \frac{k_2}{\lambda k_1} \\ \frac{k_2 k_2^*}{\lambda k_1 k_1^*} & -k_1^* & \frac{k_2^*}{\lambda k_1^*} \\ \frac{1}{\lambda} & 0 & -\frac{k_2}{\lambda k_1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Frenet apparatus and Frenet formulas of the Involute curve

Theorem

In the Euclidean $3-space$ IE^3, let $\alpha(s)$ and $\beta(s^*)$ are the arclengthed curves with the arcparametres s and s^*, respectively. . Let the first and second curvatures of the curve $\alpha(s)$ be k_1 and k_2, respectively. If the second curvature of β involute is k_2^*, If $\beta(s^*)$ is the involute of the curve $\alpha(s)$ for $\lambda = (c - s)$ then

$$k_2^* = \frac{k_1 k_2' - k_1' k_2}{\lambda k_1 \left(k_1^2 + k_2^2\right)}, \; c \text{ is constant}.$$

Frenet apparatus and Frenet formulas of the Involute curve

Corollary

If the second curvature k_2 of the curve α is equal to zero, $k_2 = 0$, then $k_2^* = 0$; that is if the curve α is a planar curve, then the involute of α is a planar curve too.

Corollary

If the second curvature k_2 of the curve $\alpha(s)$ is constant but not equal to zero, then $\dot{k}_2 = 0$. Hence we have that

$$k_2^* = -\frac{k_1^{'} k_2}{\lambda k_1 \left(k_1^2 + k_2^2\right)}.$$

Frenet apparatus and Frenet formulas of the Involute curve

Corollary

If the curve $\alpha(s)$ is a helix, then the involute $\beta(s)$ of the curve $\alpha(s)$ is a planar curve. If the curve $\alpha(s)$ is a helix

$$k_2^* = \frac{k_1 k_2' - k_1' k_2}{\lambda k_1 \left(k_1^2 + k_2^2\right)} = \frac{\left(\frac{k_2}{k_1}\right)'}{\frac{\lambda\left(k_1^2+k_2^2\right)}{k_1}} = 0$$

Involute B-scroll In the Euclidean 3-space

Definition

In the Euclidean $3-space\ IE^3$, let $\alpha(s)$ be an arclengthed curve. The equation

$$\varphi(s, u) = \alpha(s) + uV_3(s)$$

is the parametrization of the ruled surface which is called $B-scroll$ (binormal scroll) [1]. The directrix of this $B-scroll$ is the curve $\alpha(s)$. The generating space of this $B-scroll$ is spaned by binormal subvector V_3. Here $Sp\{V_1, V_2\}$ is the osculator plane of the curve $\alpha(s)$.

Involute B-scroll In the Euclidean 3-space

Definition

In the Euclidean $3-space\ IE^3$, let $\alpha(s)$ and $\beta(s^*)$ be the arclengthed curves. Let Frenet vector fields V_1, V_2, V_3 and V_1^*, V_2^*, V_3^* of α and β, respectively. If the curve $\beta(s)$ is the involute of the curve $\alpha(s)$ .The equation

$$\varphi^*(s,v) = \beta(s) + vV_3^*(s)$$

is the parametrization of the ruled surface which is called *involute* $B-scroll$ (binormal scroll) of the curve α. The directrix of this *involute* $B-scroll$ is the involute curve $\beta(s) = \alpha(s) + (c-s)V_1(s)$ of the curve $\alpha(s)$. The generating space of $B-scroll$ is spaned by binormal subvector V_3^*. Here $Sp\{V_1^*, V_2^*\}$ is the osculator plane of the curve β.

Involute B-scroll In the Euclidean 3-space

Theorem

In the Euclidean $3-space$ IE^3,let V_1, V_2, V_3, k_1, k_2 and $V_1^*, V_2^*, V_3^*, k_1^*, k_2^*$ be Frenet apparatus of the curve α and the involute curve β, respectively. The parametrization of the involute $B-scroll$ of the curve $\alpha(s)$ is

$$\begin{aligned}
\varphi^*(s,v) &= \alpha(s) + \left(\lambda + \frac{vk_2(s)}{\sqrt{k_1^2+k_2^2}}\right)V_1(s) + \frac{vk_1(s)}{\sqrt{k_1^2+k_2^2}}V_3(s) \\
\lambda &= (c-s),\ \lambda k_1 > 0.
\end{aligned}$$

Involute B-scroll In the Euclidean 3-space

Theorem

In the Euclidean $3-$ *space* IE^3*,let* k_1, k_2, V_1, V_2, V_3 *and* $k_1^*, k_2^*, V_1^*, V_2^*, V_3^*$ *be Frenet apparatus of the nonplanar curve* α *and the involute curve* β*, respectively. The intersection of the involute* $B-$ *scroll of* $\alpha(s)$ *and* $B-$ *scroll of the curve* $\alpha(s)$ *is a curve with parametrization*

$$\varphi(s) = \alpha(s) + \left(-\lambda \frac{k_1(s)}{k_2(s)}\right) V_3(s), \lambda = c - s.$$

Involute B-scroll In the Euclidean 3-space

Proof.

Under the conditions

$$\left(\lambda + \frac{vk_2}{\sqrt{k_1^2 + k_2^2}}\right) = 0 \quad \textit{and} \quad \frac{vk_1}{\sqrt{k_1^2 + k_2^2}} = u$$

we get

$$u = -\lambda \frac{k_1}{k_2}, k_2 \neq 0$$

□

Involute B-scroll In the Euclidean 3-space

Theorem

In the Euclidean $3-$ space IE^3, the Frenet vectors V_1, V_2, V_3 of curve α

$$\varphi(s,u) = \alpha(s) + uV_3(s)$$

is the parametrization of the ruled surfaces which is called $B-$ scroll (binormal scroll). Then the normal vector field [4] of ruled surface $B-$ scroll is

$$N = \frac{-uk_2V_1 - V_2}{\sqrt{1+u^2k_2^2}}.$$

Involute B-scroll In the Euclidean 3-space

Theorem

In the Euclidean $3-space$ IE^3, the normal vector field of involute $B-scroll$ of the curve $\alpha(s)$ is

$$
\begin{aligned}
N^* \;=\; & \frac{\lambda k_1^2\sqrt{k_1^2+k_2^2}}{\sqrt{\left(\lambda k_1\left(k_1^2+k_2^2\right)\right)^2+v^2\left(k_1k_2'-k_1'k_2\right)^2}}V_1 \\
& +\frac{-v\left(k_1k_2'-k_1'k_2\right)}{\sqrt{\left(\lambda k_1\left(k_1^2+k_2^2\right)\right)^2+v^2\left(k_1k_2'-k_1'k_2\right)^2}}V_2 \\
& -\frac{\lambda k_1k_2\sqrt{k_1^2+k_2^2}}{\sqrt{\left(\lambda k_1\left(k_1^2+k_2^2\right)\right)^2+v^2\left(k_1k_2'-k_1'k_2\right)^2}}V_3
\end{aligned}
$$

Involute B-scroll In the Euclidean 3-space

Proof.

We already have the equation of the *involute* $B-scroll$ of the curve $\alpha(s)$,And also it is well known that the normal vector field $N^*$of any $B-scroll$ surface [4] is

$$N^* = \frac{-vk_2^* V_1^* - V_2^*}{\sqrt{1+v^2 k_2^{*2}}}$$

so normal vector field $N^*$of the *involute* $B-scroll$ is □

Involute B-scroll In the Euclidean 3-space

Proof.

$$N^* = \frac{-v\left(k_1 k_2' - k_1' k_2\right)}{\sqrt{\left(\lambda k_1\left(k_1^2+k_2^2\right)\right)^2 + v^2\left(k_1 k_2' - k_1' k_2\right)^2}} V_2 - \frac{-k_1 V_1 + k_2 V_3}{\left(\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1\left(k_1^2+k_2^2\right)}\right)\sqrt{\left(\lambda k_1\left(k_1^2+k_2^2\right)\right)^2 + v^2\left(k_1 k_2' - k_1' k_2\right)^2}}$$

This completes the proof. □

Involute B-scroll In the Euclidean 3-space

Theorem

In the Euclidean $3-$ *space* IE^3*, let us consider the involute* $B-$ *scroll of the curve* $\alpha(s)$ *given by* $\varphi^*(s,v)=\beta(s)+vV_3^*(s)$. *if the normal vector field* N^* *of involute* $B-$ *scroll of the curve* $\alpha(s)$ *and the normal vector field* N *of* $B-$ *scroll of the curve* $\alpha(s)$ *are perpendicular to each other, then*

$$v=u\frac{\lambda k_1^2\sqrt{k_1^2+k_2^2}}{\left(k_1k_2'-k_1'k_2\right)}$$

Involute B-scroll In the Euclidean 3-space

Proof.

Lets rename the coefficients of the normal vector fields $N^*$of *involute* $B-scroll$ of the curve $\alpha(s)$ as δ, ε and η, we get $N^* = \delta V_1 + \varepsilon V_2 + \eta V_3$. Using the orthogonality condition; *If* $N^* \perp N$,then $\langle N^*, N \rangle = 0$ and

$$\begin{aligned}
\delta \frac{-uk_2}{\sqrt{1+u^2k_2^2}} &= \varepsilon \frac{1}{\sqrt{1+u^2k_2^2}}; \ \sqrt{1+u^2k_2^2} \neq 0 \\
uk_2\delta &= -\varepsilon; \sqrt{\left(\lambda k_1\left(k_1^2+k_2^2\right)\right)^2 + v^2\left(k_1k_2' - k_1'k_2\right)^2} \neq 0 \\
\frac{u}{v} &= \frac{\left(k_1k_2' - k_1'k_2\right)}{\lambda k_1^2 k_2 \sqrt{k_1^2+k_2^2}}
\end{aligned}$$

Involute B-scroll In the Euclidean 3-space

Corollary

If $\alpha(s)$ is helix , then $\left(\frac{k_2}{k_1}\right)' = 0$ the normal vector field $N^*$of involute $B-$ scroll of the curve $\alpha(s)$ and the normal vector field N of $B-$ scroll of the curve $\alpha(s)$ cant be perpendicular to each other.

$$\frac{u}{v} = \frac{1}{\lambda k_2\sqrt{k_1^2 + k_2^2}}\left(\frac{k_2}{k_1}\right)' = 0$$

$$v \neq 0 \ , \ u = 0$$

That is there are not any $B-$ scroll surfaces.

Involute B-scroll In the Euclidean 3-space

Theorem

In the Euclidean $3-space$ IE^3, if the Normal vector field N^* of involute $B-scroll$ of the curve $\alpha(s)$ and the Normal vector field N of $B-scroll$ of the curve $\alpha(s)$ can not be parallel to each other.

Involute B-scroll In the Euclidean 3-space

Proof.

Using the condition of parallellizm;

$$If \;\; N//N^*, \frac{\delta}{\frac{-uk_2}{\sqrt{1+u^2k_2^2}}} = \frac{\varepsilon}{\frac{-1}{\sqrt{1+u^2k_2^2}}} \; and \;\; \eta = 0$$

we get

$$\begin{aligned} uv &= \frac{-\lambda k_1^2\sqrt{k_1^2+k_2^2}}{k_2\left(k_1k_2' - k_1'k_2\right)} \; and \;\; \lambda k_1 k_2\sqrt{k_1^2+k_2^2} = 0 \\ uv &= 0 \end{aligned}$$

$\square$

Involute B-scroll In the Euclidean 3-space

Example

In the Euclidean $3-space\ IE^3$, along the helix $\alpha(t) = (a\cos t, a\sin t, bt)$, $a > 0$, we have the *involute* $B-scroll$ of the helix $\alpha(t)$ is

$$\begin{aligned} \varphi^*(t,v) &= \beta(t) + vV_3^*(t) \\ &= (a[(\cos t + t\sin t) - \gamma\sin t], a[(\sin t - t\cos t) + \gamma\cos t], \end{aligned}$$

Involute B-scroll In the Euclidean 3-space

Example

Then we obtain the intersection of the involute B-scroll of the helix $\alpha(s)$ and B-scroll of a nonplanar curve $\alpha(s)$ as a curve with parametrization

$$\begin{aligned}\varphi(s) &= \left(a\cos t + \frac{\lambda a \sin t}{\sqrt{(a^2+b^2)}}, a\sin t - \frac{\lambda a \cos t}{\sqrt{(a^2+b^2)}}, bt + \frac{\lambda a^2}{b\sqrt{(a^2+ \ldots\end{aligned}$$

$$t = s\left(a^2+b^2\right)^{-\frac{1}{2}}$$

Involute B-scroll In the Euclidean 3-space

Example

Along the circle $\alpha(t) = (a\cos t, a\sin t, 0)$, $a > 0$ we have the *involute* $B-$*scroll* of the circle $\alpha(t)$ with the parametrization

$$\begin{aligned} \varphi^*(t,v) &= \beta(t) + vV_3^*(t) \\ &= (a\cos t - (c-at)\sin t, a\sin t + (c-at)\cos t, v) \end{aligned}$$

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