Quantization operators and invariants of group representations

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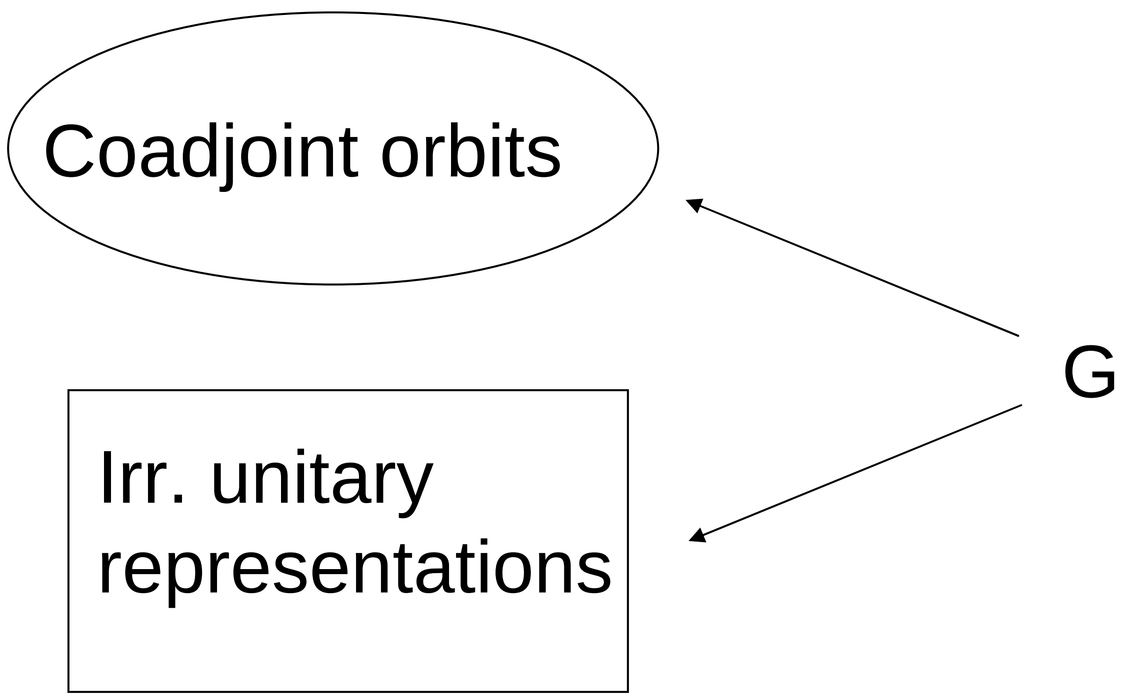
Geometry, Integrabiltity and Quantization. Varna (June 2011)

INDEX

INDEX

I. INTRODUCTION

The coadjoint action of a Lie group G gives rise to the coadjoint orbits, which are homogeneous G-spaces. On the other hand, associated with G we have its unitary dual $\widehat{G}$, (the space consisting of the irreducible unitary representations of G.)



The study of the possible relations between the set of orbits ("geometric objects") and $\widehat{G}$ (a set of "algebraic objects") is the aim of the Orbit method.

In this talk we will also describe some aspects of those relations.

INDEX

II. ABOUT THE ORBIT METHOD

Theorem (Kirillov). Let G be a nilpotent connected simply connected Lie group. Then

$$\hat{G} = \{\text{irreduc. unitary representation of } G\}$$

$$\updownarrow$$

$$\{\text{coadjoint orbits of } G\}.$$

Furthermore, Kirillov gave interpretations of facts relative to representation theory in terms of the geometry of the coadjoint orbits.

For example, if O is the coadjoint orbit of $\eta \in \mathfrak{g}^* = (Lie\ G)^*$ and π is the corresponding representation in the above bijection, then

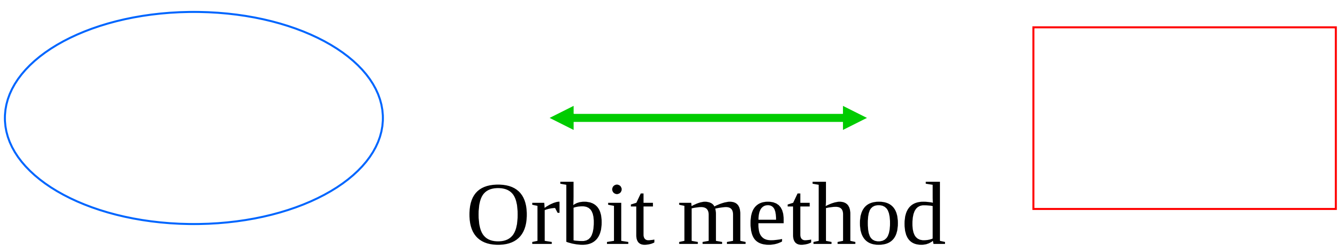
$$\chi_\pi(\exp A) = \int_O e^{2\pi i \eta(A)} d\,vol$$

Is that theorem valid for a general Lie group?

No. The complementary series of representations of $SL(2, \mathbb{R})$ are not attached to coadjoint orbits.

The orbit method is based in the idea that a bijective map similar to the preceding one there exists for any Lie group if we modify the domain and the range of the map.

{Coadjoint orbits} {Irre. unit. representations}



The physical ground of the Orbit method is related with the quantization.

Symplectic geometry is a mathematical model for classical mechanics. The phase space of a classical system is a symplectic manifold. A homogeneous G-manifold can be considered as a class. system equipped with a group G of symmetries.

A Hilbert space is a mathematical model for quantum mechanics. Thus, a representation may be regarded as a quantum system endowed with a group of symmetries.

Classical and quantum mechanics can be considered as different descriptions of “the same physical system”. So, for each classical system there should be a corresponding quantum system, and theoretically, one could construct from a classical system the respective quantum system.

When there is the action of a group G, this construction, going from the orbit (the homogeneous G-space) to an irreducible representation, is precisely what the orbit method asserts should exist.

The mathematical translation of this physical considerations is implemented by the geometric quantization.

Geometric Quantization and Borel-Weil Theorem

(N, ω) symplectic quantizable manifold, there exists a complex line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = [\omega]$.

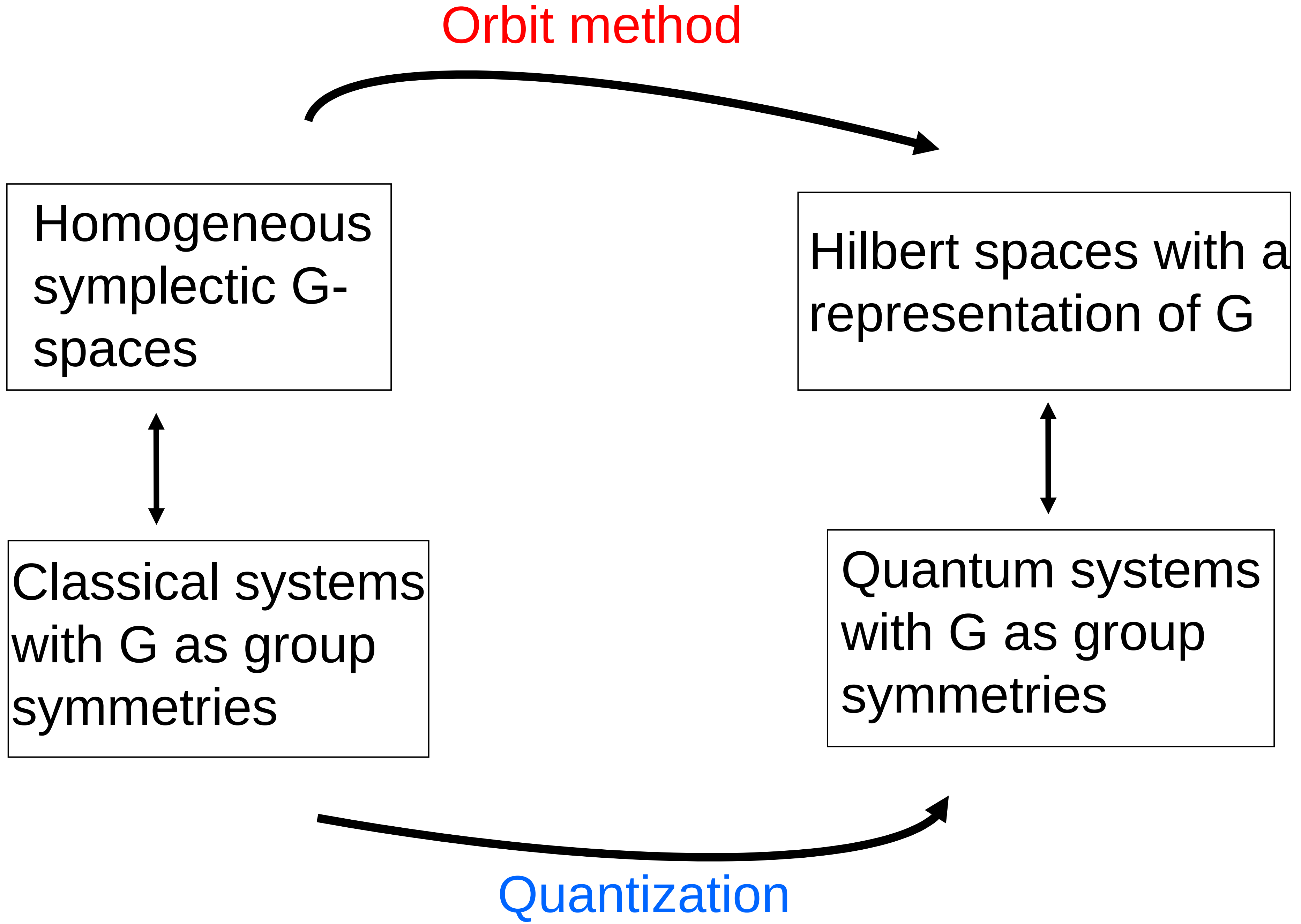
Each Hamiltonian vector field X on N has associated an operator Q_X (quantization operator) acting on the sections of $\mathcal{L}$.

If G acts on N as a group of Hamiltonian symplectomorphisms, $A \in \mathfrak{g}$ defines a vector field X_A and

$$\left\{ Q_{X_A} \right\}$$

form a representation of $\mathfrak{g}$.

When (N, ϖ) is the coadjoint orbit of an integral element of $\mathfrak{g}^*$ endowed with the Kirillov structure and G is compact, then $\mathcal{L}$ is G-equivariant. There exists a representation of G on sections of $\mathcal{L}$. The choice of a subalgebra of $\mathfrak{g}$ permits us to define polarized sections. On this space takes place an irreducible representation of G. (Borel-Weil theorem)


Orbit method
Homogeneous symplectic G-spaces
Hilbert spaces with a representation of G
Classical systems with G as group symmetries
Quantum systems with G as group symmetries
Quantization

INDEX

III. DISCRETE SERIES

Not every representation is associated to an orbit, here we will consider the discrete series representations. Firstly, the regular representation of G is the space $L^2(G)$ endowed with the left translation.

For $f \in L^2(G)$ and $g \in G$,

$$(g \bullet f)(x) = f(g^{-1}x).$$

An irreducible unitary representation π of G is said to be in the discrete series of G if it can be realized as a direct summand of the regular representation.

This is equivalent to the fact that the Plancherel measure for the decomposition of $L^2(G)$ assigns strictly positive mass to the one-point set $\{\pi\}$ in the unitary dual of G (from this property comes the name "discrete" series).

If G is compact, every irreducible representation is in the discrete series.

If G possesses discrete series representations, it contains a compact Cartan subgroup T. Kostant and Langlands conjectured the realization of the discrete series by the so-called L^2-cohomology of holomorphic line bundles over G/T (proved by Wilfried Schmid).

For G compact the conjecture reduces to the B-W theorem.

INDEX

IV. PURPOSES OF THIS TALK

In the spirit of the Orbit Method using the geometric construction of Schmid, we describe interpretations of some invariants of discrete series representations in terms of geometric concepts of the orbits.

G a linear semisimple group, $T \subset G$ compact Cartan subgroup and π in the discrete series.

(1) If $g_1 \in Z(G)$, the operator $\pi(g_1)$ commutes with the operators $\pi(h)$. By Schur's lemma $\pi(g_1)$ is a multiple of the identity.

$$\pi(g_1) = \kappa Id,$$

with $\kappa \in U(1)$.

We will give geometric interpretations of κ in terms of objects related with G/T.

For G compact, κ is the symplectic action around closed curves in G/T.

(2) The differential representation π' of $\mathfrak{g}$, defines an irreducible representation of $U(\mathfrak{g}_{\mathbb{C}})$, (universal envelopping algebra). The infinitesimal character gives the action of the centre of $U(\mathfrak{g}_{\mathbb{C}})$. It is the simplest invariant of π'.

We will relate the infinitesimal character with the quantization operators on vector bundle over G/T.

(3) Finally, we use the above results to give lower bounds for the cardinal of the fundamental group of the Hamiltonian group of G/T.

INDEX

V. GEOMETRIC FRAMEWORK

Let G be a linear semisimple Lie group, T a compact Cartan subgroup and K a maximal compact subgroup $T \subset K$.

By Δ we denote a positive root system of $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$,

$$\rho := \frac{1}{2} \sum_{\nu \in \Delta} \nu$$

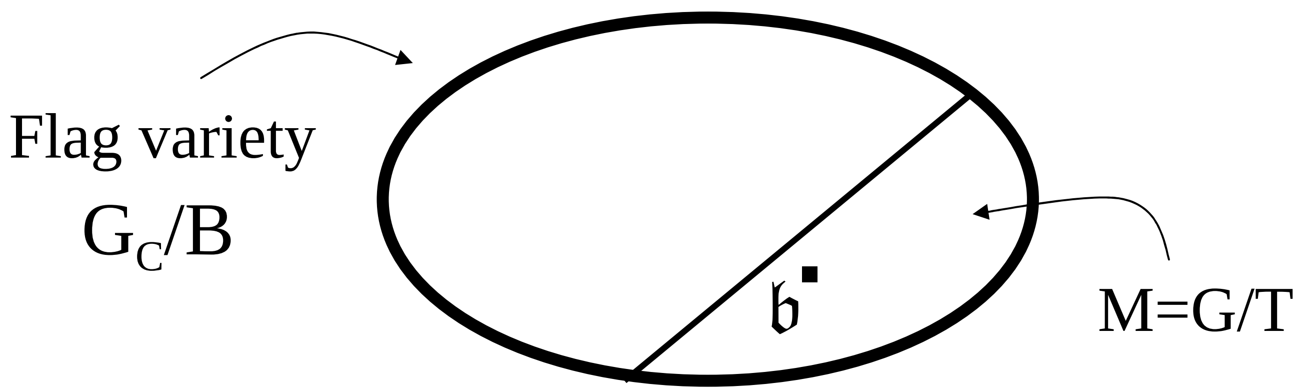
$\mathfrak{g}^{\nu} \subset \mathfrak{g}_{\mathbb{C}}$ is the root space of ν. A root ν is compact if $\mathfrak{g}^{\nu} \subset \mathfrak{k}_{\mathbb{C}}$

We define

$$\mathfrak{u} := \bigoplus_{\nu\in\Delta} \mathfrak{g}^{-\nu}$$

We put $\mathfrak{b}$ for the Borel subalgebra $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{u}$ and denote by B the Borel subgroup $G_{\mathbb{C}}$. The flag variety of $\mathfrak{g}_{\mathbb{C}}$ is diffeomorphic to $G_{\mathbb{C}} / B$.

The G-orbit of $\mathfrak{b}$ in the flag variety of $\mathfrak{g}_{\mathbb{C}}$ is a complex submanifold $M \simeq G / T$.



Let ϕ be an element of the weight lattice of $\mathfrak{t}$. ϕ induces a character Φ on B in a natural way.

Denoting by $(\cdot,\cdot)$ the Killing form on $\mathfrak{t}^*$, we put q for

$$q := \#\{\nu \in \Delta \mid \nu \text{ compact } (\phi+\rho, \nu) < 0\} + \\ \#\{\nu \in \Delta \mid \nu \text{ noncompact } (\phi+\rho, \nu) > 0\}$$

In particular, when G is compact and ϕ is dominant, $q = 0$.

We set

$$W := \mathbb{C} \otimes (\wedge^q \mathfrak{u})^*,$$

and define the representation

$$\Psi = \Phi \otimes (\wedge^q \mathrm{Ad})^* : T \to \mathrm{GL}(W)$$

With Ψ we construct

$$\mathcal{P} := G \times_\Psi \mathrm{GL}(W) \to M = G/T$$

$$\mathcal{W} := G \times_\Psi W \to M$$

The G-actions on $M = G/T$ and on $\mathcal{W}$ induce the following representation on $\Gamma(\mathcal{W})$:

$$(g \bullet \sigma)(x) = g(\sigma(g^{-1}x)).$$

If $\phi+\rho$ is regular, Schmid theory defines a subspace $\mathcal{H} \subset \Gamma(\mathcal{W})$, in which the restriction of the above representation is irreducible. This restriction is the discrete series representation π of G, associated with weight ϕ.

On $\mathcal{P}$ it is possible to define an G-invariant connection. The covariant derivative in $\mathcal{W}$ is denoted by ∇.

$\mathcal{P}, \mathcal{W}$ are the geometric framework for our developments. The vector bundle $\mathcal{W}$ plays a similar role as the prequatum bundle in geometric quantization. And the subspace $\mathcal{H}$ of $\Gamma(\mathcal{W})$ corresponds to the space of polarized sections in the formulation of Borel-Weil theory.

INDEX

VI. REPRESENTATION DIFFERENTIAL AND QUANTIZATION OPERATORS

Assume that $\phi + \rho \in i\mathfrak{t}^*$ is regular.

We denote by $\mathcal{H}_K$ the space of K-finite vectors in $\mathcal{H}$ (Harish-Chandra module of $\mathcal{H}$). π' the differential representation of π on $\mathcal{H}_K$.

The decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus_{\nu \in \Delta} (\mathfrak{g}^{\nu} \oplus \mathfrak{g}^{-\nu})$$

induces a direct sum decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{l}$. The component of $C \in \mathfrak{g}$ in $\mathfrak{t}$ is denoted C_0.

For $A \in \mathfrak{g}$, we denote by X_A (vector field on M).

$$h_A : G \to \mathfrak{gl}(W), \quad h_A(g) := \Psi'((g^{-1} \cdot A)_0).$$

$$F_A : \mathcal{W} \to \mathcal{W}, \quad F_A(\langle g, v \rangle) = \langle g,\ h_A(g)(v) \rangle$$

The differential operator $Q_A := -\nabla_{X_A} + F_A$ acting on sections of $\mathcal{W}$ is the analogue of the quantization operator.

That is, if G is compact and ϕ dominant, then $\mathcal{W}$ is a prequantum bundle and Q_A is the respective quantization operator associated to X_A by geometric quantization.

The operators Q_A will be called “quantization operators”.

Theorem 1. The correspondence $A \to Q_A$ defines a representation of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{H}_K$, which is equivalent to π'.

The universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ is defined as the quotient of the tensor algebra $T(\mathfrak{g}_{\mathbb{C}})$ by the 2-sided ideal generated by

$$XY - YX - [X, Y], \quad X, Y \in \mathfrak{g}_{\mathbb{C}}$$

The representation π' determines a representation of the associative algebra $U(\mathfrak{g}_{\mathbb{C}})$. The elements of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g}_{\mathbb{C}})$ play an important role in representation theory (among the elements of degree 2 in the centre is the Casimir).

As a consequence of the generalization of Schur's lemma (due to Dixmier), $J \in Z(\mathfrak{g})$ is a scalar operator in the representation induced π'. The resulting homomorphism

$$\chi : Z(\mathfrak{g}) \to \mathbb{C}$$

is the infinitesimal character of the $U(\mathfrak{g}_{\mathbb{C}})$-module $\mathcal{H}_K$.

Let $C_1,....,C_r$ be a basis of $\mathfrak{t}$, and E_ν a basis of $\mathfrak{g}^\nu$, then J is a polynomial $p(C_i, E_\nu)$ in the "variables" C_i, E_ν

We can prove the following theorem:

Theorem 2. The corresponding differential operator $p(Q_{C_i}, Q_{E_\nu})$ on the space $\mathcal{H}_K$ is the scalar one defined by the constant $\chi(J)$.

INDEX

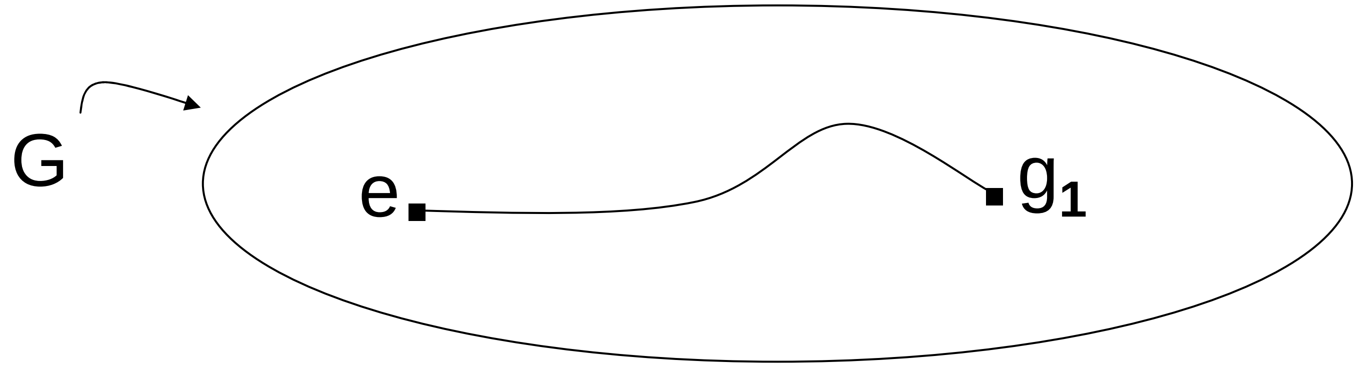
VII. INVARIANTS DEFINED BY SCHUR'S LEMMA

If $g_1 \in Z(G)$, then $\pi(g_1)\pi(h) = \pi(h)\pi(g_1), \ \forall h \in G$.
By Schur's Lemma

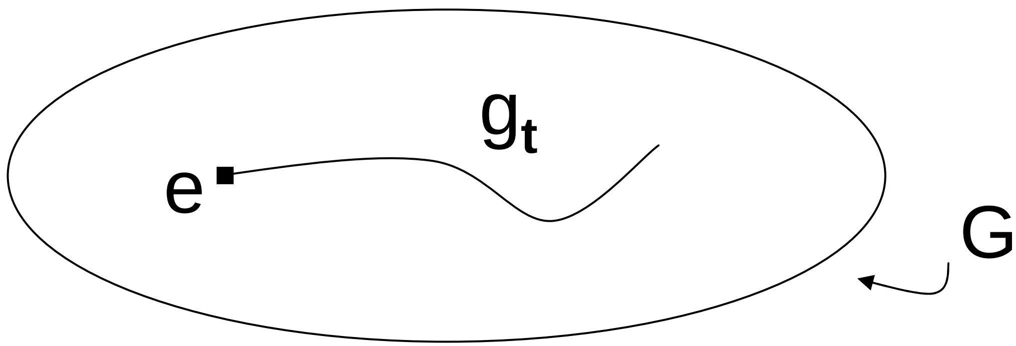
$$\pi(g_1) = \kappa \, \mathrm{Id}_{\mathcal{H}},$$

with $\kappa \in U(1)$.

To know the action of $\pi(g_1)$, we will "integrate" π' along a curve in G with initial point at e and end at g_1.



Henceforth, $\{g_t \mid t \in [0,1]\}$ stands for an *arbitrary* smooth curve in G with the initial point at e (a path in G).

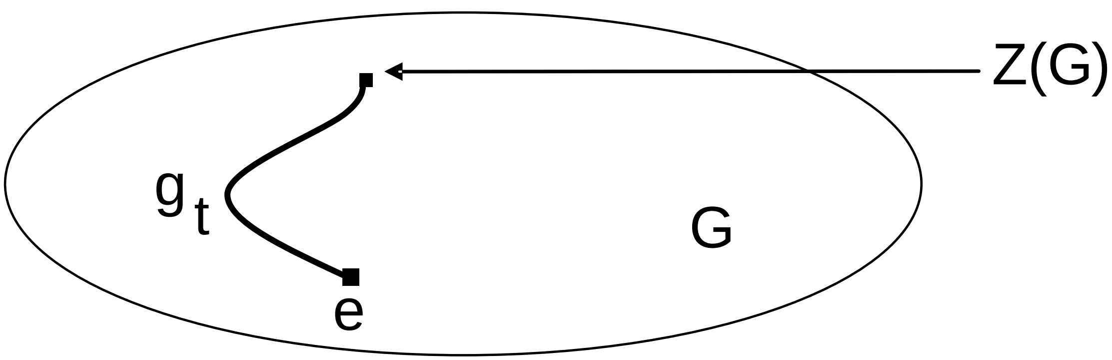


We denote by $\{A_t \in \mathfrak{g}\}$ the corresponding velocity curve,

$$A_t := \frac{dg_t}{dt} g_t^{-1}.$$

We can consider the set $\sigma_t \in \Gamma(\mathcal{W})$ defined by the following "evolution equations":

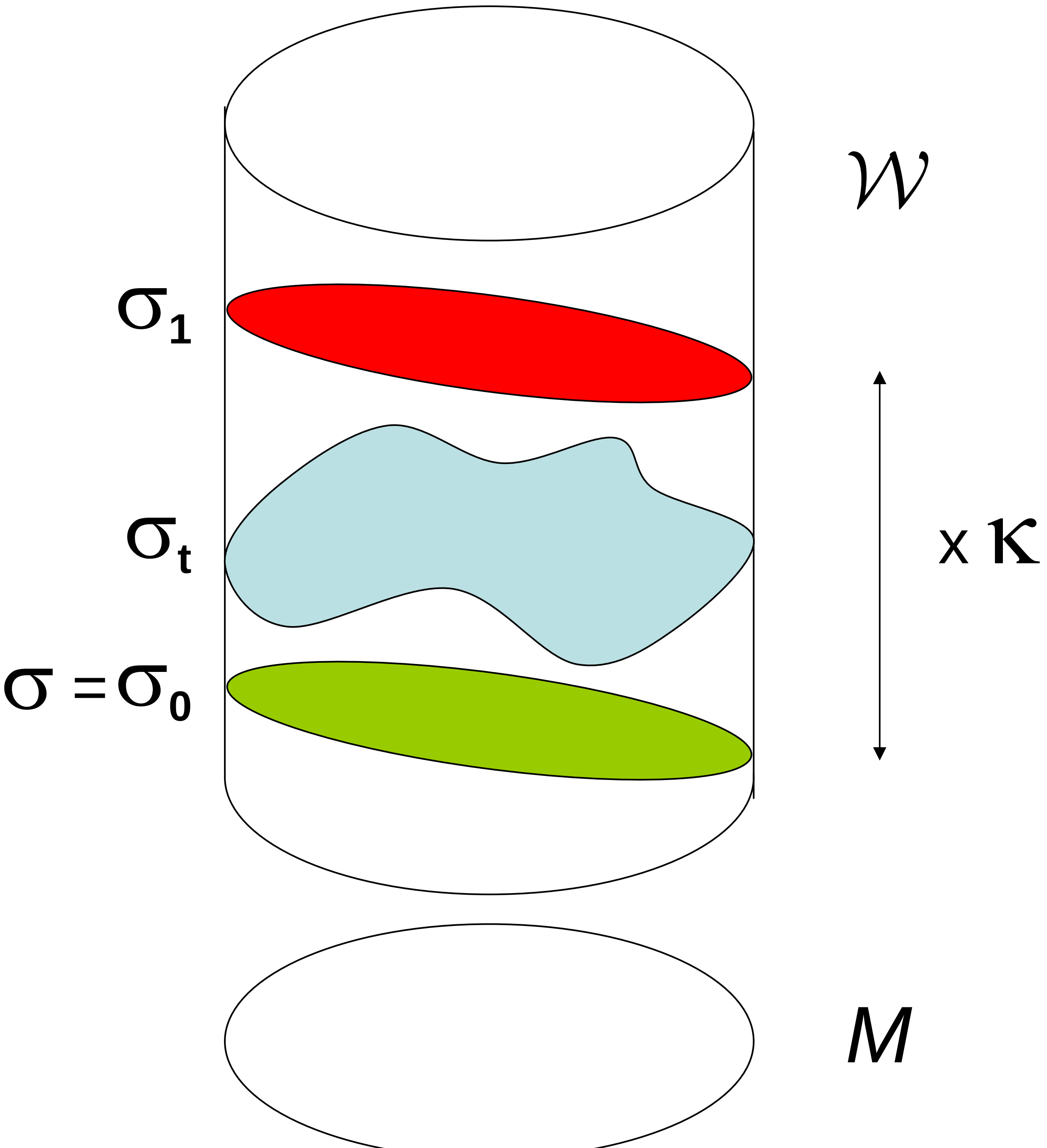
$$\frac{d\sigma_t}{dt} = Q_{A_t}(\sigma_t), \quad \sigma_0 = \sigma.$$



Theorem 3. If $g_1 \in Z(G)$, then

$$\sigma_1 = \kappa\sigma,$$

for any $\sigma \in \mathcal{H}_K$.


$\mathcal{W}$
σ_1
σ_t
$\sigma = \sigma_0$
x K
M

For each $A \in \mathfrak{g}$ the natural G-action on $\mathcal{P} = G \times_T \mathrm{GL}(W)$ determines a vector field Y_A.

So, a path g_t defines the time-dependent vector field Y_{A_t} and the corresponding flow H_t.

The following theorem gives other interpretation of κ in the context of $\mathcal{P}$.

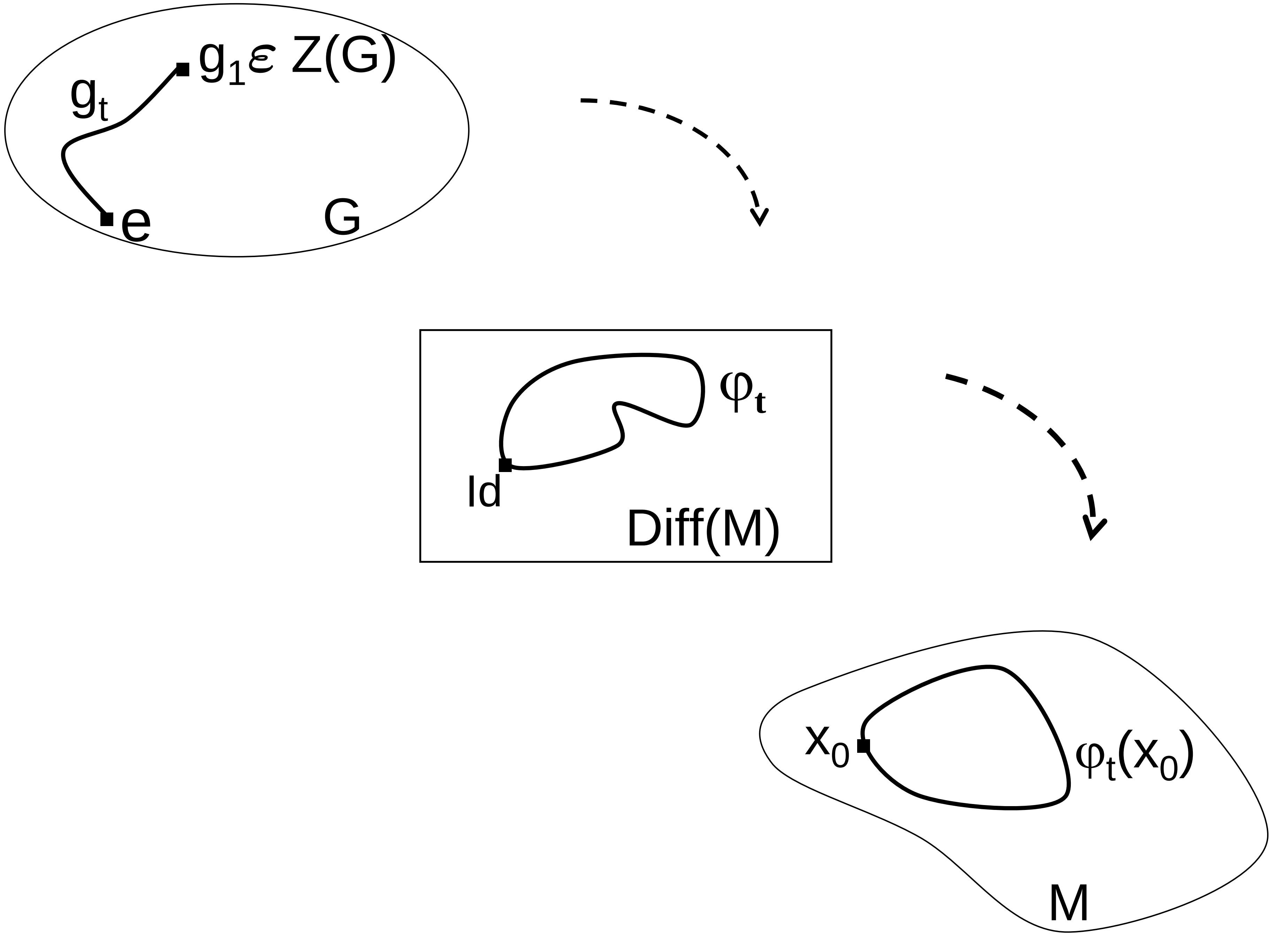
Theorem 4. If $g_1 \in Z(G)$, then H_1 is the gauge transformation

$$\mathsf{H}_1(p) = p\,\kappa.$$

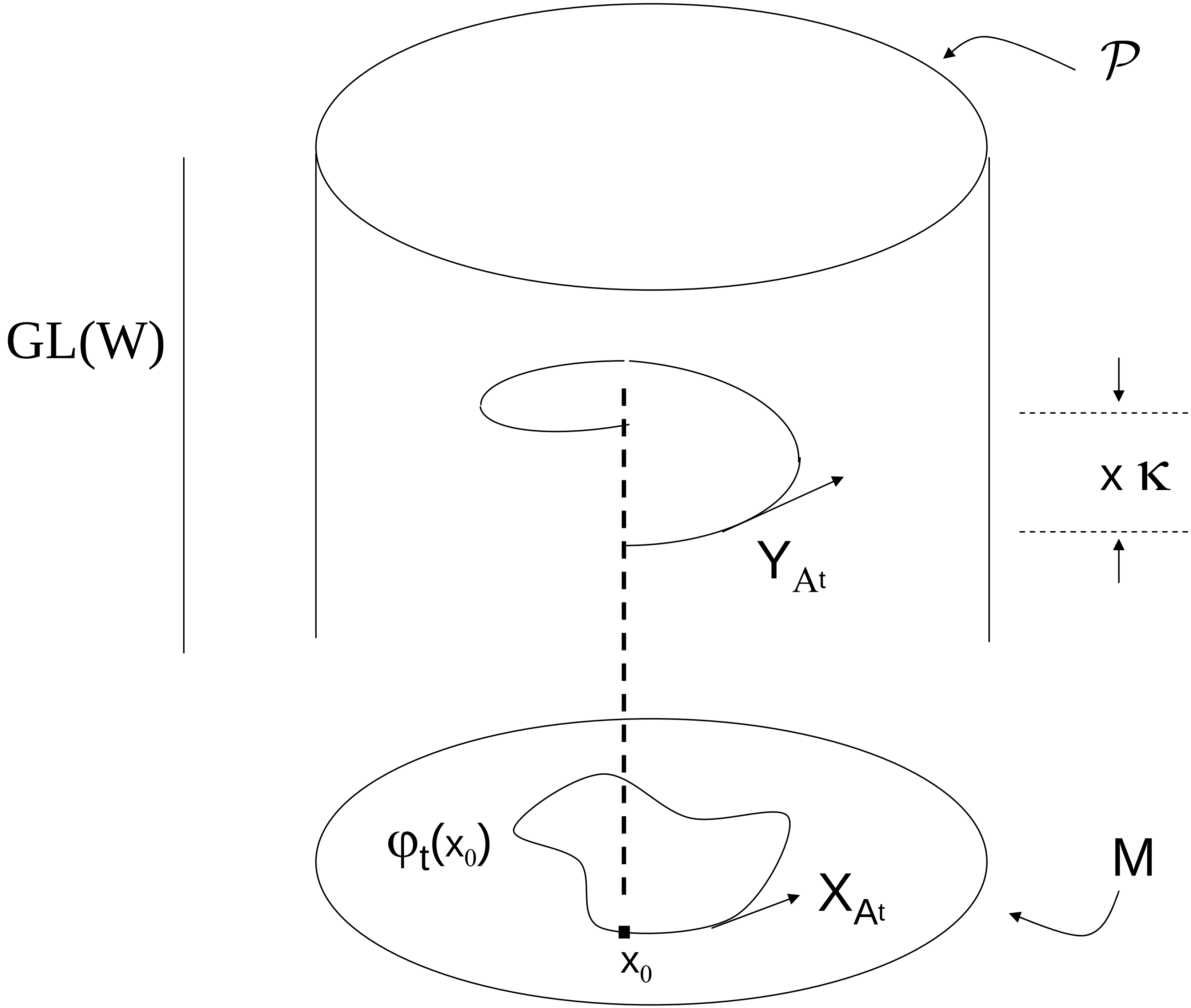
By the G-action on $M = G/T$, the path g_t determines an isotopy $\{\varphi_t \mid t \in [0,1]\}$ of M; that is,

$$\varphi_t(gT) = g_t gT.$$

If $g_1 \in Z(G)$, then $\{\varphi_t\}$ is a loop in $\mathrm{Diff}(M)$.



$g_1 \varepsilon Z(G)$
g_t
e
G
φ_t
Id
Diff(M)
x_0
$\varphi_t(x_0)$
M



$\mathcal{P}$
GL(W)
x κ
Y_{A_t}
$\varphi_t(x_0)$
M
X_{A_t}
x_0

The invariant κ also appears in the evolution of $\mathrm{GL}(W)$-equivariant W-valued functions on $\mathcal{P}$.

***Theorem 5*.** If $f_t : \mathcal{P} \to W$ is the family of equivariant maps solution of

$$\frac{d f_t}{dt} = -Y_{A_t}(f_t), \qquad f_0 = f,$$

then $f_1 = \kappa f$.

When G is compact and ϕ is a regular dominant weight, π is the representation provided by the Borel-Weil theorem.

In this case M is the flag variety of $\mathfrak{g}_{\mathbb{C}}$, i.e., a compact manifold diffeomorphic to the coadjoint orbit of $\phi \in \mathfrak{g}^*$. On M is defined the Kirillov form ϖ.

Furthermore, $\{\varphi_t\}$ is a loop in $\mathrm{Ham}(M,\varpi)$ and h_{A_t} the time-dependent Hamiltonian.

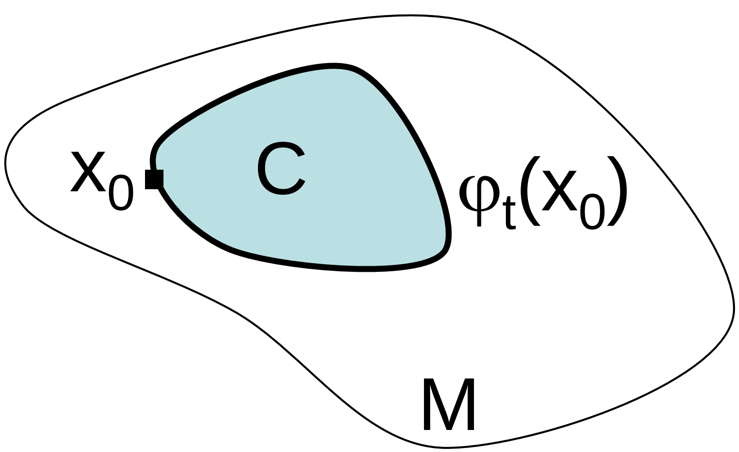
Given an arbitrary point $x_0 \in M$, the closed curve $\{\varphi_t(x_0) \mid t \in [0,1]\}$ is nullhomologous.

The symplectic action around the loop $\{\varphi_t\}$ is the element of $\mathbb{R}/\mathbb{Z}$.

$$\mathcal{SA}(\varphi) := \int_C \varpi + \int_0^1 h_{A_t}(\varphi_t(x_0))\,dt + \mathbb{Z},$$

C being a 2-chain whose boundary is

$$\{\varphi_t(x_0) \mid t \in [0,1]\}.$$

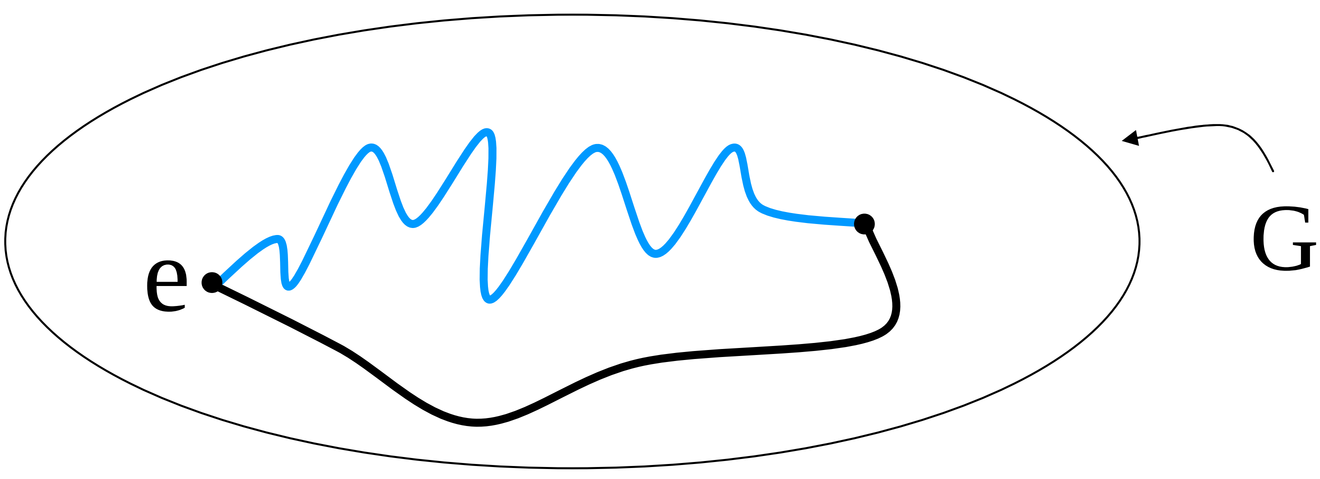


From the preceding theorem, it follows

Theorem 6. If G is compact, ϕ is a regular dominant weight and $g_1 \in Z(G)$, then

$$\kappa = \exp\left(\mathcal{SA}(\varphi)\right).$$

As a consequence, we deduce that $\exp\left(\mathcal{SA}(\varphi)\right)$ takes the same value for all the paths with end point at g_1.



INDEX

VIII. HOMOTOPY GROUP OF SUBGROUPS OF Diff(M)

Let $\mathfrak{X}(M)$ denote the Lie algebra of all vector fields on M. We will consider subalgebras $\mathfrak{X}'$ of $\mathfrak{X}(M)$, such that each $Z \in \mathfrak{X}'$ admits a lift to a vector field U on $\mathcal{P}$ satisfying $\mathfrak{L}_U \Omega = 0$, where Ω is the connection form on $\mathcal{P}$.

Let $\mathcal{G}$ be a connected Lie subgroup of $\mathrm{Diff}(M)$, which contains the isotopies associated with paths in G and such that $\mathrm{Lie}(\mathcal{G})$ is subalgebra of some $\mathfrak{X}'$.

Using the interpretation of κ as a gauge transformation which is the final point of a curve of automorphisms of $\mathcal{P}$. One can prove

Theorem 7.

$$\#\left\{\Psi(g) \mid g \in Z(G)\right\} \leq \#(\pi_1(\mathcal{G})).$$

***Corollary 8*.** If G is compact, ϕ is a regular dominant weight and $\mathcal{G}$ is any connected subgroup of $\mathrm{Ham}(M,\varpi)$ that contains G, then

$$\#\left\{\Phi(g) \mid g \in Z(G)\right\}$$

is a lower bound of $\mathrm{Card}\,(\pi_1(\mathcal{G}))$.

Example

For $G = \mathrm{SU}(2)$ the corresponding flag manifold is $\mathbb{C}P^1$.

Let ϕ be the weight of $T = U(1)$ defined by

$$\phi(\mathrm{diag}(ai, -ai)) = a.$$

The corresponding Kirillov symplectic structure ϖ is equal to $-2\pi\omega_{FS}$. So

$$\mathrm{Ham}(\mathbb{C}P^1, \varpi) \simeq \mathrm{Ham}(\mathbb{C}P^1, \omega_{FS}).$$

By the preceding Corollary

$$\#(\pi_1(\mathrm{Ham}(\mathbb{C}P^1, \varpi))) \geq 2.$$

On the other hand,

$$\pi_1(\mathrm{Ham}(\mathbb{C}P^1, \omega_{FS})) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Thus, lower bound given in the Corollary is precisely the cardinal of the homotopy group.

INDEX

IX. SCHEMATIC SUMMARY

π discrete series representation of a linear semisimple group.

(A) The differential representation π' in terms of quantization operators.

(B) Expression of the infinitesimal character as a polynomial of quantization operators.

(C) Four geometric descriptions of the invariant κ.

(D) Lower bounds for the cardinal of $\pi_1(\mathrm{Ham(M)})$.

INDEX

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