Noncommutative quantum mechanics from a Drinfel'd Twist

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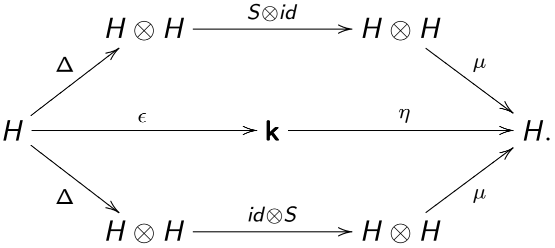
Hopf Algebras

A bialgebra with an antipode (coinverse):

- a vector space H over a field $\mathbf{k}$
- structures $\mu : H \otimes H \to H$ and $\eta : \mathbf{k} \to H$.
- costructures $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbf{k}$
- antipode $S : H \to H$ which is the inverse of the identity map with respect to the convolution operation

subject to the commutativity of the diagram



Drinfel'd Twist

Let H be cocommutative and take an element $\mathcal{F} \in H \otimes H$ which is invertible and is a *2-cocycle*, i.e.,

$$(\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F} = (\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}.$$

Notation: $\mathcal{F} = f^{\alpha} \otimes f_{\alpha}$ and $\mathcal{F}^{-1} = \bar{f}^{\alpha} \otimes \bar{f}_{\alpha}$, with α a multi-index.

$\mathcal{F}$ is called a *twist*.

Drinfel'd Twist

Defining (with $\chi = f^\alpha S(f_\alpha) \in H$)

$$\begin{aligned} \Delta^{\mathcal{F}}(a) &= \mathcal{F}\Delta(a)\mathcal{F}^{-1} \\ S^{\mathcal{F}}(a) &= \chi S(a)\chi^{-1}, \end{aligned}$$

$(H, \mu, \eta, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a triangular Hopf algebra with universal R-matrix given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$. Call it $H^{\mathcal{F}}$.

Universal Enveloping Algebra

Take a Lie algebra $\mathfrak{g}$ (with generators τ_i). Its universal enveloping algebra

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I,$$

with $T(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ the tensor algebra of $\mathfrak{g}$ and I the ideal generated by elements of the form $(x \otimes y - y \otimes x - [x, y])$, is a Hopf algebra with

$$\begin{aligned} \Delta(\tau_i) &= \tau_i \otimes \mathbf{1} + \mathbf{1} \otimes \tau_i \\ \varepsilon(\tau_i) &= 0 \\ S(\tau_i) &= -\tau_i. \end{aligned}$$

Drinfel'd Twist of the Universal Enveloping Algebra

$\mathcal{U}(\mathfrak{g})$ can be deformed into $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$.

One can ask which is the linear subspace $\mathfrak{g}^{\mathcal{F}} \subset \mathcal{U}^{\mathcal{F}}(\mathfrak{g})$, analogous to $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$.

Conditions on the generators of $\mathfrak{g}^{\mathcal{F}}$:

- $\{\tau_i^{\mathcal{F}}\}$ generates $\mathfrak{g}^{\mathcal{F}}$
- minimal deformation of the Leibniz rule: $\Delta^{\mathcal{F}}(\tau_i^{\mathcal{F}}) = \tau_i^{\mathcal{F}} \otimes \mathbf{1} + f_i^j \otimes \tau_j^{\mathcal{F}}$
- under deformed adjoint action $[\tau_i^{\mathcal{F}}, \tau_j^{\mathcal{F}}]_{\mathcal{F}} = (\tau_i^{\mathcal{F}})_1 \tau_j^{\mathcal{F}} S^{\mathcal{F}}((\tau_i^{\mathcal{F}})_2)$, the structure constants of $\mathfrak{g}$ are reproduced.[1]

[1] Sweedler indexless notation

Drinfel'd Twist of the Universal Enveloping Algebra

Take as deformed generators

$$\tau_i{}^{\mathcal{F}} = \bar{f}^{\alpha}(\tau_i)\bar{f}_{\alpha},$$

with coproduct

$$\Delta^{\mathcal{F}}(\tau_i{}^{\mathcal{F}}) = \tau_i{}^{\mathcal{F}} \otimes \mathbf{1} + \bar{R}^{\alpha} \otimes \bar{R}_{\alpha}(\tau_i{}^{\mathcal{F}}).$$

The dynamical Lie algebra

Start with the Heisenberg algebra $\mathfrak{h}_d = \{x_i, p_i, \hbar\}$ satisfying

$$[x_i, p_j] = i\hbar\delta_{ij}, \quad [\hbar, x_i] = [\hbar, p_i] = 0$$

and introduce the elements

$$\begin{aligned} H &= \frac{1}{2\hbar}\left(p_i p_i\right), \\ K &= \frac{1}{2\hbar}\left(x_i x_i\right), \\ D &= \frac{1}{4\hbar}\left(x_i p_i + p_i x_i\right), \\ L_{i_1 i_2 \cdots i_{d-2}} &= \frac{1}{\hbar}\epsilon_{i_1 i_2 \cdots i_{d-1} i_d} x_{i_{d-1}} p_{i_d}. \end{aligned}$$

Primitive vs. Composite Elements

Two-particle states $|\psi_1\rangle \otimes |\psi_2\rangle$:

- $\Delta(\vec{P}^2) = \vec{P}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \vec{P}^2$, $(\vec{P}^2_{tot} = \vec{P}^2_1 + \vec{P}^2_2)$
- $\Delta(\vec{L}^2) = \vec{L}^2 \otimes \mathbf{1} + 2\vec{L} \otimes \vec{L} + \mathbf{1} \otimes \vec{L}^2$, $(\vec{L}^2_{tot} = \vec{L}^2_1 + 2\vec{L}_1 \cdot \vec{L}_2 + \vec{L}^2_2)$

Primitiveness may be required on physical grounds.

The dynamical Lie algebra

The elements H, K, D and $L_{i_1 \ldots i_{d-2}}$ are now declared to be primitive elements of the enlarged Lie algebra

$$\mathcal{G}_d = \{\hbar, x_i, p_i, H, K, D, L_{i_1 i_2 \cdots i_{d-2}}\}, \; i = 1, ..., d.$$

The dynamical Lie algebra

For $d = 2$, the nonvanishing commutation relations read

$$\begin{aligned} [x_i, p_j] &= i\hbar\delta_{ij}, \\ [D, H] &= iH, \\ [D, K] &= -iK, \\ [K, H] &= 2iD, \\ [x_i, H] &= ip_i, \\ [x_i, D] &= \frac{i}{2}x_i, \end{aligned} \qquad \begin{aligned} [p_i, K] &= -ix_i, \\ [p_i, D] &= -\frac{i}{2}p_i, \\ [L, x_i] &= i\epsilon_{ij}x_j, \\ [L, p_i] &= i\epsilon_{ij}p_j. \end{aligned}$$

The dynamical Lie algebra

For $d = 3$, the nonvanishing commutation relations read

$$\begin{aligned} [x_i, p_j] &= i\hbar\delta_{ij}, \\ [D, H] &= iH, \\ [D, K] &= -iK, \\ [K, H] &= 2iD, \\ [x_i, H] &= ip_i, \\ [x_i, D] &= \frac{i}{2}x_i, \end{aligned}$$

$$\begin{aligned} [p_i, K] &= -ix_i, \\ [p_i, D] &= -\frac{i}{2}p_i, \\ [L_i, x_j] &= i\epsilon_{ijk}x_k, \\ [L_i, p_j] &= i\epsilon_{ijk}p_k, \\ [L_i, L_j] &= i\epsilon_{ijk}L_k. \end{aligned}$$

The dynamical Lie algebra

$\mathcal{U}(\mathcal{G}_d)$ can be deformed into $\mathcal{U}^{\mathcal{F}}(\mathcal{G}_d)$ by the Abelian twist

$$\mathcal{F} = \exp\left(i\alpha_{ij} p_i \otimes p_j\right), \quad \alpha_{ij} = -\alpha_{ji}.$$

The deformed generators are

$$\begin{aligned} x_i^{\mathcal{F}} &= x_i - \alpha_{ij} p_j \hbar, \\ K^{\mathcal{F}} &= K - \alpha_{ij} x_i p_j + \frac{\alpha_{jk}\alpha_{jl}}{2!} p_k p_l \hbar, \\ L^{\mathcal{F}}_{i_1 i_2 \cdots i_{d-2}} &= L_{i_1 i_2 \cdots i_{d-2}} - \epsilon_{i_1 i_2 \cdots i_{d-2} jk} \alpha_{jl} p_k p_l. \end{aligned}$$

The dynamical Lie algebra

This deformation yields the constant noncommutativity

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = i\Theta_{ij},$$

where

$$\Theta_{ij} = 2\alpha_{ij}\hbar^2.$$

Remark: The Jordanian twist $e^{iD\otimes \ln(\mathbf{1}+\xi H)}$ yields the Snyder noncommutativity.

The 2D harmonic oscillator

Consider the harmonic oscillator with Hamiltonian

$$\mathbf{H} = H + K.$$

The deformed Hamiltonian $\mathbf{H}^{\mathcal{F}} \in \mathcal{U}^{\mathcal{F}}(\mathcal{G}_2)$ is

$$\mathbf{H}^{\mathcal{F}} = H^{\mathcal{F}} + K^{\mathcal{F}} = H + K - \alpha x p_y + \alpha y p_x + \frac{\alpha^2}{2}\hbar(p_x^2 + p_y^2),$$

where $\alpha_{ij} = \epsilon_{ij}\alpha$.

The 2D harmonic oscillator

As usual, we introduce

$$\begin{aligned} a_i &= \frac{x_i - ip_i}{\sqrt{2}}, \\ a_i^\dagger &= \frac{x_i + ip_i}{\sqrt{2}}, \end{aligned}$$

so that

$$[a_i, a_j^\dagger] = \hbar\delta_{ij}.$$

The 2D harmonic oscillator

A change of basis will prove to be convenient:

$$
\begin{aligned}
b_{\pm} &= \frac{a_x \mp i a_y}{\sqrt{2}}, \\
b_{\pm}^{\dagger} &= \frac{a_x^{\dagger} \pm i a_y^{\dagger}}{\sqrt{2}}.
\end{aligned}
$$

They are creation and annihilation operators, because

$$
[b_{\pm}, b_{\pm}^{\dagger}] = \hbar
$$

and

$$
\begin{aligned}
[\mathbf{H}, b_{\pm}] &= -b_{\pm}, \\
[\mathbf{H}, b_{\pm}^{\dagger}] &= b_{\pm}^{\dagger}.
\end{aligned}
$$

The 2D harmonic oscillator

To calculate the single-particle spectrum, we set $\hbar = 1$.
The Hamiltonian can be written as

$$\mathbf{H} = \frac{1}{2}\sum_{i=\pm}\{b_i, b_i^\dagger\},$$

and the number operator and the angular momentum operator as

$$\begin{aligned} N &= b_+^\dagger b_+ + b_-^\dagger b_- = N_+ + N_-, \\ L &= b_+^\dagger b_+ - b_-^\dagger b_- = N_+ - N_-. \end{aligned}$$

The 2D harmonic oscillator

Since $[\mathbf{H}, L] = 0$, the $|n_+ n_-\rangle$ basis simultaneously diagonalizes both operators:

$$\begin{aligned} \mathbf{H}|n_+ n_-\rangle &= (n_+ + n_- + 1)|n_+ n_-\rangle, \\ L|n_+ n_-\rangle &= (n_+ - n_-)|n_+ n_-\rangle. \end{aligned}$$

Changing labels to $n = n_+ + n_-$ and $m = n_+ - n_-$, we have

$$\begin{aligned} \mathbf{H}|nm\rangle &= (n+1)|nm\rangle, \\ L|nm\rangle &= m|nm\rangle. \end{aligned}$$

The 2D harmonic oscillator

At the one-particle level **only**, the deformed Hamiltonian

$$\mathbf{H}^{\mathcal{F}} = H^{\mathcal{F}} + K^{\mathcal{F}} = H + K - \alpha x p_y + \alpha y p_x + \frac{\alpha^2}{2}(p_x^2 + p_y^2)$$

can be reproduced by the linear combination

$$\mathbf{H}^{\mathcal{F}} = \widetilde{\mathbf{H}} - \alpha L,$$

where

$$\widetilde{\mathbf{H}} = (1+\alpha^2)H + K$$

is just the undeformed Hamiltonian of an oscillator with frequency $\tilde{\omega} = \sqrt{1+\alpha^2}$.

The 2D harmonic oscillator

The spectrum of $\mathbf{H}^{\mathcal{F}}$ can now be easily computed:

$$\mathbf{H}^{\mathcal{F}}|nm\rangle \;=\; (\widetilde{\mathbf{H}} - \alpha L)|nm\rangle = \left[(\sqrt{1+\alpha^2})(n+1) - \alpha m\right]|nm\rangle,$$

where $m = -n, -n+2, \ldots, n-2, n$ and n is a non-negative integer.

The 2D harmonic oscillator

The energy of the first few states:

$$
\begin{aligned}
|0,0\rangle &: \sqrt{1+\alpha^2}, \\
|1,1\rangle &: 2\sqrt{1+\alpha^2}-\alpha, \\
|1,-1\rangle &: 2\sqrt{1+\alpha^2}+\alpha, \\
|2,2\rangle &: 3\sqrt{1+\alpha^2}-2\alpha, \\
|2,0\rangle &: 3\sqrt{1+\alpha^2}, \\
|2,-2\rangle &: 3\sqrt{1+\alpha^2}+2\alpha.
\end{aligned}
$$

The 2D harmonic oscillator

Since

$$\Delta^{\mathcal{F}}(\mathbf{H}^{\mathcal{F}}) \;=\; \mathcal{F} \cdot \Delta(\mathbf{H}^{\mathcal{F}}) \cdot \mathcal{F}^{-1},$$

as operators acting on $\mathcal{H} \otimes \mathcal{H}$, the undeformed and the deformed coproducts of the deformed Hamiltonian are unitarily equivalent:

$$\widehat{\Delta^{\mathcal{F}}}(\mathbf{H}^{\mathcal{F}}) \;=\; F \cdot \widehat{\Delta}(\mathbf{H}^{\mathcal{F}}) \cdot F^{-1}.$$

The same is true for the multi-particle operators of three or more particles:

$$\widehat{\Delta^{\mathcal{F}}_{(n)}}(\mathbf{H}^{\mathcal{F}}) = U_{(n)} \cdot \widehat{\Delta}(\mathbf{H}^{\mathcal{F}}) \cdot U_{(n)}^{-1}, \quad n \geq 2$$

The 2D harmonic oscillator

We are therefore entitled to use the undeformed coproduct, which is manifestly symmetric under particle exchange.

The two-particle Hamiltonian is

$$\begin{aligned}\Delta(\mathbf{H}^{\mathcal{F}}) &= \mathbf{H}^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H}^{\mathcal{F}} \\ &+\alpha(y \otimes p_x + p_x \otimes y - x \otimes p_y - p_y \otimes x) \\ &+\frac{\alpha^2}{2}\sum_{i=1}^{2}(2p_i\hbar \otimes p_i + 2p_i \otimes p_i\hbar + p_i^2 \otimes \hbar + \hbar \otimes p_i^2).\end{aligned}$$

Energy is no longer additive:

$$E_{12}^{\mathcal{F}} = E_1^{\mathcal{F}} + E_2^{\mathcal{F}} + \Omega_{12}.$$

The 2D harmonic oscillator

The three-particle Hamiltonian is explicitly given by

$$
\begin{aligned}
\Delta_{(2)}(\mathbf{H}^{\mathcal{F}}) \quad = \quad & \mathbf{H}^{\mathcal{F}} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{H}^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{H}^{\mathcal{F}} \\
& + \alpha(\mathbf{1} \otimes y \otimes p_x + y \otimes \mathbf{1} \otimes p_x + y \otimes p_x \otimes \mathbf{1}) \\
& + \alpha(\mathbf{1} \otimes p_x \otimes y + p_x \otimes \mathbf{1} \otimes y + p_x \otimes y \otimes \mathbf{1}) \\
& - \alpha(\mathbf{1} \otimes x \otimes p_y + x \otimes \mathbf{1} \otimes p_y + x \otimes p_y \otimes \mathbf{1}) \\
& - \alpha(\mathbf{1} \otimes p_y \otimes x + p_y \otimes \mathbf{1} \otimes x + p_y \otimes x \otimes \mathbf{1}) \\
& + \alpha^2 \sum_{i=1}^{2} [\mathbf{1} \otimes p_i \hbar \otimes p_i + p_i \hbar \otimes p_i \otimes \mathbf{1} + p_i \hbar \otimes p_i \otimes \mathbf{1} \\
& + \mathbf{1} \otimes p_i \otimes p_i \hbar + p_i \otimes p_i \hbar \otimes \mathbf{1} + p_i \otimes p_i \hbar \otimes \mathbf{1} \\
& + \hbar \otimes p_i \otimes p_i + p_i \otimes p_i \otimes \hbar + p_i \otimes p_i \otimes \hbar \\
& + \frac{1}{2}(\mathbf{1} \otimes \hbar \otimes p_i^2 + \hbar \otimes p_i^2 \otimes \mathbf{1} + \hbar \otimes p_i^2 \otimes \mathbf{1} \\
& + \mathbf{1} \otimes p_i^2 \otimes \hbar + p_i^2 \otimes \hbar \otimes \mathbf{1} + p_i^2 \otimes \hbar \otimes \mathbf{1})],
\end{aligned}
$$

The 2D harmonic oscillator

where the coassociativity of the coproduct

$$(id \otimes \Delta)\Delta(\mathbf{H}^{\mathcal{F}}) = (\Delta \otimes id)\Delta(\mathbf{H}^{\mathcal{F}}) \equiv \Delta_{(2)}(\mathbf{H}^{\mathcal{F}})$$

guarantees the associativity of the energy

$$E^{\mathcal{F}}_{123} \equiv E^{\mathcal{F}}_{(12)3} = E^{\mathcal{F}}_{1(23)} = E^{\mathcal{F}}_{1} + E^{\mathcal{F}}_{2} + E^{\mathcal{F}}_{3} + \Omega_{12} + \Omega_{23} + \Omega_{31} + \Omega_{123}.$$

The 3D harmonic oscillator

We can express

$$\alpha_{ij} = \epsilon_{ijk}\alpha_k$$

and then choose a reference frame where

$$\vec{\alpha} = (0, 0, \alpha).$$

The deformed Hamiltonian is then

$$\mathbf{H}^{\mathcal{F}} = H + K - \alpha(xp_y - yp_x) + \frac{\alpha^2}{2}\hbar(p_x^2 + p_y^2).$$

The 3D harmonic oscillator

To calculate the single-particle spectrum we are entitled to set $\hbar = 1$. We introduce the usual creation and annihilation operators and then perform the change of basis

$$\begin{aligned} b_{\pm} &= \frac{a_x \mp i a_y}{\sqrt{2}}, \\ b_{\pm}^{\dagger} &= \frac{a_x^{\dagger} \pm i a_y^{\dagger}}{\sqrt{2}}, \\ b_z &= a_z, \\ b_z^{\dagger} &= a_z^{\dagger}. \end{aligned}$$

The 3D harmonic oscillator

We write the Hamiltonian as

$$\mathbf{H} = \frac{1}{2}\sum_{i=\pm,z}\{b_i, b_i^\dagger\},$$

and introduce the operators

$$\begin{aligned} N_{xy} &= b_+^\dagger b_+ + b_-^\dagger b_- = N_+ + N_-, \\ N_z &= b_z^\dagger b_z, \\ L_z &= b_+^\dagger b_+ - b_-^\dagger b_- = N_+ - N_-. \end{aligned}$$

The 3D harmonic oscillator

We can use a basis labeled by the three non-negative integeres $n_\pm, n_z$, where

$$\begin{aligned} \mathbf{H}|n_+ n_- n_z\rangle &= \left(n_+ + n_- + n_z + \frac{3}{2}\right)|n_+ n_- n_z\rangle, \\ L_z|n_+ n_- n_z\rangle &= (n_+ - n_-)|n_+ n_- n_z\rangle. \end{aligned}$$

Changing to $n_{xy} = n_+ + n_-$ and $m = n_+ - n_-$, we have

$$\begin{aligned} \mathbf{H}|n_{xy} n_z m\rangle &= \left(n_{xy} + n_z + \frac{3}{2}\right)|n_{xy} n_z m\rangle, \\ L_z|nm\rangle &= m|n_{xy} n_z m\rangle. \end{aligned}$$

The 3D harmonic oscillator

We now split $\mathbf{H}$ into its xy-part and its z-part:

$$\mathbf{H} = \mathbf{H}_{xy} + \mathbf{H}_z,$$

where $\mathbf{H}_{xy} = \frac{1}{2}(x^2 + p_x^2 + y^2 + p_y^2)$ and $\mathbf{H}_z = \frac{1}{2}(z^2 + p_z^2)$.

This is feasible only at the one-particle level.

The deformation will only affect the xy-part.

The 3D harmonic oscillator

The deformed Hamiltonian

$$\mathbf{H}^{\mathcal{F}} = H + K - \alpha(xp_y - yp_x) + \frac{\alpha^2}{2}(p_x^2 + p_y^2)$$

can be written as

$$\mathbf{H}^{\mathcal{F}} = \widetilde{\mathbf{H}}_{xy} - \alpha L_z + \mathbf{H}_z,$$

where $\widetilde{\mathbf{H}}_{xy}$ a two-dimensional undeformed Hamiltonian with frequency $\tilde{\omega} = \sqrt{1+\alpha^2}$.

Isotropy is lost.

The 3D harmonic oscillator

The spectrum of $\mathbf{H}^{\mathcal{F}}$ is

$$\mathbf{H}^{\mathcal{F}}|n_{xy}n_zm\rangle = \left[\sqrt{1+\alpha^2}(n_{xy}+1) - \alpha m + \left(n_z + \frac{1}{2}\right)\right]|n_{xy}n_zm\rangle,$$

with $m = -n_{xy}, -n_{xy}+2, \ldots, n_{xy}-2, n_{xy}$.

The z-part of the Hamiltonian remains additive, so the multi-particle Hamiltonians are basically the same as in two-dimensonal case.

The 3D harmonic oscillator

First few states:

$$
\begin{aligned}
|0,0,0\rangle &: \frac{1}{2} + \sqrt{1+\alpha^2} \\
|0,1,0\rangle &: \frac{3}{2} + \sqrt{1+\alpha^2} \\
|1,0,-1\rangle &: \frac{1}{2} + 2\sqrt{1+\alpha^2} + \alpha \\
|1,0,1\rangle &: \frac{1}{2} + 2\sqrt{1+\alpha^2} - \alpha
\end{aligned}
$$

The 3D harmonic oscillator

$$
\begin{aligned}
|0,2,0\rangle &: \quad \frac{5}{2} + \sqrt{1+\alpha^2} \\
|1,1,-1\rangle &: \quad \frac{3}{2} + 2\sqrt{1+\alpha^2} + \alpha \\
|1,1,1\rangle &: \quad \frac{3}{2} + 2\sqrt{1+\alpha^2} - \alpha \\
|2,0,-2\rangle &: \quad \frac{1}{2} + 3\sqrt{1+\alpha^2} + 2\alpha \\
|2,0,0\rangle &: \quad \frac{1}{2} + 3\sqrt{1+\alpha^2} \\
|2,0,2\rangle &: \quad \frac{1}{2} + 3\sqrt{1+\alpha^2} - 2\alpha
\end{aligned}
$$

$\frac{1}{2}$ is the zero-point energy along the z-axis.

Rotational invariance in 2D

The undeformed generator of rotations on the plane satisfies

$$[L, {x_i}^{\mathcal{F}}] \;=\; i\epsilon_{ij}x_j^{\mathcal{F}}$$

and

$$[L, \mathbf{H}^{\mathcal{F}}] \;=\; 0.$$

The deformed oscillator retains its $so(2)$ invariance, even for multiparticle states:

$$\left[\Delta(\mathbf{H}^{\mathcal{F}}), \Delta(L)\right] = 0.$$

Rotational invariance in 3D

If we perform the same calculation for the L_i's in three dimensions, we obtain

$$[L_i, x_j^{\mathcal{F}}] = i\epsilon_{ijk}x_k^{\mathcal{F}} - i\hbar(\delta_{ij}\alpha p_z - p_i\alpha_j).$$

The second term on the right hand side vanishes only for $i = 3$.

Also $[\mathbf{H}^{\mathcal{F}}, L_i]$ only vanishes for $i = 3$.

So, L_z is a generator of rotational symmetry, while L_x and L_y are not, and thus the $so(3)$ invariance is broken down to an $so(2)$ invariance around the z-axis.

Rotational invariance in 3D

The same holds for multiparticle states, because

$$\begin{aligned}[\Delta(\mathbf{H}^{\mathcal{F}}), \Delta(L_i)] &= i\epsilon_{3ij}\left(\alpha L_j - 2\alpha^2 p_j p_z\right) \otimes \mathbf{1} \\ &\quad + \mathbf{1} \otimes i\epsilon_{3ij}\left(\alpha L_j - 2\alpha^2 p_j p_z\right) \\ &\quad - i\alpha(x_i \otimes p_z + p_z \otimes x_i - z \otimes p_i - p_i \otimes z)\end{aligned}$$

is zero only for $i = 3$.

Conclusions

- The single-particle spectrum of the quantum harmonic oscillator in the presence of a constant noncommutativity can be calculated in the framework of a Drinfel'd twist
- The costructures are required to unambiguously fix the multi-particle states
- Measuring multi-particle states is required to detect deformation
- The unitary equivalence between deformed and undeformed coproduct guarantees the symmetry under particle exchange
- In two dimensions, $so(2)$ invariance is retained
- In three dimensions, the $so(3)$ invariance is broken down to an $so(2)$ invariance