The Non-Uniqueness Problem of the Covariant Dirac Theory: “Conservative” vs. “Radical” Solutions

Mayeul Arminjon$^{1,2}$

$^1$ CNRS (Section of Theoretical Physics)


Geometry, Integrability & Quantization XIV, Varna (Bulgaria), June 8-13, 2012.
Experimental context

- Quantum effects in the classical gravitational field are observed on Earth for neutrons (spin $\frac{1}{2}$ particles) & atoms:
  - COW effect: gravity-induced phase shift measured by neutron (1975) and atom (1991a) interferometry;
  - Sagnac effect: Earth-rotation-induced phase shift measured by neutron (1979) and atom (1991b) interferometry;
  - Granit effect: Quantization of the energy levels proved by threshold in neutron transmission through a thin horizontal slit (2002).

- These are the only observed effects of the gravity-quantum coupling! Motivates work on curved-spacetime Dirac equation (thus first-quantized theory).
State of the art

- (Generally-)covariant rewriting of the Dirac eqn:

\[ \gamma^\mu D_\mu \Psi = -iM \Psi \quad (M \equiv mc/\hbar). \quad (1) \]

\( \gamma^\mu \): Dirac 4 \times 4 matrices. Verify anticommutation relation:

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}_4, \quad \mu, \nu \in \{0, \ldots, 3\}, \quad \mathbf{1}_4 \equiv \text{diag}(1, 1, 1, 1). \]

Here \((g^{\mu\nu}) \equiv (g_{\mu\nu})^{-1}\), with \(g_{\mu\nu}\) the components of the Lorentzian metric \(g\) on the SpaceTime manifold \(V\) in a local chart \(\chi: V \supset U \rightarrow \mathbb{R}^4\). Thus \(\gamma^\mu\) depend on \(X \in V\).

Wave function \(\psi\) is a section of a vector bundle \(E\) (“spinor bundle”) with base \(V\). \(\Psi: U \rightarrow \mathbb{C}^4\): local expression of \(\psi\) in a local frame field \((e_a)_{a=0,\ldots,3}\) on \(E\) over \(U\).

\[ D_\mu \equiv \partial_\mu + \Gamma_\mu, \text{ covariant derivatives. } \Gamma_\mu: 4 \times 4 \text{ matrices.} \]
State of the art (continued)

- For standard version (Dirac-Fock-Weyl, DFW): the field of the anticommuting Dirac matrices $\gamma^\mu$ is determined by an (orthonormal) tetrad field $(u_\alpha)$, i.e.,
  
  $V \ni X \mapsto u_\alpha(X) \in TV_X \quad (\alpha = 0, ..., 3)$.

- The tetrad field $(u_\alpha)$ may be changed by a “local Lorentz transformation” $L : V \to SO(1, 3)$, $\tilde{u}_\beta = L^{\alpha}_{\beta} u_\alpha$. Lifted to a “spin transformation” $S : V \to Spin(1, 3)$. $S$ is smooth if $V$ is topologically simple. Then the DFW eqn is covariant under changes of the tetrad field, thus the DFW eqn is unique.

- That covariance is got with the “spin connection” $D$ on the spinor bundle $E$. This connection *depends on the field of the Dirac matrices $\gamma^\mu$*, thus it depends on the tetrad field.
State of the art (end)

- DFW has been investigated in physical situations, notably
  - in a uniformly rotating frame in Minkowski SpaceTime
  - in a uniformly accelerating frame in Minkowski ST
  - in a static, or stationary, weak gravitational field.

- Differences with non-relativistic Schrödinger eqn with Newtonian potential: not currently measurable.

- First expected new effect with respect to non-relativistic Schrödinger eqn with Newtonian potential: “Spin-rotation coupling” in a rotating frame (Mashhoon 1988, Hehl-Ni 1990). Would affect the energy levels of a Dirac particle.

NB. For a physically relevant spacetime $V$, there are two explicit realizations of a spinor bundle $E$:

- $E = V \times \mathbb{C}^4$ (wave function is a complex four-scalar)
- $E = T_C V$ (wave function is a complex four-vector).

Surprising recent results

- Ryder (Gen. Rel. Grav. 2008) considered uniform rotation w.r.t. inertial frame in Minkowski ST. Found in this particular case:

  Mashhoon’s term in the DFW Hamiltonian operator $H$ is there for one tetrad field $(u_\alpha)$, is not for another one $(\tilde{u}_\alpha)$.

- Independently we identified in the most general case the relevant scalar product for the covariant Dirac eqn (M.A. & F. Reifler, arXiv:0807.0570 (gr-qc)/ Braz. J. Phys. 2010). And:

  Hermiticity of $H$ w.r.t. that scalar product depends on the choice of the admissible field $\gamma^\mu$. 
Surprising recent results (continued)

- This fact (instability of the hermiticity of $H$ under admissible changes of the $\gamma^{\mu}$ field) led us to a general study of the non-uniqueness problem of the covariant Dirac theory.

- As for this fact, we did that study for DFW, and for alternative versions of the covariant Dirac eqn.

- Found that, for any of these versions (standard, alternative), in any given reference frame:
  - The Hamiltonian operator $H$ is non-unique.
  - So is also the energy operator $E$ (Hermitian part of $H$)
  - The Dirac energy spectrum ($=\text{of } E$) is non-unique.
Local similarity (or gauge) transformations

Recall: in a curved spacetime \((V, g)\), the Dirac matrices \(\gamma^\mu\) depend on \(X \in V\).

If one changes from one admissible field \((\gamma^\mu)\) to another one \((\tilde{\gamma}^\mu)\), the new field obtains by a local similarity transformation (or local gauge transformation):

\[ \exists S = S(X) \in \text{GL}(4, \mathbb{C}) : \quad \tilde{\gamma}^\mu(X) = S^{-1} \gamma^\mu(X) S, \quad \mu = 0, ..., 3. \quad (2) \]

For the standard Dirac eq (DFW), the gauge transformations are restricted to the spin group \(\text{Spin}(1, 3)\), because they are got by lifting a local Lorentz transformation \(L(X)\) applied to a tetrad field. For the alternative eqs, they are general: \(S(X) \in \text{GL}(4, \mathbb{C})\).
The general Dirac Hamiltonian

Rewriting the covariant Dirac eqn in the “Schrödinger” form:

\[
i \frac{\partial \Psi}{\partial t} = H \Psi, \quad (t \equiv x^0),
\]

(3)

- \( H \) depends on the coordinate system, or more exactly on the reference frame — an equivalence class of charts defined on a given open set \( U \subset V \) and exchanging by

\[
x'^0 = x^0, \quad x'^j = f^j((x^k)) \quad (j, k = 1, 2, 3).
\]

(4)
Invariance condition of the Hamiltonian under a local gauge transformation

When does a gauge transfo. \( S(X) \), applied to the field of Dirac matrices \( \gamma^\mu \), leave \( H \) invariant? I.e., when do we have

\[
\tilde{H} = S^{-1} H S
\]

(5)

E.g. if the Dirac eqn is covariant under the local gauge transformation \( S \) (case of DFW), it is easy to see that we have (5) iff \( S(X) \) is time-independent, \( \partial_0 S = 0 \), independently of the explicit form of \( H \). (Other conditions for alternative eqs.)

In the general case \( g_{\mu\nu,0} \neq 0 \), any possible field \( \gamma^\mu \) depends on \( t \), and so does \( S \). Thus the Dirac Hamiltonian is not unique and one also proves that the energy operator and its spectrum are not unique. (M.A. & F. Reifler, Ann. der Phys. 2011)
Basic reason for the non-uniqueness

- Thus, in a given general reference frame or even in a given coordinate system, the Hamiltonian and energy operators associated with the generally-covariant Dirac eqn depend on the choice of the field of Dirac matrices \( X \mapsto \gamma^\mu(X) \).

- In contrast, in a given inertial reference frame or in a given Cartesian coordinate system, the Hamiltonian operator associated with the original Dirac eqn of special relativity is Hermitian and does not depend on the choice of the constant set of Dirac matrices \( \gamma^\#_\alpha \).


- Clearly, the non-uniqueness means there is too much choice for the field \( \gamma^\mu \) — too much gauge freedom.
Tetrad fields adapted to a reference frame

- The data of a reference frame $F$ fixes a unique four-velocity field $v_F$: the unit tangent vector to the world lines

$$X \in U, \quad x^0(X) \text{ variable,} \quad x^j(X) = \text{constant for } j = 1, 2, 3.$$ (6)

These world lines (invariant under an internal change (4)) are the trajectories of the particles constituting the reference frame $\Rightarrow$ a chart has physical content after all!

- Natural to impose on the tetrad field $(u_\alpha)$ the condition: time-like vector of the tetrad = four-velocity of the reference frame: $u_0 = v_F$.

- Then the spatial triad $(u_p)$ ($p = 1, 2, 3$) can only be rotating w.r.t. the reference frame. (Outline follows.)
Space manifold and spatial tensor fields

- Let $F$ be a reference frame, with its domain $U \subset V$. The set $M$ of the world lines is endowed with a natural structure of differential manifold: for any chart $\chi \in F$, its spatial part $\tilde{\chi} : M \ni x \mapsto (x^j)_{j=1,2,3}$ is a chart on $M$.

- Space manifold $M$ is frame-dependent and is not a 3-D submanifold of the spacetime manifold $V$!
  

- One then defines spatial tensor fields depending on the spacetime position, e.g. a spatial vector field:
  
  $U \ni X \mapsto u(X) \in TM_{x(X)}$, where, for $X \in U$, $x(X)$ = unique world line $x \in M$, s.t. $X \in x$. (See Eq. (6).)
Rotation rate tensor field of the spatial triad

- Again a reference frame $F$ is given. $\forall X \in U$, there is a canonical isomorphism between four-vectors $\perp v_F$ and spatial vectors:

$$H_X \equiv \{u_X \in TV_X \ ; \ g(u_X, v_F(X)) = 0\} \Rightarrow TM_x(X), \quad (7)$$

$u$ (with components $u^\mu, \mu = 0, \ldots, 3$ in some $\chi \in F$)

$\mapsto u$ (with components $u^j, j = 1, 2, 3$ in $\tilde{\chi}$).

(Independent of $\chi \in F$.)

- Then, $\exists$ one natural time-derivative for spatial vector fields. This allows one to geometrically define the rotation rate field $\Omega$ of the spatial triad field $(u_p)$ $(p = 1, 2, 3)$ associated with a tetrad field $(u_\alpha)$ $(\alpha = 0, \ldots, 3)$. MA, Ann. der Phys. 2011
Tetrad fields adapted to a reference frame (end)

> Two tetrad fields \((u_\alpha)\) and \((\tilde{u}_\alpha)\) s.t. \(u_0 = \tilde{u}_0 = v_F\), and with the same rotation rate \(\Xi = \tilde{\Xi}\), exchange by a time-independent Lorentz transformation. Hence they give rise in \(F\) to equivalent Hamiltonian operators and to equivalent energy operators.

> Two natural ways to fix the tensor field \(\Xi\) are: i) \(\Xi = \Omega\), where \(\Omega\) is the unique rotation rate field of the given reference frame \(F\), and ii) \(\Xi = 0\).

> Either choice, i) or ii), thus provides a solution to the non-uniqueness problem. These two solutions are not equivalent, so that experiments would be required to decide between the two. Moreover, each solution is valid only in a given reference frame.
Getting unique Hamiltonian & energy operators in any reference frame at once?

- The invariance condition of the Hamiltonian $H$ after a gauge transfo. for DFW: $\partial_0 S = 0$, is coordinate-dependent. This condition implies also the invariance of the energy operator $E$ for DFW.

$\Rightarrow$ The stronger condition $\partial_\mu S = 0 \ (\mu = 0, \ldots, 3)$ implies the invariance of both $H$ and $E$ simultaneously in any chart (hence in any reference frame), for DFW.
Getting unique Hamiltonian & energy operators in any reference frame at once? (continued)

- Alternative versions of covariant Dirac eqn: the invariance conditions of $H$ and $E$ contain $D_\mu S$. But, for the “QRD–0” version, we define the connection matrices to be

$$\Gamma_{\mu} = 0 \quad \text{in the canonical frame field } (E_a) \text{ of } V \times \mathbb{C}^4, \quad (8)$$

so we have by construction $\partial_\mu S = D_\mu S$ for QRD–0.

- Thus, if we succeed in restricting the choice of the $\gamma^\mu$ field so that any two choices exchange by a constant gauge transfo. ($\partial_\mu S = 0$), we solve the non-uniqueness problem simultaneously in any reference frame — for both DFW and QRD–0, and only for them.
Fixing one tetrad field in each chart

In a chart, a tetrad \((u_{\alpha})\) is defined by a matrix \(a \equiv (a^{\mu}_{\alpha})\), s.t. \(u_{\alpha} = a^{\mu}_{\alpha} \partial_{\mu}\). Orthonormality of the tetrad in the metric with matrix \(G \equiv (g_{\mu\nu}) = G(X)\) \((X \in V)\):

\[
    b^T \eta b = G \quad [b \equiv a^{-1}, \quad \eta \equiv \text{diag}(1, -1, -1, -1)]. \tag{9}
\]

Generalized Cholesky decomposition (Reifler 2008): \(\exists! \ b = C\): lower triangular solution of (9) with \(C^{\mu}_{\mu} > 0, \ \mu = 0, \ldots, 3\).

\rightarrow a unique tetrad in a given chart: “Cholesky prescription”. One other known prescription (Kibble 1963) has this property. Both coincide for a “diagonal metric”:

\(G = \text{diag}(d_{\mu}) \Rightarrow u_{\alpha} \equiv \delta^{\mu}_{\alpha} \partial_{\mu}/\sqrt{|d_{\mu}|}, “\text{diagonal tetrad}”.\)
The reference frame, not the chart, is physically given

- What is physically given is the reference frame: a three-dimensional congruence of time-like world lines.

- Given a reference frame $F$, there remains a whole functional space of different choices for a chart $\chi \in F$. 
Fixing one tetrad field in each chart is not enough

Consider a prescription (e.g. “Cholesky“): \( \chi \mapsto a \mapsto (u_\alpha) \). For two different charts \( \chi, \chi' \in F \), we get two tetrad fields \( (u_\alpha), (u'_\alpha) \) with matrices \( a, a' \). We have \( u'_\beta = L^\alpha_\beta u_\alpha \), with

\[
L = b \, P \, a', \quad b \equiv a^{-1}, \quad P^\mu_\nu \equiv \frac{\partial x^\mu}{\partial x'^\nu}.
\] (10)

\( b \) and \( a' \) depend on \( t \equiv x^0 = x'^0 \) as do \( G \) and \( G' \). Since \( \chi, \chi' \in F \), the matrix \( P \) doesn’t depend on \( t \), Eq. (4).

In general, the dependences on \( t \) of \( b \) and \( a' \) don’t cancel each other in Eq. (10).

Thus in general the Lorentz transformation \( L \) depends on \( t \).

\( \Rightarrow \) \( L \) is lifted to a gauge transformation \( S \) depending on \( t \).

\( \Rightarrow \) \( H \) and \( H' \) not equivalent: The non-uniqueness still there.
The case with a diagonal metric

Consider the Cholesky prescription applied to a “diagonal metric”: \( G = \text{diag}(d_\mu) \) \( (d_0 > 0, \; d_j < 0, \; j = 1, 2, 3) \). Some algebra gives us

\[
\frac{\partial}{\partial t} \left( L_p^3 \right) \propto P_3^p (P_3^j)^2 \frac{\partial}{\partial t} \left( \frac{d_j}{d_p} \right) \quad \text{(no sum on } p = 1, 2, 3),
\]

with a non-zero proportionality factor. Thus in general \( \frac{\partial}{\partial t} \left( L_p^3 \right) \neq 0 \), non-uniqueness of \( H \) and \( E \) still there.

Exception: \( d_j(X) = d_j^0 \; h(X) \) with \( d_j^0 \) constant \( (d_j^0 < 0 \text{ with } h > 0) \). Then after changing \( x'^j = x^j \sqrt{-d_j^0} \), we get

\[
d_j' = -h \; (j = 1, 2, 3), \quad \text{or}
\]

\[
G \equiv (g_{\mu \nu}) = \text{diag}(f, -h, -h, -h), \quad f > 0, \; h > 0. \quad (12)
\]
Space-isotropic diagonal metric

**Theorem** (M.A., arXiv:1205.3386). Let the metric have the space-isotropic diagonal form (12) in some chart \( \chi \). Let \( \chi' \) belong to the same reference frame \( R \).

(i) The metric has the form (12) also in \( \chi' \), iff \((x^j) \mapsto (x'^j)\) is a constant rotation, combined with a constant homothecy.

(ii) If one applies the “diagonal tetrad” prescription in each of the two charts, the two tetrads obtained thus are related together by a constant Lorentz transformation \( L \), hence give rise, in any reference frame \( F \), to equivalent Hamiltonian operators as well to equivalent energy operators — for the DFW and QRD–0 versions of the Dirac equation.
Uniformly rotating frame in flat spacetime

Let $\chi' : X \mapsto (ct', x', y', z')$ be a Cartesian chart in the Minkowski spacetime, thus $g'_{\mu\nu} = \eta_{\mu\nu}$. Defines inertial frame $F'$.

Go from $\chi'$ to $\chi : X \mapsto (ct, x, y, z)$ defining uniformly rotating ref. frame $F$ ($\omega = \text{constant}$):

\[
t = t', \quad x = x' \cos \omega t + y' \sin \omega t, \quad y = -x' \sin \omega t + y' \cos \omega t, \quad z = z'.
\]

(13)

With $\rho \equiv \sqrt{x^2 + y^2}$, the Minkowski metric writes in the chart $\chi$:

\[
g_{00} = 1 - \left(\frac{\omega \rho}{c}\right)^2, \quad g_{01} = -g_{02} = \frac{\omega}{c}, \quad g_{03} = 0, \quad g_{jk} = -\delta_{jk}.
\]

(14)

4-velocity of $F$ : $v = \partial_0 / \sqrt{g_{00}} \Rightarrow g(v, \partial_j) \neq 0$.

Each of Ryder’s (2008) two tetrads has $u_0 = v' \neq v$:

Each is adapted to the inertial frame, not to the rotating frame.
A tetrad adapted to the rotating frame

Adopt the “rotating cylindrical” chart $\chi^\circ$, also belonging to the rotating frame $F$. Related to the “rotating Cartesian” chart (13):

$$
\chi^\circ : X \mapsto (ct, \rho, \varphi, z) \quad \text{with} \quad x = \rho \cos \varphi, \quad y = \rho \sin \varphi.
$$

(15)

Define $u_0 \equiv v$, $u_p \equiv \Pi \partial_p / \parallel \Pi \partial_p \parallel$, where $\Pi = \perp$ projection onto the hyperplane $\perp v$. This is an orthonormal tetrad adapted to $F$, because for the chart $\chi^\circ$ we have $g(u_p, u_q) = 0$, $1 \leq p \neq q \leq 3$.

Rotation rate tensor of $(u_p)$: $\Xi_{pq} = -c \frac{d\tau}{dt} \gamma_{pq0}$. Here $\Xi_{pq} = 0$ except for

$$
\Xi_{21} = -\Xi_{12} = \frac{\omega}{\sqrt{1 - (\omega \rho)^2 / c^2}}.
$$

(16)

Differs from rotation rate tensor $\Omega$ of the rotating frame $F$ only by $O(V^2/c^2)$ terms ($V \equiv \omega \rho \ll c$).
Explicit expression of the Dirac Hamiltonian operator

Hamiltonian operator for the generally-covariant Dirac eqn (1):

\[ H = mc^2 \alpha^0 - i\hbar c \alpha^j D_j - i\hbar c \Gamma_0, \]  
(17)

where

\[ \alpha^0 \equiv \gamma^0 / g^{00}, \quad \alpha^j \equiv \gamma^0 \gamma^j / g^{00}. \]  
(18)

Spin connection matrices with an orthonormal tetrad field \((u_\alpha)\):

\[ \Gamma^\#_c = \frac{1}{8} \gamma_{\alpha\beta\epsilon} \left[ \gamma^\#_{\alpha}, \gamma^\#_{\beta} \right]. \quad (\gamma^\#_{\alpha} = \text{“flat” Dirac matrices}) \]  
(19)

Spin connection matrices with the natural basis \((\partial_\mu = b^\alpha_\mu u_\alpha)\):

\[ \Gamma_\mu = b^\alpha_\mu \Gamma^\#_\alpha. \]  
(20)
Hamiltonian for adapted rotating tetrad

Using the foregoing expressions, it is straightforward to compute \( H \) in the rotating frame \( F \) with the adapted rotating tetrad. We find that the spin connection matrices \( \Gamma_\mu \) do involve spin operators made with the Pauli matrices \( \sigma^j \). In particular, we have for \( V \equiv \omega \rho \ll c \):

\[
\Gamma_0 = -\frac{i}{2} \frac{\omega}{c} \Sigma^3 \left[ 1 + O\left(\frac{V}{c}\right) \right], \quad \Sigma^j \equiv \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix},
\]

where

\[
\Sigma^j \equiv \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix},
\]

for which \( -i\hbar c \Gamma_0 \) is the usual “spin-rotation coupling” term in \( H \).

Also the \( \Gamma_j \) matrices \((j = 1, 2, 3)\) contain spin operators. Likely to come from the fact that the adapted rotating tetrad involves projecting the natural tetrad of the rotating coordinates.
$H$ for rotating frame with $\gamma^\mu$ matrices from Minkowski tetrad (“gauge freedom restricted” soln)

Defining the $\gamma^\mu$ matrices from the “diagonal tetrad” prescription in the Cartesian chart $\chi'$, and transforming them to the rotating chart $\chi$, gives after a simple calculation:

$$H = H' - i\hbar\omega(y\partial_x - x\partial_y) = H' - \omega \cdot L,$$

(22)

with $H' \equiv$ special-relativistic Dirac Hamiltonian in the inertial frame $F'$, and $L \equiv r \wedge (-i\hbar\nabla)$: angular momentum operator.

NB. The same $H$ applies, whether DFW or QRD–0 is chosen. (The spin connection matrices are zero.)

Thus, there is no spin-rotation coupling with the “gauge freedom restriction” solution of the non-uniqueness problem.
Conclusion

• Non-unique Hamiltonian and energy operators in covariant Dirac theory: due to gauge freedom in choice of $\gamma^\mu$ matrices. (Yet standard covariant Dirac eqn is unique by construction.)

• “Conservative” way of restricting the gauge freedom: fix vector $u_0$, then fix rotation rate of triad $(u_p)$. Applies to a given reference frame. Uneasy to implement. Spin-rotation coupling.

• “Radical” way: arrange that same gauge freedom applies as in special relativity — constant gauge transformations. Needs diagonal space-isotropic metric. (Always valid in “scalar ether theory”. Other metrics?) Applies independently of reference frame. Easy to implement. No spin-rotation coupling.