Matrix generalizations of integrable systems with Lax integro-differential representations

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We consider linear space $\zeta$ of micro-differential operators over the field $\mathbb{C}$ of the following form:

$$L \in \zeta = \left\{ \sum_{i=-\infty}^{n(L)} a_i D^i : n(L) \in \mathbb{Z} \right\},$$

where coefficients $a_i$ are functions dependent on spatial variable $x = t_1$ and evolution parameters $t_2, t_3, \ldots$. Coefficients $a_i(t), t = (t_1, t_2, \ldots)$, are supposed to be smooth functions of vector variable $t$ that has a finite number of elements that belong to some functional space $A$. This space is a differential algebra under arithmetic operations. An operator of differentiation is denoted in the following way: $D := \frac{\partial}{\partial x}$. 
Addition and multiplication of operators by scalars (elements of the field \( C \)) are introduced in the following way:

\[
\lambda_1 L_1 \pm \lambda_2 L_2 = \sum_{i=-\infty}^{N_1} \lambda_1 a_1 i D^i \pm \sum_{i=-\infty}^{N_2} \lambda_2 a_2 i D^i = \max<\!N_1, N_2\!> \sum_{i=-\infty} \left( \lambda_1 a_1 i \pm \lambda_2 a_2 i \right) D^i, \lambda_1, \lambda_2 \in C.
\]

The structure of Lie algebra on a linear space \( \zeta \) (1) is defined by the commutator \([\cdot, \cdot] : \zeta \times \zeta \to \zeta\), \([L_1, L_2] = L_1 L_2 - L_2 L_1\), where the composition of micro-differential operators \( L_1 \) and \( L_2 \) is induced by general Leibniz rule:
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\[ D^n f := \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)} D^{n-j}, \]

\[ n \in \mathbb{Z}, \ f \in A \subset \zeta, \ f^{(j)} := \frac{\partial^j f}{\partial x^j} \in A \subset \zeta, \ D^n D^m = D^m D^n = D^{n+m}, \]

\[ n, m \in \mathbb{Z}, \text{ where } \binom{n}{0} := 1, \binom{n}{j} := \frac{n(n-1)\ldots(n-j+1)}{j!} . \]

Formula (2) defines the composition of the operator \( D^n \in \zeta \) and the operator of multiplication by function \( f \in A \subset \zeta \) in contradistinction to the denotation \( D^k \{ f \} := \frac{\partial^k f}{\partial x^k} \in A, \ k \in \mathbb{Z}_+ \).
Consider a microdifferential Lax operator:

\[ L := WDW^{-1} = D + \sum_{i=1}^{\infty} U_i D^{-i}, \]  

which is parametrized by the infinite number of dynamic variables \( U_i = U_i(t_1, t_2, t_3, \ldots), \) \( i \in \mathbb{N}, \) which depend on an arbitrary (finite) number of independent variables \( t_1 := x, t_2, t_3, \ldots \) All dynamic variables \( U_i \) can be expressed in terms of functional coefficients of formal dressing Zakharov-Shabat operator:

\[ W = I + \sum_{i=1}^{\infty} w_i D^{-i}, \]  

The inverse of formal operator \( W \) has the form:

\[ W^{-1} = I + \sum_{i=1}^{\infty} a_i D^{-i}. \]
In scalar case, Kadomtsev-Petviashvili hierarchy is a commuting family of evolution Lax equations for the operator $L$ (3)

$$\alpha_i L_{t_i} = [B_i, L] := B_i L - LB_i,$$

(6)

where $\alpha_i \in \mathbb{C}$, $i \in \mathbb{N}$, the operator $B_i := (L^i)_+$ is a differential part of the $i$-th power of microdifferential symbol $L$. By symbol $L_{t_i}$ we will denote the following operator:

$$L_{t_i} := (WDW^{-1})_{t_i} = \sum_{j=1}^{\infty} (U_j)_{t_i} D^{-j}.$$

(7)

Formally transposed and conjugated operators $L^\tau, L^*$ have the form:
Introduction

\[ L^\tau := -D + \sum_{j=1}^{\infty} (-1)^j D^{-j} U_j, \quad L^* := \bar{L}^\tau. \]  

(8)

Zakharov-Shabat equations are consequences of the commutativity of two arbitrary flows in (6) with \( i = m \) and \( i = n \)

\[ L_{tmtn} = L_{tntm} \Rightarrow \]

\[ \Rightarrow [\alpha_n \partial_{tn} - B_n, \alpha_m \partial_{tm} - B_m] = \alpha_m B_{ntm} - \alpha_n B_{mtn} + [B_n, B_m] = 0. \]  

(9)
Symmetry reductions of the KP–hierarchy

References


Symmetry reductions of the KP–hierarchy

References

- A. M. Samoilenko, V. G. Samoilenko and Yu. M. Sidorenko
Symmetry reductions of the KP–hierarchy

Consider a symmetry reduction of the KP-hierarchy, which is a generalization of the Gelfand-Dickey k-reduction:

\[(L^k)_- := (L^k)_{<0} = q \mathbb{M}_0 D^{-1} r^\top = \]
\[= \int_x^\infty q(x, t_2, t_3, \ldots) \mathbb{M}_0 r^\top (s, t_2, t_3, \ldots) \cdot ds, \quad (10)\]

where Mat_{l \times l}(\mathbb{C}) \ni \mathbb{M}_0 is a constant matrix, and functions \(q = (q_1, \ldots, q_l), \ r = (r_1, \ldots, r_l)\) are fixed solutions of the following system of differential equations:

\[
\begin{cases}
\alpha_n q_{tn} = B_n \{q\}, \\
\alpha_n r_{tn} = -B_n^\top \{r\},
\end{cases}
\quad (11)
\]

where \(n \in \mathbb{N}\).
Symmetry reductions of the KP–hierarchy

Reduced flows (6), (10), (11) admit Lax representation

\[
[L_k, M_n] = 0, \quad L_k = B_k + q M_0 D^{-1} r^\top, \quad M_n = \alpha_n \partial_t n - B_n. \quad (12)
\]

Equation (12) is equivalent to the \((1 + 1)\)-dimensional integrable systems for functional coefficients \(U_i, i = 1, k - 1\) and functions \(q, r\):

\[
\begin{cases}
U_{itn} = P_{in}[U_1, U_2, ..., U_{k-1}, q, r], \\
q_{tn} = B_n[U_i, q, r]\{q\}, \quad r_{tn} = -B_n^\top[U_i, q, r]\{r\},
\end{cases} \quad (13)
\]

where \(i = 1, k - 1\), \(P_{in}\) and \(B_n\) are differential polynomials with respect to dynamic variables that are indicated in square brackets.
Symmetry reductions of the KP–hierarchy

(2+1)-dimensional generalizations of Lax representations (12) have the form:

$$[L_k, M_n] = 0, \quad (14)$$

where $L_k$ is (2+1)-dimensional integro-differential operator:

$$L_k = \alpha \partial_y - B_k - q M_0 D^{-1} r^\top, \quad (15)$$

and $M_n$ in (14) is evolutional differential operator of $n$-th order with respect to spatial variable $x$:

$$M_n = \alpha_n \partial_{t_n} - \sum_{j=1}^{n} v_j D^j \quad (16)$$
Consider examples of equations (12)-(13) and their generalizations (14)-(16) for some \( k \) and \( n \):

1. \( k = 1, n = 2 \):

\[
L_1 = D + q\mathcal{M}_0 D^{-1} q^* , M_2 = \alpha_2 \partial_{t_2} - D^2 - 2q\mathcal{M}_0 q^*,
\]

where \( \alpha_2 \in i\mathbb{R}, \mathcal{M}_0^* = \mathcal{M}_0 \).

Equation \([L_1, M_2] = 0\) is equivalent to nonlinear Schrodinger equation (NLS):

\[
\alpha_2 q_{t_2} = q_{xx} + 2(q\mathcal{M}_0 q^*) q. \tag{17}
\]
Symmetry reductions of the KP–hierarchy

Now let us consider spatially two-dimensional generalizations of the operators $L_1$, $M_2$:

$$L_1 = \partial_y - q M_0 D^{-1} q^*, \quad M_2 = \alpha_2 \partial_{t_2} - c_1 D^2 - 2c_1 S_1,$$

where $\alpha_2 \in i\mathbb{R}$, $S_1 = S_1(x, y, t_2) = \overline{S}_1(x, y, t_2), c_1 \in \mathbb{R}$

Lax equation $[L_1, M_2] = 0$ is equivalent to Davey-Stewartson system DS-III:

$$\begin{cases} 
\alpha_2 q_{t_2} = c_1 q_{xx} - 2c_1 S_1 q \\
S_1_y = (q M_0 q^*)_x 
\end{cases}$$

System (19) is spatially two-dimensional $l$–component generalization of NLS.
2. $k = 2$, $n = 2$:
\[ L_2 = D^2 + 2u + qM_0D^{-1}q^*, \quad M_2 = \alpha_2 \partial_t - D^2 - 2u, \]
where $M_0^* = -M_0$, $u = \bar{u}$, $\alpha_2 \in i\mathbb{R}$.
Operator equation $[L_2, M_2] = 0$ is equivalent to Yajima-Oikawa system:

\[
\begin{cases}
\alpha_2 q_{t_2} = q_{xx} + 2uq, \\
\alpha_2 u_{t_2} = (qM_0q^*)_x.
\end{cases}
\] (20)
(2+1)-dimensional generalization of the operators $L_2, M_2$ have a form:

$$L_2 = i \partial_y - D^2 - 2u - qM_0D^{-1}q^*,$$
$$M_2 = \alpha_2 \partial_{t_2} - D^2 - 2u,$$

where $\alpha_2 \in i\mathbb{R}$, $M_0 = -M^*_0$, $u = \bar{u}$.

Equation $[L_2, M_2] = 0$ can be represented in the following way:

$$\begin{cases} 
\alpha_2 u_{t_2} = iu_y + (qM_0q^*)_x \\
\alpha_2 q_{t_2} = q_{xx} + 2uq.
\end{cases}$$

System (21) is $l$–component spatially two-dimensional generalization of the Yajima-Oikawa system.
Symmetry reductions of the KP–hierarchy

3. \( k = 2, n = 3 \):

\[ L_2 = D^2 + 2u + qM_0 D^{-1} q^*, \]

\[ M_3 = \alpha_3 \partial_t - D^3 - 3uD - \frac{3}{2}(u_x + qM_0 q^*), \]

where \( M_0 = -M_0^* \), \( u = \bar{u} \), \( \alpha_3 \in \mathbb{R} \).

Equation \([L_2, M_3] = 0\) is equivalent to the system:

\[
\begin{aligned}
\alpha_3 q_{t_3} &= q_{xxx} + 3uq_x + \frac{3}{2}u_x q + \frac{3}{2}qM_0 q^* q, \\
\alpha_3 u_{t_3} &= \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}(q_x M_0 q^* - q M_0 q^*_x)_x.
\end{aligned}
\] (22)
Symmetry reductions of the KP–hierarchy

Spatially two-dimensional generalizations of $L_2$ and $M_3$ have the form:

\[
L_2 = i \partial_y - D^2 - 2u - qM_0D^{-1}q^* ,
\]
\[
M_3 = \alpha_3 \partial_{t_3} - D^3 - 3uD - \frac{3}{2} (u_x + iD^{-1}\{u_y\} + qM_0q^*) ,
\]

where $\alpha_3 \in \mathbb{R}$, $M_0^* = -M_0$, $u = \bar{u}$. Equation $[L_2, M_3] = 0$ is equivalent to the system:

\[
\begin{cases}
\alpha_3 q_{t_3} = q_{xxx} + 3uq_x + \frac{3}{2} (u_x + iD^{-1}\{u_y\} + qM_0q^*) q , \\
[\alpha_3 u_{t_3} - \frac{1}{4} u_{xxx} - 3uu_x + \frac{3}{4} (qM_0q_x^* - q_xM_0q^*)_x + \\
-\frac{3}{4} i (qM_0q^*)_y]_x = -\frac{3}{4} u_{yy} .
\end{cases}
\]

Equations (24) generalize Mel’nikov system.
Symmetry reductions of the KP–hierarchy

References

Symmetry reductions of the KP–hierarchy

In other cases we will obtain:

4. Vector generalization of the modified Korteweg-de Vries equation ($k = 1, n = 3$):

\[ \alpha_3 q_{t_3} = q_{xxx} + 3 (q M_0 q^*) q_x + 3 (q_x M_0 q^*) q, \quad M_0^* = M_0. \] (25)

5. Generalization of the Boussinesq equation ($k = 3, n = 2$):

\[
\begin{cases}
3 \alpha_2^2 u_{t_2 t_2} = (-u_{xx} - 6 u^2 + 4(q M_0 q^*))_{xx}, \quad M_0^* = M_0, \\
\alpha_2 q_{t_2} - q_{xx} - 2 u q = 0,
\end{cases}
\] (26)

6. Vector generalization of the Drinfeld-Sokolov system ($k = 3, n = 3$):

\[
\begin{cases}
\alpha_3 q_{t_3} = q_{xxx} + 3 u q_x + \frac{3}{2} u_x q, \quad q = \bar{q}, \quad M_0^* = M_0, \\
\alpha_3 u_{t_3} = (q M_0 q^\top)_x,
\end{cases}
\] (27)
Exact solutions of some nonlinear models from the KP-hierarchy

In this section we will consider the construction of exact solutions of the integrable systems from the KP-hierarchy. For this reason we will use invariant transformations for linear integro-differential operators from the previous section. Consider the integro-differential operator:

\[ L := \alpha \partial_t - \sum_{i=0}^{n} u_i D^i + q M_0 D^{-1} r^T, \quad \alpha \in i\mathbb{R} \cup \mathbb{R} \quad (28) \]

with \((N \times N)\)-matrix coefficients \( u_i = u_i(x, t); \Lambda, \tilde{\Lambda}\) and \(M_0\) are \((K \times K)\) and \((l \times l)\)-matrices correspondingly; \(q, r\) are \((N \times l)\)-matrices. Assume that \((N \times K)\)-matrix functions \(\varphi, \psi\) satisfy linear problems: \(L\{\varphi\} = \varphi \Lambda, L^\tau\{\psi\} = \psi \tilde{\Lambda}.\)
Define the binary Darboux-type transformation (BT) as:

\[ W = I - \varphi \left( C + D^{-1}\{\psi^\top\varphi}\right)^{-1} D^{-1}\psi^\top \]  

(29)

The following theorem holds:

**Theorem 1**

Let functions \( f \) and \( g \) be \((N \times 1)\)-solutions of the linear systems

\[ L\{f\} = f\lambda, \quad L^\tau\{g\} = g\tilde{\lambda}. \]

Then, functions \( F = W\{f\} \), \( G = W^{-1,\tau}\{g\} \) satisfy equations

\[ \hat{L}\{F\} = F\lambda, \quad \hat{L}^{-1,\tau}\{G\} = G\tilde{\lambda} \]

with the operator

\[ \hat{L} = \alpha \partial_t - \sum_{i=0}^{n} \hat{u}_i D^i + \hat{q} M_0 D^{-1} \hat{r}^\top + \Phi M D^{-1} \psi^\top, \]  

(30)

where \( M = C\Lambda - \tilde{\Lambda}^\top C \), \( \Phi = \varphi \left( C + D^{-1}\{\psi^\top\varphi}\right)^{-1} \),

\( \psi^\top = \left( C + D^{-1}\{\psi^\top\varphi}\right)^{-1} \psi^\top \), \( \hat{q} = W\{q\} \), \( \hat{r} = W^{-1,\tau}\{r\} \).
Exact solutions of some nonlinear models from the KP-hierarchy

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Consider two possible realizations of the integration operator $D^{-1}$ in BT (29):

$$W^+ = I - \varphi \left( C + \int_{-\infty}^{x} \psi^\top(s)\varphi(s) ds \right)^{-1} \int_{-\infty}^{x} \psi^\top(s) \cdot ds, \quad (31)$$

$$W^- = I - \varphi \left( C + \int_{-\infty}^{x} \psi^\top(s)\varphi(s) ds \right)^{-1} \int_{+\infty}^{x} \psi^\top(s) \cdot ds, \quad (32)$$

under assumption that the components of $(N \times K)$-matrix functions $\varphi$ and $\psi$ decrease rapidly at both infinities. A composition of operators $(W^+)^{-1}$ and $W^-$ gives Fredholm operator:

$$S_R = (W^+)^{-1}W^- = I + \varphi C^{-1} \int_{-\infty}^{+\infty} \psi^\top(s) \cdot ds. \quad (33)$$
Assume that integral part in $L$ is equal to zero. Using the equalities $L\{\varphi\} = \varphi \Lambda$, $L^\tau\{\psi\} = \psi \tilde{\Lambda}$ we obtain that the commutator of $S_R$ and $L$ has the form:

$$[L, S_R] = \varphi C^{-1} \mathcal{M} \int_{-\infty}^{+\infty} C^{-1} \psi^\top(s) \cdot ds, \quad \mathcal{M} = C \Lambda - \tilde{\Lambda}^\top C$$

Using $W^+$, $W^-$ as the dressing operators for $L$ we obtain that:

$$\hat{L}_1 = W^+ L (W^+)^{-1} = (\hat{L}_1)_+ + \Phi \mathcal{M} \int_{-\infty}^{X} \psi^\top(s) \cdot ds,$$
$$\hat{L}_2 = W^- L (W^-)^{-1} = (\hat{L}_2)_+ + \Phi \mathcal{M} \int_{+\infty}^{X} \psi^\top(s) \cdot ds, \quad (34)$$
$$\quad (\hat{L}_1)_+ = (\hat{L}_2)_+$$
If we put $\Lambda = \tilde{\Lambda} = 0$ and consider the differential operator $L$ ($M_0 = 0$), then using transformations $W^+$ or $W^-$ we obtain the differential operator $\hat{L}$. In this case, $[L, S_R] = 0$. Thus, we obtain dressing due to Zakharov-Shabat.

**References**

Now we will consider realizations of integral transformation $W$ (29) and construction of the solutions for integrable systems from the KP-hierarchy. At first we will consider the scalar NLS

$$iq_t = q_{xx} + 2\mu|q|^2q, \; \mu = \pm 1,$$

(35)

and its vector generalization – Manakov system ($l$ components):

$$i(q_j)_t = (q_j)_{xx} + 2\left(\sum_{s=1}^l \mu_s |q_s|^2\right)q_j, \; \mu_s = \pm 1, j = 1, \ldots, l.$$  

(36)
References

Proposition 1

Let function \( \varphi := (\varphi_1, \ldots, \varphi_K) \) be a fixed solution of the system:

\[
\begin{aligned}
\varphi_x &= \varphi \Lambda, \\
i\varphi_t &= \varphi_{xx},
\end{aligned}
\]

(37)

where \( \Lambda \in \text{Mat}_{K \times K}(\mathbb{C}) \).

Let \( f := (f_1, \ldots, f_l) \) be an arbitrary solution of the problem

\[
i f_t = f_{xx}.
\]

(38)

Then functions \( F := f - \varphi(C + \Omega[\varphi, \varphi])^{-1}\Omega[\varphi, f], \Phi = \varphi(C + \Omega[\varphi, \varphi])^{-1}, \) where

\[
\Omega[\varphi, \varphi] = \int_{(x_0, t_0)}^{(x, t)} \varphi^* \varphi dx + i(\varphi_x^* \varphi - \varphi^* \varphi_x) dt,
\]

\[
\Omega[\varphi, f] = \int_{(x_0, t_0)}^{(x, t)} \varphi^* f dx + i(\varphi_x^* f - \varphi^* f_x) dt, \quad C = C^* \in \text{Mat}_{K \times K}(\mathbb{C})
\]
satisfy equations:
Exact solutions of some nonlinear models from the KP-hierarchy

\[ iF_t = F_{xx} + 2\Phi \hat{\mathcal{M}} \Phi^* F, \quad (39) \]

\[ i\Phi_t = \Phi_{xx} + 2\Phi \hat{\mathcal{M}} \Phi^* \Phi, \quad (40) \]

where \( \hat{\mathcal{M}} = \mathcal{C}\Lambda + \Lambda^* \mathcal{C} - (\varphi^* \varphi)(x_0, t_0) \)
Using proposition 1, we can obtain K-soliton solution of NLS $(\mu = 1)$ of the following structure:

$$q = \frac{\det \left( \begin{array}{cc} \Delta_2 & \vec{1} \\ \varphi & 0 \end{array} \right)}{\det(\Delta_2)},$$

where $\varphi_j = \gamma_j e^{\lambda_j x + i\lambda_j^2 t}$, $\gamma_j, \lambda_j \in \mathbb{C}$, $j = \overline{1,K}$; $\vec{1}$ is a row-vector (K-components) consisting of 1,

$$\Delta_2 = \left( \frac{1}{\lambda_j + \bar{\lambda}_s (\varphi_j \varphi_s + 1)} \right)^K_{j,s=1}$$

Animation 1 describes the behavior of 3-soliton solution ($|q|$ and $\text{Re}(q)$) with $\lambda_1 = 1.5 + i$, $\lambda_2 = 1 + 2i$, $\lambda_3 = 2.5 + 3.5i$ and $\gamma_1 = e$, $\gamma_2 = e^{10}$, $\gamma_3 = e^5$. 
We can also use Proposition 1 for obtaining other kinds of solutions (e.g. bound states) for NLS and constructing solutions of vector generalization of NLS.

Animation 2 describes the behavior of NLS solution, consisting of 1 bound state and 1 soliton.

Animation 3 represents the absolute value of the solution $(\lambda_1 = 2 - 3i, \lambda_2 = 1 + 2i, \gamma_1 = e^{100}, \gamma_2 = e^{10})$ for 2-component NLS generalization of the form:

$$i(q_j)_t = (q_j)_{xx} + 2\left(|q_1|^2 - |q_2|^2\right)q_j, \ j = 1, 2$$  \hspace{1cm} (41)
Exact solutions of some nonlinear models from the KP-hierarchy

Similar types of solutions for other integrable systems of the KP-hierarchy can also be constructed. In particular, one of bound-state solutions of the Yajima-Oikawa system

\[
\begin{align*}
    iq_{t_2} &= q_{xx} + 2uq, \\
    iu_{t_2} &= (\mu |q|^2)_x;
\end{align*}
\]  

(42)

in case \( \mu = -i \) is presented on animation 4 (\( \lambda = 3 + i, \gamma = e^5 \)). Animation 5 describes the behavior of 2-soliton solution of Drinfeld-Sokolov system:

\[
\begin{align*}
    q_{t_3} &= q_{xxx} + 3uq_x + \frac{3}{2}u_xq, \\
    u_{t_3} &= (q^2)_x.
\end{align*}
\]  

(43)
Exact solutions of some nonlinear models from the KP-hierarchy

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Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Consider the generalizations of operators $L_1$, $M_2$ (18):

$$L_1 = \partial_y - q M_0 D^{-1} q^*,$$

$$M_2 = \alpha_2 \partial_{t_2} - c_1 D^2 - c_2 \partial^2_y + 2 c_1 S_1 + 2 c_2 q M_0 D^{-1} \partial_y q^*, \quad (44)$$

where $c_1, c_2 \in \mathbb{R}$, $\alpha_2 \in i\mathbb{R}$, $q = q(x, y, t)$ and $S_1 = S_1(x, y, t) = S_1^*(x, y, t)$ are matrix functions with dimensions $N \times l$ and $N \times N$ respectively; $M_0 = M_0^*$ is a constant $(l \times l)$-dimensional matrix.

Lax equation $[L_1, M_2] = 0$ is equivalent to the system:

$$\begin{cases}
\alpha_2 q_{t_2} = c_1 q_{xx} + c_2 q_{yy} - 2 c_1 S_1 q - 2 c_2 q M_0 S_2, \\
S_1 y = (q M_0 q^*)_x, \quad S_2 x = (q^* q)_y.
\end{cases} \quad (45)$$
Integro-differential Lax representations for Davey-Stewartson and
Chen-Lee-Liu equations

In scalar case \((N = 1, l = 1)\), by taking
\[ S = c_1 S_1 + c_2 S_2, \]
\[ \mu := \mathcal{M}_0 = 1, \]
we obtain the following differential consequence from (45):

\[
\begin{align*}
\alpha_2 q_{t_2} &= c_1 q_{xx} + c_2 q_{yy} - 2S q, \\
S_{xy} &= c_1 |q|^2_{xx} + c_2 |q|^2_{yy}.
\end{align*}
\]

(46)

If \(c_1 = -c_2 = c \in \mathbb{R}\) we obtain Davey-Stewartson system (DS-I)
from (46).
Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Consider the following pair of operators:

\[ L_1 = \partial \bar{z} - q D_z^{-1} \bar{q}, \]
\[ M_2 = \alpha_2 \partial_{t_2} - c D_{zz}^2 + c \partial_{\bar{z}z}^2 + 2c S_1 - 2c q D_z^{-1} \bar{q} \bar{z} - 2c q D_z^{-1} \bar{q} \partial \bar{z}, \]  

(47)

where \( \alpha_2, c \in i \mathbb{R} \);
\( q \) and \( S_1 \) are \( (N \times N) \)-matrices, \( z = x + iy \). Lax equation \([L_1, M_2] = 0\) is equivalent to the system:

\[
\begin{align*}
\alpha_2 q_{t_2} &= -icq_{xy} - 2c S_1 q + 2c q \bar{S}_1, \\
S_1 x + iS_1 y &= (q \bar{q})_x - i(q \bar{q})_y. 
\end{align*}
\]

(48)
Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

In scalar case ($N = 1$) we obtain the following differential consequence from system (48):

\[
\begin{align*}
\alpha_2 q_{t_2} &= -ic q_{xy} - 4ic \tilde{S}q, \\
\tilde{S}_{xx} + \tilde{S}_{yy} &= -4|q|^2_{x_2y_2}.
\end{align*}
\]  

(49)

System (49) is Davey-Stewartson system (DS-II).
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Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Consider the following pair of operators:

\[ L_1 = \partial_y - q \mathcal{M}_0 D^{-1} r^\top D, \tag{50} \]

\[ M_2 = \alpha_2 \partial_{t_2} - c_1 D^2 - c_2 \partial_y^2 + 2c_1 S_1 D + 2c_2 q \mathcal{M}_0 D^{-1} \partial_y r^\top D, \tag{51} \]

where \( q = q(x, y, t_2), \ r = r(x, y, t_2) \) and \( S_1 = S_1(x, y, t_2) \) are matrix functions with dimensions \((N \times M)\) and \((N \times N)\) respectively; \( \mathcal{M}_0 \) is a constant \((M \times M)\)-dimensional matrix. Equation \([L_1, M_2] = 0\) is equivalent to the following system:

\[
\begin{align*}
\alpha_2 q_{t_2} - c_1 q_{xx} - c_2 q_{yy} + 2c_1 S_1 q_x - 2c_2 q \mathcal{M}_0 S_2 + & + 2c_2 q \mathcal{M}_0 (r^\top q)_y = 0, \\
\alpha_2 r_{t_2}^\top + c_1 r_{xx}^\top + c_2 r_{yy}^\top + 2c_1 r_x^\top S_1 + 2c_2 S_2 \mathcal{M}_0 r^\top = 0, \\
S_{1y} = (q \mathcal{M}_0 r^\top)_x + [q \mathcal{M}_0 r^\top, S_1], & S_{2x} = (r^\top q)_y.
\end{align*}
\tag{52} \]
Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

a). Under additional conditions $\alpha_2 \in i\mathbb{R}$, $c_1, c_2 \in \mathbb{R}$, $M_0 = -M_0^*$, $r^\top = q^*$, $S_1 = S_1^*$ operators $L_1$ (50) and $M_2$ (51) are $D$-skew-Hermitian ($L_1^* = -DL_1D^{-1}$) and $D$-Hermitian ($M_2^* = D M_2 D^{-1}$). System (52) has a form:

$$\begin{align*}
\left\{ \begin{array}{l}
\alpha_2 q_{t_2} - c_1 q_{xx} - c_2 q_{yy} + 2c_1 S_1 q_x + 2c_2 q M_0 S_2 = 0, \\
S_{1y} = (q M_0 q^*)_x + [q M_0 q^*, S_1], \\
S_{2x} = (q^* q_x)_y.
\end{array} \right.
\end{align*} \tag{53}$$

Consider a scalar case of equation (53) ($N = 1, M = 1$) and take $c_2 = 0$, $y = x$, $\mu := M_0$. Then we obtain Chen-Lee-Liu equation (DNLS-II) from (53):

$$\alpha_2 q_{t_2} - c_1 q_{xx} + 2c_1 \mu |q|^2 q_x = 0. \tag{54}$$
Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

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b). We will put $\mathcal{M}_0 r^\top(x, y, t_2) = \nu$, where $\nu$ is $(M \times N)$-dimensional constant matrix. After the change $u := q \nu$ system (52) takes the form:

$$\begin{align*}
\alpha_2 u_{t_2} - c_1 u_{xx} - c_2 u_{yy} + 2c_1 S_1 u_x + 2c_2 uu_y &= 0, \\
S_1 y &= u_x + [u, S_1].
\end{align*} \tag{55}$$

System (55) is $(2+1)$-dimensional matrix generalization of Burgers equation. It can be generalized onto $(n+1)$-dimensional case:

$$\begin{align*}
\alpha_2 u_{t_2} &= \Delta u - 2S \nabla u, \\
\frac{\partial S_i}{\partial x_i} &= \frac{\partial u}{\partial x_i} + [u, S_1], \quad i = 1, n, \\
\end{align*} \tag{56}$$

where $S = (S_1, \ldots S_n)$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. 
Proposition

Let \( T := T(x, y, t_2) \) be \((N \times N)\)-matrix function that satisfies equation:

\[
\alpha_2 T_{t_2} = c_1 T_{xx} + c_2 T_{yy}
\]  

(57)

Then \((N \times N)\)-matrix functions

\[
u := -T^{-1} T_y, \quad S_1 = -T^{-1} T_x.
\]  

(58)

satisfy system (55).
Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Remark

It can be checked that functions $u$, $S_1$ defined by formula (58) satisfy another version of (2+1)-dimensional generalization of matrix Burgers equation:

\[
\begin{align*}
\alpha_2 u_{t_2} - c_1 u_{xx} - c_2 u_{yy} + 2c_1 S_1 u_x + 2c_2 uu_y &= 0, \\
\alpha_2 S_1_{t_2} - c_1 S_1_{xx} - c_2 S_1_{yy} + 2c_1 S_1 S_1_x + 2c_2 uS_1_y &= 0
\end{align*}
\]  
(59)
It is also constructed the integro-differential representation for the equation:

$$\alpha_3 q_{t3} + c_1 q_{xxx} - c_2 q_{yyy} - 3c_1 \mu q_x \int |q|^2_x dy +$$

$$3c_2 \mu q_y \int |q|^2_y dx + 3c_2 \mu q \int (\bar{q} q_y)_y dx - 3c_1 \mu q \int (q_x q)_x dy = 0. \tag{60}$$

where $\alpha_3, \mu, c_1, c_2 \in \mathbb{R}$, which can be reduced to the mKdV equation:

$$\alpha_3 q_{t3} + q_{xxx} - 6\mu q^2 q_x = 0. \tag{61}$$
Lax integro-differential representation was also constructed for the following system:

\[ \begin{align*}
\alpha_3 q_{t_3} + c_1 q_{xxx} - c_2 q_{yyy} - 3c_1 v_1 q_{xx} - 3c_1 v_3 q_x + 3\mu c_2 q_y D^{-1}\{\bar{q} q_x\}_y + \\
+ 3c_2 \mu q D^{-1}\{\bar{q} q_{xy}\}_y - 3c_2 \mu^2 q D^{-1}\{|q|^2 \bar{q} q_x\}_y = 0,
\end{align*} \]

\[ v_{1y} = \mu(|q|^2)_x, \]

\[ v_{3y} = \mu(q_x \bar{q})_x - 2\mu v_1(|q|^2)_x, \quad (62) \]

where \( \alpha_3, c_1, c_2 \in \mathbb{R}, \mu \in i\mathbb{R}, v_1 = v_1^*, v_3 + v_3^* = v_{1x} \), which reduces to the higher Chen-Lee-Liu equation (\( c_1 = 1, c_2 = 0 \)):

\[ \alpha_3 q_{t_3} + q_{xxx} - 3\mu|q|^2 q_{xx} - 3\mu \bar{q} q_x^2 + 3\mu^2|q|^4 q_x = 0. \]
Thank you for your attention!