Matrix generalizations of integrable systems with Lax integro-differential representations

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Contents

Introduction

We consider linear space ζ of micro-differential operators over the field C of the following form:

$$L \in \zeta = \left\{ \sum_{i=-\infty}^{n(L)} a_i D^i : \; n(L) \in \mathbf{Z} \right\}, \tag{1}$$

where coefficients a_i are functions dependent on spatial variable $x = t_1$ and evolution parameters t_2, t_3 .... Coefficients $a_i(t)$, $t = (t_1, t_2, \ldots)$, are supposed to be smooth functions of vector variable t that has a finite number of elements that belong to some functional space A. This space is a differential algebra under arithmetic operations. An operator of differentiation is denoted in the following way: $D := \frac{\partial}{\partial x}$.

Introduction

Addition and multiplication of operators by scalars (elements of the field C) are introduced in the following way:

$$\lambda_1 L_1 \pm \lambda_2 L_2 = \sum_{i=-\infty}^{N_1} \lambda_1 a_{1i} D^i \pm \sum_{i=-\infty}^{N_2} \lambda_2 a_{2i} D^i =$$

$$\sum_{i=-\infty}^{\max<N_1,N_2>} (\lambda_1 a_{1i} \pm \lambda_2 a_{2i}) D^i, \lambda_1, \lambda_2 \in \mathbf{C}.$$

The structure of Lie algebra on a linear space ζ (1) is defined by the commutator $[\cdot,\cdot] : \zeta \times \zeta \to \zeta$, $[L_1, L_2] = L_1 L_2 - L_2 L_1$, where the composition of micro-differential operators L_1 and L_2 is induced by general Leibniz rule:

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$$D^n f := \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)} D^{n-j}, \qquad (2)$$

$n \in \mathbf{Z}$, $f \in A \subset \zeta$, $f^{(j)} := \frac{\partial^j f}{\partial x^j} \in A \subset \zeta$, $D^n D^m = D^m D^n = D^{n+m}$, $n, m \in \mathbf{Z}$, where $\binom{n}{0} := 1$, $\binom{n}{j} := \frac{n(n-1)\dots(n-j+1)}{j!}$.
Formula (2) defines the composition of the operator $D^n \in \zeta$ and the operator of multiplication by function $f \in A \subset \zeta$ in contradistinction to the denotation $D^k\{f\} := \frac{\partial^k f}{\partial x^k} \in A$, $k \in \mathbf{Z}_+$.

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Consider a microdifferential Lax operator:

$$L := WDW^{-1} = D + \sum_{i=1}^{\infty} U_i D^{-i}, \tag{3}$$

which is parametrized by the infinite number of dynamic variables $U_i = U_i(t_1, t_2, t_3, ...)$, $i \in \mathbb{N}$, which depend on an arbitrary (finite) number of independent variables $t_1 := x$, t_2, t_3, ... All dynamic variables U_i can be expressed in terms of functional coefficients of formal dressing Zakharov-Shabat operator:

$$W = I + \sum_{i=1}^{\infty} w_i D^{-i}, \tag{4}$$

The inverse of formal operator W has the form:

$$W^{-1} = I + \sum_{i=1}^{\infty} a_i D^{-i}. \tag{5}$$

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In scalar case, Kadomtsev -Petviashvili hierarchy is a commuting family of evolution Lax equations for the operator L (3)

$$\alpha_i L_{t_i} = [B_i, L] := B_i L - L B_i, \tag{6}$$

where $\alpha_i \in \mathbb{C}$, $i \in \mathbb{N}$, the operator $B_i := (L^i)_+$ is a differential part of the i-th power of microdifferential symbol L.
By symbol L_{t_i} we will denote the following operator:

$$L_{t_i} := (WDW^{-1})_{t_i} = \sum_{j=1}^{\infty} (U_j)_{t_i} D^{-j}. \tag{7}$$

Formally transposed and conjugated operators L^τ, L^* have the form:

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$$L^\tau := -D + \sum_{j=1}^{\infty}(-1)^j D^{-j} U_j, L^* := \bar{L}^\tau. \quad (8)$$

Zakharov-Shabat equations are consequences of the commutativity of two arbitrary flows in (6) with $i = m$ and $i = n$

$$L_{t_m t_n} = L_{t_n t_m} \Rightarrow$$

$$\Rightarrow [\alpha_n \partial_{t_n} - B_n, \alpha_m \partial_{t_m} - B_m] = \alpha_m B_{n t_m} - \alpha_n B_{m t_n} + [B_n, B_m] = 0. \quad (9)$$

Symmetry reductions of the KP–hierarchy

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Symmetry reductions of the KP–hierarchy

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Symmetry reductions of the KP–hierarchy

Consider a symmetry reduction of the KP-hierarchy, which is a generalization of the Gelfand-Dickey k-reduction:

$$(L^k)_- := (L^k)_{<0} = \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top = \\ = \int^x \mathbf{q}(x, t_2, t_3, \dots)\mathcal{M}_0\mathbf{r}^\top(s, t_2, t_3, \dots)\cdot ds, \quad (10)$$

where $\mathrm{Mat}_{l\times l}(\mathbb{C}) \ni \mathcal{M}_0$ is a constant matrix, and functions $\mathbf{q} = (q_1, ..., q_l)$, $\mathbf{r} = (r_1, ..., r_l)$ are fixed solutions of the following system of differential equations:

$$\begin{cases} \alpha_n \mathbf{q}_{t_n} = B_n\{\mathbf{q}\}, \\ \alpha_n \mathbf{r}_{t_n} = -B_n^\tau\{\mathbf{r}\}, \end{cases} \quad (11)$$

where $n \in \mathbb{N}$.

Symmetry reductions of the KP–hierarchy

Reduced flows (6), (10), (11) admit Lax representation

$$[L_k, M_n] = 0, \; L_k = B_k + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top, \; M_n = \alpha_n \partial_{t_n} - B_n. \quad (12)$$

Equation (12) is equivalent to the $(1+1)$-dimensional integrable systems for functional coefficients U_i , $i = \overline{1, k-1}$ and functions $\mathbf{q}$, $\mathbf{r}$:

$$\begin{cases} U_{it_n} = P_{in}[U_1, U_2, ..., U_{k-1}, \mathbf{q}, \mathbf{r}], \\ \mathbf{q}_{t_n} = B_n[U_i, \mathbf{q}, \mathbf{r}]\{\mathbf{q}\}, \; \mathbf{r}_{t_n} = -B_n^\tau[U_i, \mathbf{q}, \mathbf{r}]\{\mathbf{r}\}, \end{cases} \quad (13)$$

where $i = \overline{1, k-1}$, P_{in} and B_n are differential polynomials with respect to dynamic variables that are indicated in square brackets.

Symmetry reductions of the KP–hierarchy

(2+1)-dimensional generalizations of Lax representations (12) have the form:

$$[L_k, M_n] = 0, \tag{14}$$

where L_k is (2+1)-dimensional integro-differential operator:

$$L_k = \alpha \partial_y - B_k - \mathbf{q} \mathcal{M}_0 D^{-1} \mathbf{r}^\top, \tag{15}$$

and M_n in (14) is evolutional differential operator of n-th order with respect to spatial variable x:

$$M_n = \alpha_n \partial_{t_n} - \sum_{j=1}^{n} v_j \mathcal{D}^j \tag{16}$$

Symmetry reductions of the KP–hierarchy

Consider examples of equations (12)-(13) and their generalizations (14)-(16) for some k and n:
1. $k = 1, n = 2$:
$L_1 = D + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*$,$M_2 = \alpha_2\partial_{t_2} - D^2 - 2\mathbf{q}\mathcal{M}_0\mathbf{q}^*$,
where $\alpha_2 \in i\mathbf{R}$, $\mathcal{M}_0^* = \mathcal{M}_0$.
Equation $[L_1, M_2] = 0$ is equivalent to nonlinear Schrodinger equation (NLS):

$$\alpha_2\mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2\left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q}. \tag{17}$$

Symmetry reductions of the KP–hierarchy

Now let us consider spatially two-dimesional generalizations of the operators L_1, M_2:

$$\begin{aligned} L_1 &= \partial_y - \mathbf{q}\mathcal{M}_0\mathcal{D}^{-1}\mathbf{q}^*, \\ M_2 &= \alpha_2\partial_{t_2} - c_1\mathcal{D}^2 - 2c_1S_1, \end{aligned} \qquad (18)$$

where $\alpha_2 \in i\mathbb{R}$, $S_1 = S_1(x, y, t_2) = \bar{S}_1(x, y, t_2)$,$c_1 \in \mathbb{R}$
Lax equation $[L_1, M_2] = 0$ is equivalent to Davey-Stewartson system DS-III:

$$\begin{cases} \alpha_2\mathbf{q}_{t_2} = c_1\mathbf{q}_{xx} - 2c_1S_1\mathbf{q} \\ S_{1\,y} = (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x \end{cases} \qquad (19)$$

System (19) is spatially two-dimensional l–component generalization of NLS.

Symmetry reductions of the KP–hierarchy

2. $k = 2, n = 2$:
$L_2 = D^2 + 2u + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*$, $M_2 = \alpha_2\partial_{t_2} - D^2 - 2u$,
where $\mathcal{M}_0^* = -\mathcal{M}_0$, $u = \bar{u}$, $\alpha_2 \in i\mathbf{R}$.
Operator equation $[L_2, M_2] = 0$ is equivalent to Yajima-Oikawa system:

$$\begin{cases} \alpha_2\mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2u\mathbf{q}, \\ \alpha_2 u_{t_2} = (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x. \end{cases} \quad (20)$$

Symmetry reductions of the KP–hierarchy

(2+1)-dimensional generalization of the operators L_2, M_2 have a form:

$$L_2 = i\partial_y - \mathcal{D}^2 - 2u - \mathbf{q}\mathcal{M}_0\mathcal{D}^{-1}\mathbf{q}^*,$$

$$M_2 = \alpha_2\partial_{t_2} - \mathcal{D}^2 - 2u,$$

where $\alpha_2 \in i\mathbb{R}$, $\mathcal{M}_0 = -\mathcal{M}_0^*$, $u = \bar{u}$.
Equation $[L_2, M_2] = 0$ can be represented in the following way:

$$\begin{cases} \alpha_2 u_{t_2} = iu_y + (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x \\ \alpha_2\mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2u\mathbf{q}. \end{cases} \tag{21}$$

System (21) is l–component spatially two-dimensional generalization of the Yajima-Oikawa system.

Symmetry reductions of the KP–hierarchy

3. $k = 2, n = 3$:
$L_2 = D^2 + 2u + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*$,
$M_3 = \alpha_3\partial_{t_3} - D^3 - 3uD - \frac{3}{2}(u_x + \mathbf{q}\mathcal{M}_0\mathbf{q}^*)$,
where $\mathcal{M}_0 = -\mathcal{M}_0^*$, $u = \bar{u}$, $\alpha_3 \in \mathbf{R}$.
Equation $[L_2, M_3] = 0$ is equivalent to the system:

$$\begin{cases} \alpha_3\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3u\mathbf{q}_x + \frac{3}{2}u_x\mathbf{q} + \frac{3}{2}\mathbf{q}\mathcal{M}_0\mathbf{q}^*\mathbf{q}, \\ \alpha_3 u_{t_3} = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}\left(\mathbf{q}_x\mathcal{M}_0\mathbf{q}^* - \mathbf{q}\mathcal{M}_0\mathbf{q}_x^*\right)_x . \end{cases} \tag{22}$$

Symmetry reductions of the KP–hierarchy

Spatially two-dimensional generalizations of L_2 and M_3 have the form:

$$L_2 = i\partial_y - D^2 - 2u - \mathbf{q}\mathcal{M}_0\mathcal{D}^{-1}\mathbf{q}^*,$$
$$M_3 = \alpha_3\partial_{t_3} - D^3 - 3uD - \frac{3}{2}\left(u_x + iD^{-1}\{u_y\} + \mathbf{q}\mathcal{M}_0\mathbf{q}^*\right), \quad (23)$$

wher $\alpha_3 \in \mathbb{R}$, $\mathcal{M}_0^* = -\mathcal{M}_0$, $u = \bar{u}$. Equation $[L_2, M_3] = 0$ is equivalent to the system:

$$\begin{cases} \alpha_3\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3u\mathbf{q}_x + \frac{3}{2}\left(u_x + iD^{-1}\{u_y\} + \mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q}, \\ \left[\alpha_3 u_{t_3} - \frac{1}{4}u_{xxx} - 3uu_x + \frac{3}{4}\left(\mathbf{q}\mathcal{M}_0\mathbf{q}_x^* - \mathbf{q}_x\mathcal{M}_0\mathbf{q}^*\right)_x + \right. \\ \qquad \left. -\frac{3}{4}i\left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)_y\right]_x = -\frac{3}{4}u_{yy}. \end{cases} \quad (24)$$

Equations (24) generalize Mel'nikov system.

Symmetry reductions of the KP–hierarchy

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Symmetry reductions of the KP–hierarchy

In other cases we will obtain:

4. Vector generalization of the modified Korteweg-de Vries equation ($k = 1, n = 3$):

$$\alpha_3 \mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3\left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q}_x + 3\left(\mathbf{q}_x\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q}, \;\; \mathcal{M}_0^* = \mathcal{M}_0. \quad (25)$$

5. Generalization of the Boussinesq equation ($k = 3, n = 2$):

$$\begin{cases} 3\alpha_2^2 u_{t_2 t_2} = (-u_{xx} - 6u^2 + 4(\mathbf{q}\mathcal{M}_0\mathbf{q}^*))_{xx}, \;\; \mathcal{M}_0^* = \mathcal{M}_0, \\ \alpha_2 \mathbf{q}_{t_2} - \mathbf{q}_{xx} - 2u\mathbf{q} = 0, \end{cases} \quad (26)$$

6. Vector generalization of the Drinfeld-Sokolov system ($k = 3, n = 3$):

$$\begin{cases} \alpha_3 \mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3u\mathbf{q}_x + \frac{3}{2}u_x\mathbf{q}, \;\; \mathbf{q} = \bar{\mathbf{q}}, \;\; \mathcal{M}_0^* = \mathcal{M}_0, \\ \alpha_3 u_{t_3} = (\mathbf{q}\mathcal{M}_0\mathbf{q}^\top)_x, \end{cases} \quad (27)$$

Exact solutions of some nonlinear models from the KP-hierarchy

In this section we will consider the construction of exact solutions of the integrable systems from the KP-hierarchy. For this reason we will use invariant transformations for linear integro-differential operators from the previous section. Consider the integro-differential operator:

$$L := \alpha\partial_t - \sum_{i=0}^{n} u_i \mathcal{D}^i + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top, \ \alpha \in i\mathbb{R} \cup \mathbb{R} \qquad (28)$$

with $(N \times N)$-matrix coefficients $u_i = u_i(x,t)$; Λ, $\tilde{\Lambda}$ and $\mathcal{M}_0$ are $(K \times K)$ and $(l \times l)$-matrices correspondingly; $\mathbf{q}$, $\mathbf{r}$ are $(N \times l)$-matrices. Assume that $(N \times K)$-matrix functions φ, ψ satisfy linear problems: $L\{\varphi\} = \varphi\Lambda$, $L^\tau\{\psi\} = \psi\tilde{\Lambda}$.

Define the binary Darboux-type transformation (BT) as:

$$W = I - \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1} D^{-1}\psi^\top \qquad (29)$$

The following theorem holds:

Theorem 1

Let functions f and g be $(N \times 1)$-solutions of the linear systems $L\{f\} = f\lambda$, $L^\tau\{g\} = g\tilde{\lambda}$.
Then, functions $F = W\{f\}$, $G = W^{-1,\tau}\{g\}$ satisfy equations $\hat{L}\{F\} = F\lambda$, $\hat{L}^{-1,\tau}\{G\} = G\tilde{\lambda}$
with the operator

$$\hat{L} = \alpha\partial_t - \sum_{i=0}^{n} \hat{u}_i\mathcal{D}^i + \hat{\mathbf{q}}\mathcal{M}_0 D^{-1}\hat{\mathbf{r}}^\top + \Phi\mathcal{M}D^{-1}\Psi^\top, \qquad (30)$$

where $\mathcal{M} = C\Lambda - \tilde{\Lambda}^\top C$, $\Phi = \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1}$,
$\Psi^\top = \left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top$, $\hat{\mathbf{q}} = W\{\mathbf{q}\}$, $\hat{\mathbf{r}} = W^{-1,\tau}\{\mathbf{r}\}$.

Exact solutions of some nonlinear models from the KP-hierarchy

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Exact solutions of some nonlinear models from the KP-hierarchy

Consider two possible realizations of the integration operator D^{-1} in BT (29):

$$W^{+} = I - \varphi\left(C + \int_{-\infty}^{x} \psi^{\top}(s)\varphi(s)ds\right)^{-1} \int_{-\infty}^{x} \psi^{\top}(s)\cdot ds, \quad (31)$$

$$W^{-} = I - \varphi\left(C + \int_{-\infty}^{x} \psi^{\top}(s)\varphi(s)ds\right)^{-1} \int_{+\infty}^{x} \psi^{\top}(s)\cdot ds, \quad (32)$$

under assumption that the components of $(N \times K)$-matrix functions φ and ψ decrease rapidly at both infinities. A composition of operators $(W^{+})^{-1}$ and W^{-} gives Fredholm operator:

$$S_R = (W^{+})^{-1}W^{-} = I + \varphi C^{-1} \int_{-\infty}^{+\infty} \psi^{\top}(s)\cdot ds. \quad (33)$$

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Assume that integral part in L is equal to zero. Using the equalities $L\{\varphi\} = \varphi\Lambda$, $L^{\tau}\{\psi\} = \psi\tilde{\Lambda}$ we obtain that the commutator of S_R and L has the form:

$$[L, S_R] = \varphi C^{-1}\mathcal{M}\int_{-\infty}^{+\infty} C^{-1}\psi^{\top}(s)\cdot ds,\ \mathcal{M} = C\Lambda - \tilde{\Lambda}^{\top}C$$

Using W^{+}, W^{-} as the dressing operators for L we obtain that:

$$\begin{array}{c} \hat{L}_1 = W^{+}L(W^{+})^{-1} = (\hat{L}_1)_{+} + \Phi\mathcal{M}\int_{-\infty}^{x}\Psi^{\top}(s)\cdot ds, \\ \hat{L}_2 = W^{-}L(W^{-})^{-1} = (\hat{L}_2)_{+} + \Phi\mathcal{M}\int_{+\infty}^{x}\Psi^{\top}(s)\cdot ds, \\ (\hat{L}_1)_{+} = (\hat{L}_2)_{+} \end{array} \qquad (34)$$

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If we put $\Lambda = \tilde{\Lambda} = 0$ and consider the differential operator L ($\mathcal{M}_0 = 0$), then using transformations W^+ or W^- we obtain the differential operator $\hat{L}$. In this case, $[L, S_R] = 0$. Thus, we obtain dressing due to Zakharov-Shabat.

References

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Exact solutions of some nonlinear models from the KP-hierarchy

Now we will consider realizations of integral transformation W (29) and construction of the solutions for integrable systems from the KP-hierarchy. At first we will consider the scalar NLS

$$iq_t = q_{xx} + 2\mu|q|^2 q, \ \mu = \pm 1, \tag{35}$$

and its vector generalization – Manakov system (l components):

$$i(q_j)_t = (q_j)_{xx} + 2\left(\sum_{s=1}^{l} \mu_s |q|_s^2\right) q_j, \ \ \mu_s = \pm 1, j = \overline{1, l}. \tag{36}$$

Exact solutions of some nonlinear models from the KP-hierarchy

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- S.V. Manakov. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. Sov. Phys. JETP 38:2 (1974) 248–253.

Exact solutions of some nonlinear models from the KP-hierarchy

Proposition 1

Let function $\varphi := (\varphi_1, \ldots, \varphi_K)$ be a fixed solution of the system:

$$\begin{cases} \varphi_x = \varphi\Lambda, \\ i\varphi_t = \varphi_{xx}, \end{cases} \tag{37}$$

where $\Lambda \in Mat_{K\times K}(\mathbb{C})$.
Let $f := (f_1, \ldots, f_l)$ be an arbitrary solution of the problem

$$if_t = f_{xx}. \tag{38}$$

Then functions $F := f - \varphi(C + \Omega[\bar{\varphi}, \varphi])^{-1}\Omega[\bar{\varphi}, f]$,
$\Phi = \varphi(C + \Omega[\bar{\varphi}, \varphi])^{-1}$, where
$\Omega[\bar{\varphi}, \varphi] = \int_{(x_0,t_0)}^{(x,t)} \varphi^*\varphi dx + i(\varphi_x^*\varphi - \varphi^*\varphi_x)dt$,
$\Omega[\bar{\varphi}, f] = \int_{(x_0,t_0)}^{(x,t)} \varphi^* f dx + i(\varphi_x^* f - \varphi^* f_x)dt$, $C = C^* \in Mat_{K\times K}(\mathbb{C})$
satisfy equations:

Exact solutions of some nonlinear models from the KP-hierarchy

$$iF_t = F_{xx} + 2\Phi\hat{\mathcal{M}}\Phi^* F, \tag{39}$$

$$i\Phi_t = \Phi_{xx} + 2\Phi\hat{\mathcal{M}}\Phi^*\Phi, \tag{40}$$

where $\hat{\mathcal{M}} = C\Lambda + \Lambda^* C - (\varphi^*\varphi)(x_0, t_0)$

Exact solutions of some nonlinear models from the KP-hierarchy

Using proposition 1, we can obtain K-soliton solution of NLS ($\mu = 1$) of the following structure:

$$q = \frac{\det\begin{pmatrix} \Delta_2 & \overrightarrow{1} \\ \varphi & 0 \end{pmatrix}}{\det(\Delta_2)},$$

where $\varphi_j = \gamma_j e^{\lambda_j x + i\lambda_j^2 t}$, $\gamma_j, \lambda_j \in \mathbb{C}$, $j = \overline{1, K}$; $\overrightarrow{1}$ is a row-vector (K-components) consisting of 1,

$$\Delta_2 = \left(\frac{1}{\lambda_s + \bar{\lambda}_j}(\bar{\varphi}_j \varphi_s + 1)\right)_{j,s=1}^{K}$$

Animation 1 describes the behavior of 3-soliton solution ($|q|$ and $Re(q)$) with $\lambda_1 = 1.5 + i$, $\lambda_2 = 1 + 2i$, $\lambda_3 = 2.5 + 3.5i$ and $\gamma_1 = e$, $\gamma_2 = e^{10}$, $\gamma_3 = e^5$.

Exact solutions of some nonlinear models from the KP-hierarchy

We can also use Proposition 1 for obtaining other kinds of solutions (e.g. bound states) for NLS and constructing solutions of vector generalization of NLS.
Animation 2 describes the behavior of NLS solution, consisting of 1 bound state and 1 soliton.
Animation 3 represents the absolute value of the solution ($\lambda_1 = 2 - 3i$, $\lambda_2 = 1 + 2i$, $\gamma_1 = e^{100}$, $\gamma_2 = e^{10}$) for 2-component NLS generalization of the form:

$$i(q_j)_t = (q_j)_{xx} + 2\left(|q_1|^2 - |q_2|^2\right) q_j,\ j = 1, 2 \qquad (41)$$

Exact solutions of some nonlinear models from the KP-hierarchy

Similar types of solutions for other integrable systems of the KP-hierarchy can also be constructed. In particular, one of bound-state solutions of the Yajima-Oikawa system

$$\begin{cases} iq_{t_2} = q_{xx} + 2uq. \\ iu_{t_2} = (\mu|q|^2)_x; \end{cases} \tag{42}$$

in case $\mu = -i$ is presented on animation 4 $(\lambda = 3 + i, \gamma = e^5)$. Animation 5 describes the behavior of 2-soliton solution of Drinfeld-Sokolov system:

$$\begin{cases} q_{t_3} = q_{xxx} + 3uq_x + \frac{3}{2}u_x q, \\ u_{t_3} = (q^2)_x. \end{cases} \tag{43}$$

Exact solutions of some nonlinear models from the KP-hierarchy

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Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Consider the generalizations of operators L_1, M_2 (18):

$$L_1 = \partial_y - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*,$$
$$M_2 = \alpha_2\partial_{t_2} - c_1 D^2 - c_2\partial_y^2 + 2c_1 S_1 + 2c_2\mathbf{q}\mathcal{M}_0 D^{-1}\partial_y\mathbf{q}^*, \quad (44)$$

where $c_1, c_2 \in \mathbb{R}$, $\alpha_2 \in i\mathbb{R}$, $\mathbf{q} = \mathbf{q}(x, y, t)$ and $S_1 = S_1(x, y, t) = S_1^*(x, y, t)$ are matrix functions with dimensions $N \times l$ and $N \times N$ respectively; $\mathcal{M}_0 = \mathcal{M}_0^*$ is a constant $(l \times l)$-dimensional matrix.

Lax equation $[L_1, M_2] = 0$ is equivalent to the system:

$$\begin{cases} \alpha_2\mathbf{q}_{t_2} = c_1\mathbf{q}_{xx} + c_2\mathbf{q}_{yy} - 2c_1 S_1\mathbf{q} - 2c_2\mathbf{q}\mathcal{M}_0 S_2, \\ S_{1y} = (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x, \; S_{2x} = (\mathbf{q}^*\mathbf{q})_y. \end{cases} \quad (45)$$

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

In scalar case ($N = 1, l = 1$), by taking $S = c_1 S_1 + c_2 S_2$, $\mu := \mathcal{M}_0 = 1$, we obtain the following differential consequence from (45):

$$\begin{cases} \alpha_2 q_{t_2} = c_1 q_{xx} + c_2 q_{yy} - 2Sq, \\ S_{xy} = c_1 |q|^2_{xx} + c_2 |q|^2_{yy}. \end{cases} \qquad (46)$$

If $c_1 = -c_2 = c \in \mathbb{R}$ we obtain Davey-Stewartson system (DS-I) from (46).

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Consider the following pair of operators:

$$L_1 = \partial_{\bar{z}} - \mathbf{q} D_z^{-1} \bar{\mathbf{q}},$$

$$M_2 = \alpha_2 \partial_{t_2} - c D_{zz}^2 + c \partial_{\bar{z}\bar{z}}^2 + 2cS_1 - 2c\mathbf{q} D_z^{-1} \bar{\mathbf{q}}_{\bar{z}} - 2c\mathbf{q} D_z^{-1} \bar{\mathbf{q}} \partial_{\bar{z}}, \quad (47)$$

where α_2, $c \in i\mathbb{R}$;
$\mathbf{q}$ and S_1 are $(N \times N)$-matrices, $z = x + iy$. Lax equation $[L_1, M_2] = 0$ is equivalent to the system:

$$\begin{cases} \alpha_2 \mathbf{q}_{t_2} = -ic\mathbf{q}_{xy} - 2cS_1\mathbf{q} + 2c\mathbf{q}\bar{S}_1, \\ S_{1x} + iS_{1y} = (\mathbf{q}\bar{\mathbf{q}})_x - i(\mathbf{q}\bar{\mathbf{q}})_y. \end{cases} \quad (48)$$

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

In scalar case ($N = 1$) we obtain the following differential consequence from system (48):

$$\left\{ \begin{array}{c} \alpha_2 q_{t_2} = -icq_{xy} - 4ic\tilde{S}q, \\ \tilde{S}_{xx} + \tilde{S}_{yy} = -4|q|^2_{xy}. \end{array} \right. \tag{49}$$

System (49) is Davey-Stewartson system (DS-II).

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

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Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Consider the following pair of operators:

$$L_1 = \partial_y - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \tag{50}$$

$$M_2 = \alpha_2\partial_{t_2} - c_1 D^2 - c_2\partial_y^2 + 2c_1 S_1 D + 2c_2\mathbf{q}\mathcal{M}_0 D^{-1}\partial_y \mathbf{r}^\top D, \tag{51}$$

where $\mathbf{q} = \mathbf{q}(x, y, t_2)$, $\mathbf{r} = \mathbf{r}(x, y, t_2)$ and $S_1 = S_1(x, y, t_2)$ are matrix functions with dimensions $(N \times M)$ and $(N \times N)$ respectively; $\mathcal{M}_0$ is a constant $(M \times M)$-dimensional matrix. Equation $[L_1, M_2] = 0$ is equivalent to the following system:

$$\left\{ \begin{array}{c} \alpha_2\mathbf{q}_{t_2} - c_1\mathbf{q}_{xx} - c_2\mathbf{q}_{yy} + 2c_1 S_1\mathbf{q}_x - 2c_2\mathbf{q}\mathcal{M}_0 S_2 + \\ +2c_2\mathbf{q}\mathcal{M}_0(\mathbf{r}^\top\mathbf{q})_y = 0, \\ \alpha_2\mathbf{r}^\top_{t_2} + c_1\mathbf{r}^\top_{xx} + c_2\mathbf{r}^\top_{yy} + 2c_1\mathbf{r}^\top_x S_1 + 2c_2 S_2\mathcal{M}_0\mathbf{r}^\top = 0, \\ S_{1y} = (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)_x + [\mathbf{q}\mathcal{M}_0\mathbf{r}^\top, S_1], \;\; S_{2x} = (\mathbf{r}^\top_x\mathbf{q})_y. \end{array} \right. \tag{52}$$

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

a). Under additional conditions $\alpha_2 \in i\mathbb{R}$, $c_1, c_2 \in \mathbb{R}$, $\mathcal{M}_0 = -\mathcal{M}_0^*$, $\mathbf{r}^\top = \mathbf{q}^*$, $S_1 = S_1^*$ operators L_1 (50) and M_2 (51) are D-skew-Hermitian ($L_1^* = -DL_1D^{-1}$) and D-Hermitian ($M_2^* = DM_2D^{-1}$). System (52) has a form:

$$\begin{cases} \alpha_2 \mathbf{q}_{t_2} - c_1 \mathbf{q}_{xx} - c_2 \mathbf{q}_{yy} + 2c_1 S_1 \mathbf{q}_x + 2c_2 \mathbf{q} \mathcal{M}_0 S_2 = 0, \\ S_{1y} = (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x + [\mathbf{q}\mathcal{M}_0\mathbf{q}^*, S_1], \;\; S_{2x} = (\mathbf{q}^*\mathbf{q}_x)_y. \end{cases} \tag{53}$$

Consider a scalar case of equation (53) $(N = 1, M = 1)$ and take $c_2 = 0$, $y = x$, $\mu := \mathcal{M}_0$. Then we obtain Chen-Lee-Liu equation (DNLS-II) from (53):

$$\alpha_2 q_{t_2} - c_1 q_{xx} + 2c_1 \mu |q|^2 q_x = 0. \tag{54}$$

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

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Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

b). We will put $\mathcal{M}_0 \mathbf{r}^\top(x, y, t_2) = \nu$, where ν is $(M \times N)$-dimensional constant matrix. After the change $u := \mathbf{q}\nu$ system (52) takes the form:

$$\begin{cases} \alpha_2 u_{t_2} - c_1 u_{xx} - c_2 u_{yy} + 2c_1 S_1 u_x + 2c_2 u u_y = 0, \\ S_{1y} = u_x + [u, S_1]. \end{cases} \qquad (55)$$

System (55) is (2+1)-dimensional matrix generalization of Burgers equation. It can be generalized onto $(n+1)$-dimensional case:

$$\begin{cases} \alpha_2 u_{t_2} = \Delta u - 2\mathbf{S}\nabla u, \\ \frac{\partial S_i}{\partial x_1} = \frac{\partial u}{\partial x_i} + [u, S_1], \ i = \overline{1, n}, \end{cases} \qquad (56)$$

where $\mathbf{S} = (S_1, \ldots S_n)$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Proposition

Let $T := T(x, y, t_2)$ be $((N \times N))$-matrix function that satisfies equation:

$$\alpha_2 T_{t_2} = c_1 T_{xx} + c_2 T_{yy} \tag{57}$$

Then $(N \times N)$-matrix functions

$$u := -T^{-1} T_y, \; S_1 = -T^{-1} T_x. \tag{58}$$

satisfy system (55).

Integro-differential Lax representations for Davey-Stewartson and Chen-Lee-Liu equations

Remark

It can be checked that functions u, S_1 defined by formula (58) satisfy another version of (2+1)-dimensional generalization of matrix Burgers equation:

$$\begin{cases} \alpha_2 u_{t_2} - c_1 u_{xx} - c_2 u_{yy} + 2c_1 S_1 u_x + 2c_2 u u_y = 0, \\ \alpha_2 S_{1t_2} - c_1 S_{1xx} - c_2 S_{1yy} + 2c_1 S_1 S_{1x} + 2c_2 u S_{1y} = 0 \end{cases} \tag{59}$$

It is also constructed the integro-differential representation for the equation:

$$\alpha_3 q_{t_3} + c_1 q_{xxx} - c_2 q_{yyy} - 3c_1 \mu q_x \int |q|_x^2 dy +$$

$$3c_2 \mu q_y \int |q|_y^2 dx + 3c_2 \mu q \int (\bar{q} q_y)_y dx - 3c_1 \mu q \int (q_x q)_x dy = 0. \quad (60)$$

where $\alpha_3, \mu, c_1, c_2 \in \mathbb{R}$, which can be reduced to the mKdV equation:

$$\alpha_3 q_{t_3} + q_{xxx} - 6\mu q^2 q_x = 0. \quad (61)$$

Lax integro-differential representation was also constructed for the following system:

$$\alpha_3 q_{t_3} + c_1 q_{xxx} - c_2 q_{yyy} - 3c_1 v_1 q_{xx} - 3c_1 v_3 q_x + 3\mu c_2 q_y D^{-1}\{\bar{q} q_x\}_y +$$
$$+3c_2 \mu q D^{-1}\{\bar{q} q_{xy}\}_y - 3c_2 \mu^2 q D^{-1}\{|q|^2 \bar{q} q_x\}_y = 0,$$
$$v_{1y} = \mu(|q|^2)_x,$$
$$v_{3y} = \mu(q_x \bar{q})_x - 2\mu v_1 (|q|^2)_x, \tag{62}$$

where $\alpha_3,\ c_1, c_2 \in \mathbb{R},\ \mu \in i\mathbb{R},\ v_1 = v_1^*,\ v_3 + v_3^* = v_{1x}$, which reduces to the higher Chen-Lee-Liu equation ($c_1 = 1,\ c_2 = 0$):

$$\alpha_3 q_{t_3} + q_{xxx} - 3\mu |q|^2 q_{xx} - 3\mu \bar{q} q_x^2 + 3\mu^2 |q|^4 q_x = 0.$$

Thank you for your attention!