A Möbius geometric interpretation of the Lawson correspondence for minimal surfaces

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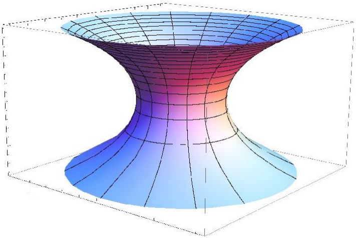
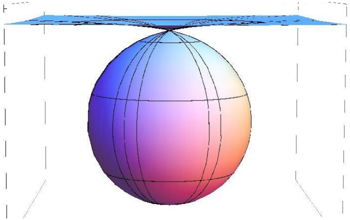
- But there are important differences too..

A cousin pair in the upper-half space model of $\mathbb{H}^3$

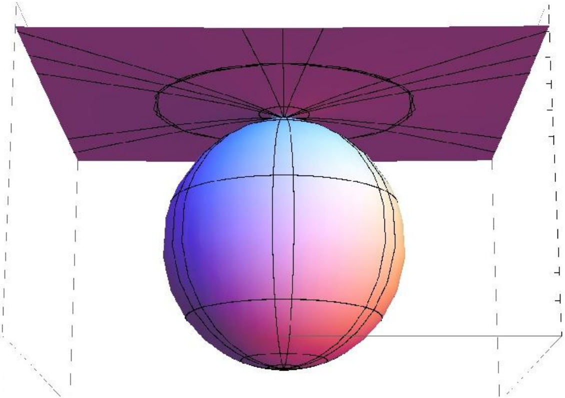


Catenoid



Cateniod cousin

Different view of the Catenoid cousin



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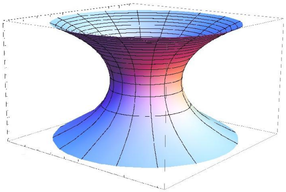
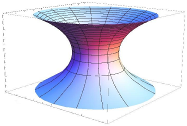
- We do not regard conformal deformation as "integrable," in the sense that it cannot be computed explicitly in general.

Conformal deformation, $\lambda < 1$

-  

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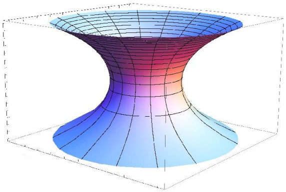
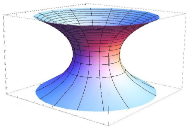
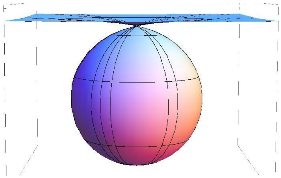
-  

  $\downarrow$ $\downarrow$

- 

  ???
  (but not a surface of revolution!)

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- Similarly, CMC1 $\hat{x}(z) = \pi \circ F(z)$, where F is constructed from:

$$\tilde{F} = \begin{pmatrix} x_1 & \frac{\dot{x}_1 - g\eta x_1}{\eta} \\ x_2 & \frac{\dot{x}_2 - g\eta x_2}{\eta} \end{pmatrix} = \begin{pmatrix} \frac{\dot{y}_1 + g\eta y_1}{g^2\eta} & y_1 \\ \frac{\dot{y}_2 + g\eta y_2}{g^2\eta} & y_2 \end{pmatrix}$$

where x_1, x_2 and y_1, y_2 are pairs of lin. indep. solutions of

construction..

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- This $\hat{x}$ is the Bryant cousin of the minimal surface $x = \pi \circ \gamma$.

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- The first step is to "complexify" Möbius geometry...

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  - Viewing the sphere as the projective null cone in Minkowski space $\left\{ v \in \mathbb{R}^{n+1,1} \,|\, v \cdot v = 0 \right\} / v \sim \lambda v$ leads to the isomorphism $\mathcal{M}ob_n \simeq SO_0^+(n+1, 1)$.

Complex Möbius geometry

- Complexify: The null cone S^n, projective Minkowski space $\mathbb{R}P^{n+1}$, and the Möbius group $SO_0^+(n+1,1)$ complexify (plus Wick rotation) to the standard quadric $Q_n = \left\{v \in \mathbb{C}^{n+2} \,|\, v \cdot v = 0\right\} / v \sim \lambda v$, projective space $\mathbb{C}P^{n+1}$, and Möbius group $\mathcal{M}ob_n^{\mathbb{C}} = SO_{n+2}\mathbb{C}$, respectively.

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- Complexify: The null cone S^n, projective Minkowski space $\mathbb{R}P^{n+1}$, and the Möbius group $SO_0^+(n+1,1)$ complexify (plus Wick rotation) to the standard quadric $Q_n = \left\{ v \in \mathbb{C}^{n+2} \,|\, v \cdot v = 0 \right\} / v \sim \lambda v$, projective space $\mathbb{C}P^{n+1}$, and Möbius group $\mathcal{M}ob_n^{\mathbb{C}} = SO_{n+2}\mathbb{C}$, respectively.
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  - Differentiating $v \cdot v = 0$, can describe the tangent bundle as $TQ_n = \{[v,w] \,|\, [v] \in Q_n,\ v \cdot w = 0\} / \sim$.
  - The scalar product induces a conformal structure on the tangent spaces, *or* specifying a holomorphic distribution of "null cones" $C_p = T_pQ_n \cap Q_n \subset p^\perp \cap Q_n = T_pQ_n$.
  - The *Möbius group* $\mathcal{M}ob_n^{\mathbb{C}}$ is the set of maps $\mu : Q_n \to Q_n$ preserving the null cone distribution.
- To obtain a Liouville-type theorem, we need Clifford algebras...

Clifford algebra

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- The main automorphism splits $Cl_B(\mathbb{C}^n)$ into ± 1 eigenspaces $Cl_B^0(\mathbb{C}^n) \oplus Cl_B^1(\mathbb{C}^n)$ (*even* and *odd* subspaces) and defines a $\mathbb{Z}_2$-grading.

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- Now split $\mathbb{C}^n = \mathbb{C}^{n-2} \oplus \mathbb{C}^2$ so that $B(v, v) = \tilde{B}(w, w) + xy$. Consider $S : \mathbb{C}^n \to M_{2x2}(Cl_{\tilde{B}}(\mathbb{C}^{n-2}))$ given by

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- Since $S(v)^2 = B(v,v)I$, it extends to an isomorphism $Cl_B(\mathbb{C}^n) \simeq M_{2x2}(Cl_{\tilde{B}}(\mathbb{C}^{n-2}))$. The image of $Spin_n^{\mathbb{C}}$ turns out to be...

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- What's good about that? Take a null vector of the form $(w, -w^2, 1)$ and look at the *projective* image under $S(w)$:

$$[S(w)] = \begin{bmatrix} w & -w^2 \\ 1 & -w \end{bmatrix} = \begin{bmatrix} w \\ 1 \end{bmatrix} \begin{bmatrix} 1 & w^* \end{bmatrix}$$

LFT form of $\mathcal{M}ob_n^{\mathbb{C}}$ action

- Then $\mu[S(w)]\mu^{-1} = [\mu S(w)\mu^*]$ can be rewritten
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- Any *local* conformal transformation $\phi : U \to V$ between open subsets $U, V \subset \mathbb{C}^{n-2}$ is a restriction of $S \circ \mu \circ S^{-1}$, where $S = S|_{\{(w, -w^2, 1)\}}$, whose image omits exactly $\infty = (0, 1, 0)$ plus the "null cone at infinity" $C_\infty = T_\infty Q_{n-2} \cap Q_{n-2}$.

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- As a map to the "standard" quadric in $\mathbb{C}P^{n-1}$, the restriction $S : \mathbb{C}^{n-2} \to Q_{n-2} - C_\infty \subset \mathbb{C}P^{n-1}$ is inverse stereo proj from the hyperplane $T_\infty Q_{n-2}$:
$$w \mapsto \begin{bmatrix} \frac{1}{2}(1-w^2) \\ \frac{i}{2}(1+w^2) \\ w \end{bmatrix}$$

Minimal surfaces of arbitrary codimension

- Let $x : M^2 \to \mathbb{R}^n$ be an immersion. The Gauss map is the distribution of tangent planes, a map into the Grassmannian $M^2 \to G_2(\mathbb{R}^n)$.

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- x is minimal iff the Gauss map is *holomorphic* [Chern].

Weierstrass rep and the transform

- Can now use inverse stereographic projection to give a Weierstrass representation:

$$x(z) = \mathrm{Re} \int_{z_0}^{z} \begin{pmatrix} \frac{1}{2}(1-g^2) \\ \frac{i}{2}(1+g^2) \\ g \end{pmatrix} \eta$$

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- Definition
Let $\mu \in \mathcal{M}ob^{\mathbb{C}}_{n-2}$ and $x : M^2 \to \mathbb{R}^n$ a minimal surface with Weierstrass data $\{g, \eta\}$. Define x_μ to be the surface determined by data $\{g_\mu, \eta_\mu\} = \left\{(ag + b)(cg + d)^{-1}, \frac{\eta}{(cg+d)^*(cg+d)}\right\}$.

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- Given a fixed minimal surface $x : M^2 \to \mathbb{R}^n$, the group G acts on x by deformation $x \mapsto x_{Ad(\mu)}$, and the moduli space of such deformations is $\mathcal{H}_x^n = G/K$.

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- Identifying each tangent space $T_p\mathcal{H}^n_x \simeq i\mathfrak{k} = i\,\mathrm{Lie}(K)$, then B defines a Riemannian metric on $\mathcal{H}^n_x$, i.e. for the symmetric pair (G, K, τ, B), with $\tau : G \to G$ the involution fixing K.

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Given a minimal immersion $x : M^2 \to \mathbb{R}^n$, define the **canonical cousin** *to be $\hat{x} : M^2 \to G/K$ such that $\hat{x}(z_0) = I$ and $\hat{x} = \pi \circ F$, where $F : M^2 \to G$ is a solution to $F^{-1}dF = \phi(\partial x) = \phi(x_z dz)$.*

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- Theorem (Bryant)

When $G = SL_2\mathbb{C}$, the canonical cousin is the Bryant cousin.

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- Definition

  Given a surface $f = \pi(F) : M^2 \to \mathcal{H}^n$ and $\mu \in G$, define the transform $f_\mu = \pi(F\mu^{-1})$.

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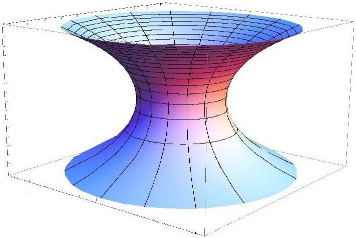
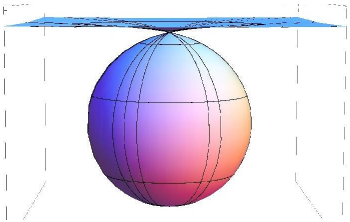
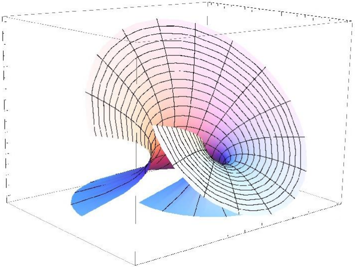
  - In the case $n = 3$, Clifford operations are trivial, and $Ad : SL_2\mathbb{C} \to Spin_3^{\mathbb{C}}$ is an *isomorphism*, so we get *all* Möbius deformations.

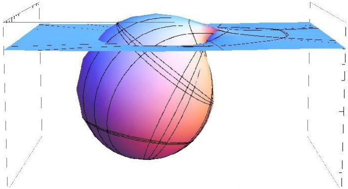
Catenoid cousins: $M^2 \simeq \mathbb{C} - \{0\}$ and $(g, \eta) = (\frac{1}{z}, kdz)$





deformation: $\mu = \begin{pmatrix} 1 & \frac{i}{2} \\ 0 & 1 \end{pmatrix}$





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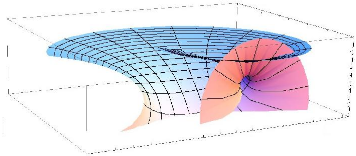
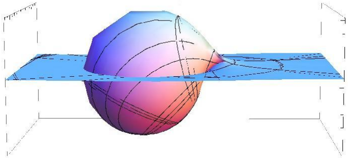
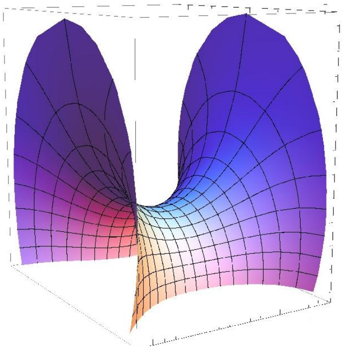
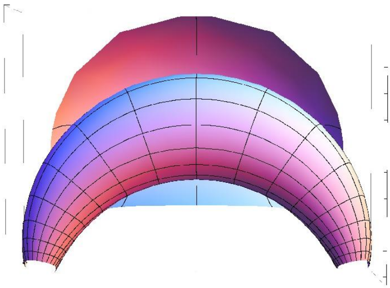


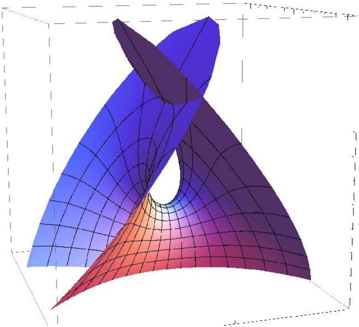


Voss cousins: $M^2 \simeq \mathbb{C} - \{\pm 1\}$ and $(g, \eta) = (\, z, \, (z-1)^{-1}(z+1)^{-1} dz)$

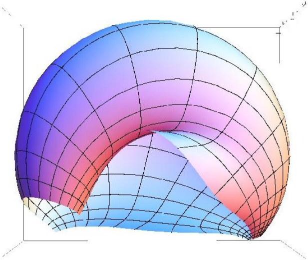




deformation: $\mu = \begin{pmatrix} 1+i & 0 \\ 0 & (1+i)^{-1} \end{pmatrix}$

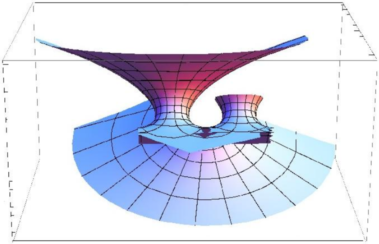


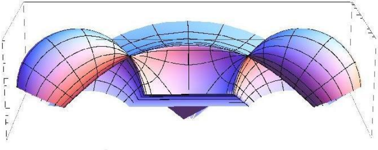
a = 1+1



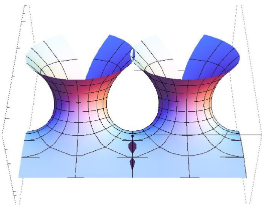
a = 1+1

Bessel cousins: $M^2 \simeq \mathbb{C} - \{0\}$ and $(g, \eta) = (z^2, \frac{dz}{z})$





different views of the minimal surface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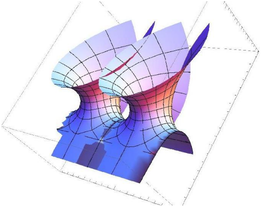
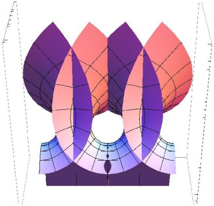

Side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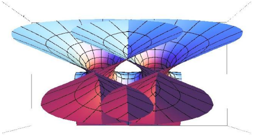
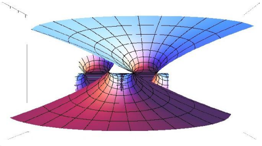
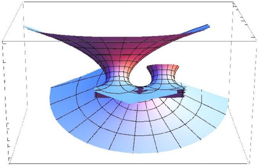
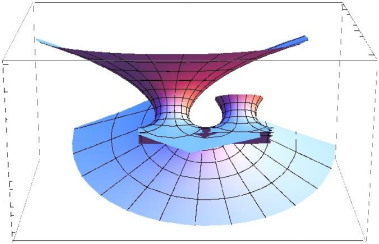
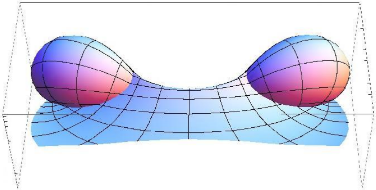

Front


Front


Back

deformation: $\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$





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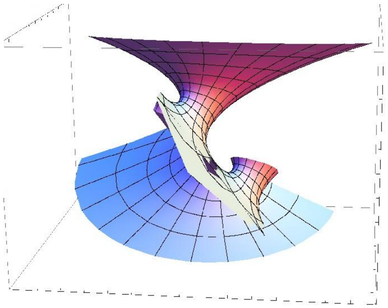
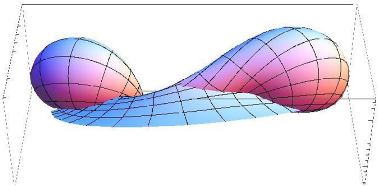
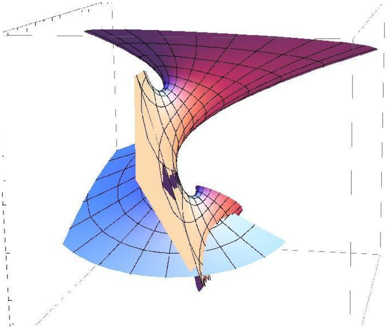
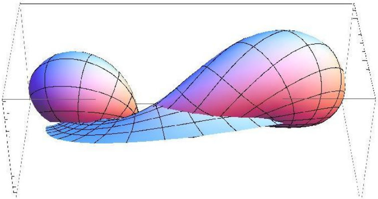




deformation: $\mu = \begin{pmatrix} 1 & \frac{3i}{4} \\ 0 & 1 \end{pmatrix}$





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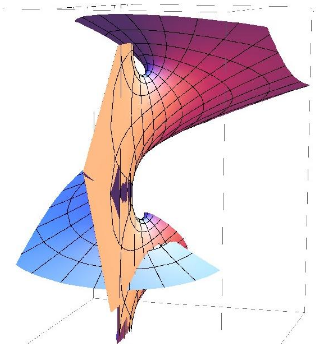

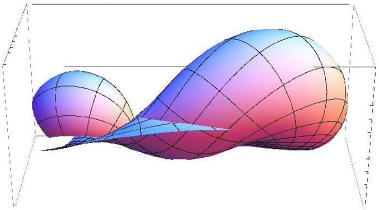

Thank you.