Reductions and New types of Integrable models in two and more dimensions

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PLAN

- Reductions of spinor models
- RHP with canonical normalization
- Reductions of polynomial bundles
- New $N$-wave equations ($k = 2$) in 2 and more dimensions
- Conclusions and open questions
1 Integrable 2-dimensional spinor models

The integrability of the 2-dimensional versions of the Nambu–Jona-Lasinio–Vaks–Larkin (NJLVL), Gross–Neveu model (GN) and Zakharov and Mikhailov – see Zakharov and Mikhailov (1981). NJLVL models are related to $su(N)$ algebras, Gross–Neveu models – to $sp(N)$ and Zakharov–Mikhailov (ZM) models – to $so(N)$.

Lax representations of these models:

$$
\Psi_\xi = U(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \quad \Psi_\eta = U(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda),
$$

$$
U(\xi, \eta, \lambda) = \frac{U_1(\xi, \eta)}{\lambda - a}, \quad V(\xi, \eta, \lambda) = \frac{V_1(\xi, \eta)}{\lambda + a},
$$

where $\eta = t + x$, $\xi = t - x$ and $a$ is a real number.

We also impose the $\mathbb{Z}_2$-reduction:

$$
U^\dagger(x, t, \lambda) = -U(x, t, \lambda^*), \quad V^\dagger(x, t, \lambda) = -V(x, t, \lambda^*).
$$
The compatibility condition of the above linear problems reads:

\[ U\eta - V\xi + [U, V] = 0, \]

which is equivalent to

\[ U_{1,\eta} + \frac{1}{2a} [U_1, V_1(\xi, \eta)] = 0, \quad V_{1,\xi} - \frac{1}{2a} [V_1, U_1(\xi, \eta)] = 0. \]

i) Nambu-Jona-Lasinio-Vaks-Larkin models. Here we choose \( g \simeq su(N) \). Then \( \psi(\xi, \eta) \) and \( \phi(\xi, \eta) \) are elements of the group \( SU(N) \) and by definition \( \hat{\psi}(\xi, \eta) = \psi^\dagger(\xi, \eta), \hat{\phi}(\xi, \eta) = \phi^\dagger(\xi, \eta) \). Next we choose \( J = \text{diag}(1, 0, \ldots, 0) \) and as a result only the first columns \( \phi^{(1)}, \psi^{(1)} \) and the first rows \( \hat{\phi}^{(1)}, \hat{\psi}^{(1)} \) enter into the systems. If we introduce the notations:

\[ \phi_\alpha(\xi, \eta) = \phi^{(1)}_{\alpha,1}, \quad \psi_\alpha(\xi, \eta) = \psi^{(1)}_{\alpha,1}, \]
then the explicit form of the system is:

\[ \frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{2a} \psi_\alpha \sum_{\beta=1}^{N} \psi^*_\beta \phi_\beta, \]

\[ \frac{\partial \psi_\alpha}{\partial \xi} = \frac{i}{2a} \phi_\alpha \sum_{\beta=1}^{N} \phi^*_\beta \psi_\beta. \]

The functional of the action is:

\[ A_{NJLVL} = \int_{-\infty}^{\infty} dx \, dt \left( i \sum_{\alpha=1}^{N} \left( \phi^*_\alpha \frac{\partial \phi_\alpha}{\partial \eta} + \psi^*_\alpha \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left| \sum_{\alpha=1}^{N} (\psi^*_\alpha \phi_\alpha) \right|^2 \right). \]

ii) **Gross-Neveu models.** Here we choose \( g \simeq sp(2N, \mathbb{R}) \); then \( \psi(\xi, \eta) \) and \( \phi(\xi, \eta) \) are elements of the group \( \mathfrak{g} \simeq SP(2N, \mathbb{R}) \). Following [?] we use the standard definition of symplectic group elements:

\[ \hat{\psi}(\xi, \eta) = \mathfrak{J} \psi^T(\xi, \eta) \mathfrak{J}, \quad \hat{\phi}(\xi, \eta) = \mathfrak{J} \phi^T(\xi, \eta) \mathfrak{J}, \quad \mathfrak{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]
Then the corresponding Lie algebraic elements acquire the following block-matrix structure:

\[
U_1(\xi, \eta) = \begin{pmatrix}
A & B \\
C & -A^T
\end{pmatrix},
\]

where \(A, B, C\) are arbitrary real \(N \times N\) matrices. Next we choose

\[
J = \begin{pmatrix}
0 & B_0 \\
0 & 0
\end{pmatrix}, \quad B_0 = \text{diag}(1, 0, \ldots, 0, 0).
\]

As a consequence again only the first columns \(\phi^{(1)}, \psi^{(1)}\) and the first rows \(\hat{\phi}^{(1)}, \hat{\psi}^{(1)}\) enter into the systems. If we introduce the \(N\)-component complex vectors:

\[
\phi_\alpha(\xi, \eta) = \frac{1}{2}(\phi^{(1)}_{\alpha, 1} + i\phi^{(1)}_{N+\alpha, 1}), \quad \psi_\alpha(\xi, \eta) = \frac{1}{2}(\psi^{(1)}_{\alpha, 1} + i\psi^{(1)}_{N+\alpha, 1})
\]
then the explicit form of the system is:

\[
\frac{\partial \phi_\alpha}{\partial \eta} = -\frac{i}{a} \psi_\alpha \sum_{\beta=1}^{N} (\psi_\beta \phi_\beta^* - \phi_\beta \psi_\beta),
\]

\[
\frac{\partial \psi_\alpha}{\partial \xi} = -\frac{i}{a} \phi_\alpha \sum_{\beta=1}^{N} (\phi_\beta \psi_\beta^* - \psi_\beta \phi_\beta).
\]

The functional of the action is:

\[
A_{GN} = \int_{-\infty}^{\infty} dx \, dt \left( i \sum_{\alpha=1}^{N} \left( \phi_\alpha^* \frac{\partial \phi_\alpha}{\partial \eta} + \psi_\alpha^* \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left( \sum_{\alpha=1}^{N} (\psi_\alpha^* \phi_\alpha - \phi_\alpha^* \psi_\alpha) \right)^2 \right).
\]

iii) Zakharov–Mikhailov models. Now we choose \( g \simeq so(N, \mathbb{R}) \); then \( \psi(\xi, \eta) \) and \( \phi(\xi, \eta) \) are elements of the group \( \mathfrak{g} \simeq SO(N, \mathbb{R}) \). Following [?] we use the standard definition of orthogonal group elements:

\[
\hat{\psi}(\xi, \eta) = \psi^T(\xi, \eta), \quad \hat{\phi}(\xi, \eta) = \phi^T(\xi, \eta).
\]
Now we choose

\[ J = E_{1,N} - E_{N,1}, \]

where the \( N \times N \) matrices \( E_{kp} \) are defined by \( (E_{kp})_{nm} = \delta_{kn}\delta_{pm} \).

As a consequence now the first and the last columns \( \phi^{(1)}, \phi^{(N)}, \psi^{(1)}, \psi^{(N)} \) and the first and the last rows \( \hat{\phi}^{(1)}, \hat{\phi}^{(N)}, \hat{\psi}^{(1)}, \hat{\psi}^{(N)} \) enter into the systems. If we introduce the \( N \)-component complex vectors:

\[ \phi_\alpha(\xi, \eta) = \frac{1}{2}(\phi^{(1)}_\alpha + i\phi^{(N)}_\alpha), \quad \psi_\alpha(\xi, \eta) = \frac{1}{2}(\psi^{(1)}_\alpha + i\psi^{(N)}_\alpha) \]

then the explicit form of the system becomes:

\[
\begin{align*}
    i \frac{\partial \psi_\alpha}{\partial \xi} &= \frac{i}{a} \sum_{\beta=1}^{N} (\phi^*_\alpha \phi_\beta \psi_\beta - \phi_\alpha \phi^*_\beta \psi_\beta), \\
    i \frac{\partial \phi_\alpha}{\partial \eta} &= \frac{i}{a} \sum_{\beta=1}^{N} (\psi^*_\alpha \psi_\beta \phi_\beta - \psi_\alpha \psi^*_\beta \phi_\beta),
\end{align*}
\]
The functional of the action is:

$$A_{ZM} = \int_{-\infty}^{\infty} dx \; dt \left( i \sum_{\alpha=1}^{N} \left( \phi_{\alpha}^{*} \frac{\partial \phi_{\alpha}}{\partial \eta} + \psi_{\alpha}^{*} \frac{\partial \psi_{\alpha}}{\partial \xi} \right) \right)$$

$$- \frac{1}{2a} \left( \sum_{\alpha, \beta=1}^{N} \left( \phi_{\alpha}^{*} \phi_{\beta} - \phi_{\beta}^{*} \phi_{\alpha} \right) \left( \psi_{\alpha}^{*} \psi_{\beta} - \psi_{\beta}^{*} \psi_{\alpha} \right) \right).$$
2 \( \mathbb{Z}_2 \)-Reductions of the spinor models

Apply the idea of the reduction group – Mikhailov (1980) and obtain new types of spinor models generalizing the previous ones.

Start with the Lax representation:

\[
\Psi_\xi = U_R(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \quad \Psi_\eta = V_R(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda),
\]

\[
U_R(\xi, \eta, \lambda) = \frac{U_1(\xi, \eta)}{\lambda - a} + \frac{CU_1(\xi, \eta)C^{-1}}{\epsilon\lambda^{-1} - a}, \quad V_R(\xi, \eta, \lambda) = \frac{V_1(\xi, \eta)}{\lambda + a} + \frac{CV_1(\xi, \eta)C^{-1}}{\epsilon\lambda^{-1} + a},
\]

where \( \epsilon = \pm 1, \ a \neq 1 \) is a real number and \( C \) is an involutive automorphism of \( \mathfrak{g} \). It satisfy also:

\[
U_R(\xi, \eta, \lambda) = CU_R(\xi, \eta, \epsilon\lambda^{-1})C^{-1}, \quad V_R(\xi, \eta, \lambda) = CV_R(\xi, \eta, \epsilon\lambda^{-1})C^{-1},
\]

The new Lax representation is:

\[
\frac{\partial U_R}{\partial \eta} - \frac{\partial V_R}{\partial \xi} + [U_R, V_R] = 0,
\]

which is equivalent to

\[
U_1,\eta + [U_1, V_R(\xi, \eta, a)] = 0, \quad V_1,\xi + [V_1, U_R(\xi, \eta, -a)] = 0.
\]
In the same way as above we get:

**i) \( \mathbb{Z}_2\)-NJLVL models.** Here \( \mathfrak{G} \simeq SU(N) \) and the system takes the form:

\[
\begin{align*}
    i \frac{\partial \vec{\phi}}{\partial \eta} &+ \frac{1}{2a} \vec{\psi}(\vec{\psi}^\dagger \vec{\phi}) + \frac{1}{\epsilon a^{-1} + a} C \vec{\psi}(\vec{\psi}^\dagger \hat{C} \vec{\phi})(\xi, \eta) = 0, \\
    i \frac{\partial \vec{\psi}}{\partial \xi} &+ \frac{1}{2a} \vec{\phi}(\vec{\phi}^\dagger \vec{\psi}) + \frac{1}{\epsilon a^{-1} + a} C \vec{\phi}(\vec{\phi}^\dagger \hat{C} \vec{\psi})(\xi, \eta) = 0.
\end{align*}
\]

where \( \vec{\psi} = (\psi_{\alpha,1}, \ldots, \psi_{\alpha,N})^T \) and \( \vec{\phi} = (\phi_{\alpha,1}, \ldots, \phi_{\alpha,N})^T \).

For the automorphism \( C \) of the \( SU(N) \) group we may have

a) \( C_N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_N), \quad \epsilon_j = \pm 1 \), \quad b) \( C'_N = \begin{pmatrix} 1 & 0 \\ 0 & C_{N-1} \end{pmatrix} \).

where \( C_{N-1} \) belongs to the Weyl group of \( SU(N-1) \) and is such that \( C^2_{N-1} = 1 \). These two special choices of \( C \) are such that

\( \lim_{\xi \to \pm \infty} U_R(\xi, \eta) = \lim_{\xi \to \pm \infty} C U_R(\xi, \eta) \hat{C}. \)
ii) $\mathbb{Z}_2$-GN models. Here $\mathcal{G} \simeq SP(2N, \mathbb{R})$. Two typical choices of $C$ are given by:

\begin{align*}
a) \quad C &= \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}, \\
b) \quad C' &= \begin{pmatrix} 0 & C_2 \\ C_2 & 0 \end{pmatrix},
\end{align*}

where $C_1^2 = C_2^2 = 1$.

$$
\frac{\partial \vec{\phi}}{\partial \eta} = -\frac{i}{a} \vec{\psi} \left((\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi})\right) - \frac{2i}{a + \epsilon a^{-1}} C_1 \vec{\psi} \left((\vec{\psi}^\dagger C_1 \vec{\phi}) - (\vec{\phi}^\dagger C_1 \vec{\psi})\right),
$$

$$
\frac{\partial \vec{\psi}}{\partial \xi} = \frac{i}{a} \vec{\phi} \left((\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi})\right) + \frac{2i}{a + \epsilon a^{-1}} C_1 \vec{\phi} \left((\vec{\psi}^\dagger C_1 \vec{\phi}) - (\vec{\phi}^\dagger C_1 \vec{\psi})\right).
$$

The corresponding action can be written as follows:

$$
A_{\mathbb{Z}_2, GN} = \int_{-\infty}^{\infty} dx \, dt \left( i \left( \vec{\phi}^\dagger \frac{\partial \vec{\phi}}{\partial \eta} + \vec{\psi}^\dagger \frac{\partial \vec{\psi}}{\partial \xi} \right) - \frac{1}{2a} \left((\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi})\right)^2 \\
- \frac{1}{\epsilon a^{-1} + a} \left((\vec{\psi}^\dagger C_1 \vec{\phi}) - (\vec{\phi}^\dagger C_1 \vec{\psi})\right)^2 \right).
$$
The second $\mathbb{Z}_2$-reduced GN-system is:

\[
\frac{\partial \vec{\phi}}{\partial \eta} = -\frac{i}{a} \vec{\psi} \left( (\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C_2 \vec{\psi}^* \left( (\vec{\psi}^T C_2 \vec{\phi}) + (\vec{\psi}^\dagger C_2 \vec{\phi}^*) \right),
\]

\[
\frac{\partial \vec{\psi}}{\partial \xi} = \frac{i}{a} \vec{\phi} \left( (\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C_2 \vec{\phi}^* \left( (\vec{\phi}^T C_2 \vec{\psi}) + (\vec{\phi}^\dagger C_2 \vec{\psi}^*) \right).
\]

These equations can be obtained from the action:

\[
A_{\mathbb{Z}_2, GN_b} = \int_{-\infty}^{\infty} dx \, dt \left( i \left( \vec{\phi}^\dagger \frac{\partial \vec{\phi}}{\partial \eta} + \vec{\psi}^\dagger \frac{\partial \vec{\psi}}{\partial \xi} \right) - \frac{1}{2a} \left( (\vec{\psi}^\dagger, \vec{\phi}) - (\vec{\phi}^\dagger, \vec{\psi}) \right)^2 \right.
\]

\[
- \frac{1}{\epsilon a^{-1} + a} \left( (\vec{\phi}^\dagger C_2 \vec{\psi}^*) + (\vec{\phi}^T C_2 \vec{\psi}) \right)^2 \right).
\]

iii) $\mathbb{Z}_2$-ZM models. Here $\mathfrak{g} \simeq SO(N, \mathbb{R})$. Again we used $N$-component
vectors to cast the $\mathbb{Z}_2$-reduced ZM systems in the form:

\[
\frac{\partial \vec{\psi}}{\partial \xi} = \frac{i}{a} \left( \vec{\phi}^\star (\vec{\phi}^T, \vec{\psi}) - \vec{\phi}(\vec{\phi}^\dagger, \vec{\psi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C \left( \vec{\phi}^\star (\vec{\phi}^T C \vec{\psi}) - \vec{\phi}(\vec{\phi}^\dagger C \vec{\psi}) \right),
\]

\[
\frac{\partial \vec{\phi}}{\partial \eta} = \frac{i}{a} \left( \vec{\psi}^\star (\vec{\psi}^T, \vec{\phi}) - \vec{\psi}(\vec{\psi}^\dagger, \vec{\phi}) \right) + \frac{2i}{a + \epsilon a^{-1}} C \left( \vec{\psi}^\star (\vec{\psi}^T \hat{C} \vec{\phi}) - \vec{\psi}(\vec{\psi}^\dagger \hat{C} \vec{\phi}) \right),
\]

where the involutive automorphism $C$ can be chosen as one of the type:

a) \( C = \text{diag} (\epsilon_1, \epsilon_2, \ldots, \epsilon_2, \epsilon_1), \quad \epsilon_j = \pm 1, \quad \)  

b) \( C' = \begin{pmatrix} 1 & 0 \\ 0 & C_3 \end{pmatrix} \),

with \( C_3^2 = 1 \). For these choices of $C$ we have \( \lim_{\xi \to \pm \infty} U_R(\xi, \eta) = \lim_{\xi \to \pm \infty} CU_R(\xi, \eta) \hat{C} \).
The action for the reduced ZM models is provided by:

\[
A_{Z_2, ZM} = \int_{-\infty}^{\infty} dx \, dt \left( i \left( \phi^\dagger \frac{\partial \phi}{\partial \eta} + \psi^\dagger \frac{\partial \psi}{\partial \xi} \right) + \frac{1}{a} \left( (\psi^\dagger, \phi^*) (\phi^T, \psi) - (\phi^\dagger, \psi)(\psi^\dagger, \phi) \right) + \frac{2}{\epsilon a^{-1} + a} \left( (\psi^\dagger C \phi^*) (\phi^TC \psi) - (\phi^\dagger C \psi)(\psi^\dagger C \phi) \right) \right).
\]

3 RHP with canonical normalization

\[
\xi^+ (\vec{x}, t, \lambda) = \xi^- (\vec{x}, t, \lambda) G(\vec{x}, t, \lambda), \quad \lambda^k \in \mathbb{R}, \quad \lim_{\lambda \to \infty} \xi^+ (\vec{x}, t, \lambda) = 1,
\]

\[
\xi^\pm (\vec{x}, t, \lambda) \in \mathcal{G}
\]

Consider particular type of dependence \( G(\vec{x}, t, \lambda) \):

\[
i \frac{\partial G}{\partial x_s} - \lambda^k [J_s, G(\vec{x}, t, \lambda)] = 0, \quad i \frac{\partial G}{\partial t} - \lambda^k [K, G(\vec{x}, t, \lambda)] = 0.
\]
where $J_s \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP:

$$
\xi^\pm(\vec{x}, t, \lambda) = \exp Q(\vec{x}, t, \lambda), \quad Q(\vec{x}, t, \lambda) = \sum_{k=1}^{\infty} Q_k(\vec{x}, t) \lambda^{-k}.
$$

where all $Q_k(\vec{x}, t) \in \mathfrak{g}$. However,

$$
\mathcal{J}_s(\vec{x}, t, \lambda) = \xi^\pm(\vec{x}, t, \lambda) J_s \hat{\xi}^\pm(\vec{x}, t, \lambda), \quad \mathcal{K}(\vec{x}, t, \lambda) = \xi^\pm(\vec{x}, t, \lambda) K \hat{\xi}^\pm(\vec{x}, t, \lambda),
$$

belong to the algebra $\mathfrak{g}$ for any $J$ and $K$ from $\mathfrak{g}$. If in addition $K$ also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$
[\mathcal{J}_s(\vec{x}, t, \lambda), \mathcal{K}(\vec{x}, t, \lambda)] = 0.
$$

Zakharov-Shabat theorem

**Theorem 1.** Let $\xi^\pm(x, t, \lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables $\vec{x}$ and $t$ as above. Then $\xi^\pm(x, t, \lambda)$
are fundamental solutions of the following set of differential operators:

\[ L_s \xi^{\pm} \equiv i \frac{\partial \xi^{\pm}}{\partial x_s} + U_s(\vec{x}, t, \lambda) \xi^{\pm}(\vec{x}, t, \lambda) - \lambda^k [J_s, \xi^{\pm}(\vec{x}, t, \lambda)] = 0, \]

\[ M \xi^{\pm} \equiv i \frac{\partial \xi^{\pm}}{\partial t} + V(\vec{x}, t, \lambda) \xi^{\pm}(\vec{x}, t, \lambda) - \lambda^k [K, \xi^{\pm}(\vec{x}, t, \lambda)] = 0. \]

**Proof.** Introduce the functions:

\[ g^{\pm}_s(\vec{x}, t, \lambda) = i \frac{\partial \xi^{\pm}}{\partial x_s} \hat{\xi}^{\pm}(\vec{x}, t, \lambda) + \lambda^k \xi^{\pm}(\vec{x}, t, \lambda) J_s \hat{\xi}^{\pm}(\vec{x}, t, \lambda), \]

\[ g^{\pm}(\vec{x}, t, \lambda) = i \frac{\partial \xi^{\pm}}{\partial t} \hat{\xi}^{\pm}(\vec{x}, t, \lambda) + \lambda^k \xi^{\pm}(\vec{x}, t, \lambda) K \hat{\xi}^{\pm}(\vec{x}, t, \lambda), \]

and prove that

\[ g^+_s(\vec{x}, t, \lambda) = g^-_s(\vec{x}, t, \lambda), \quad g^+(\vec{x}, t, \lambda) = g^-(\vec{x}, t, \lambda), \]

which means that these functions are analytic functions of \( \lambda \) in the whole complex \( \lambda \)-plane. Next we find that:

\[ \lim_{\lambda \to \infty} g^+_s(\vec{x}, t, \lambda) = \lambda^k J_s, \quad \lim_{\lambda \to \infty} g^+(\vec{x}, t, \lambda) = \lambda^k K. \]
and make use of Liouville theorem to get

\[
g^+_s(\vec{x}, t, \lambda) = g^-_s(\vec{x}, t, \lambda) = \lambda^k J_s - \sum_{l=1}^{k} U_{s;l}(\vec{x}, t) \lambda^{k-l},
\]

\[
g^+(\vec{x}, t, \lambda) = g^-(\vec{x}, t, \lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(\vec{x}, t) \lambda^{k-l}.
\]

We shall see below that the coefficients \( U_{s;l}(\vec{x}, t) \) and \( V_l(\vec{x}, t) \) can be expressed in terms of the asymptotic coefficients \( Q_s \).

**Lemma 1.** The set of operators \( L_s \) and \( M \) commute, i.e. the following set of equations hold:

\[
i \frac{\partial U_s}{\partial x_j} - i \frac{\partial U_j}{\partial x_s} + [U_s(\vec{x}, t, \lambda) - \lambda^k J_s, U_j(\vec{x}, t, \lambda) - \lambda^k J_j] = 0,
\]

\[
i \frac{\partial U_s}{\partial t} - i \frac{\partial V}{\partial x_s} + [U_s(\vec{x}, t, \lambda) - \lambda^k J_s, V(\vec{x}, t, \lambda) - \lambda^k K] = 0.
\]
where

\[ U_s(\vec{x}, t, \lambda) = \sum_{l=1}^{k} U_{s;l}(\vec{x}, t) \lambda^{k-l}, \quad V(\vec{x}, t, \lambda) = \sum_{l=0}^{k} V_{l}(\vec{x}, t) \lambda^{k-l}. \]

Proof. The set of the operators \( L_s \) and \( M \) have a common FAS, i.e. they all must commute.

\[ \square \]

4 Jets of order \( k \)

Consider the jets of order \( k \) of \( \mathcal{J}(x, \lambda) \) and \( \mathcal{K}(x, \lambda) \):

\[ \mathcal{J}_s(\vec{x}, t, \lambda) \equiv \left( \lambda^k \xi^\pm(\vec{x}, t, \lambda) J_l \xi^\pm(\vec{x}, t, \lambda) \right)_+ = \lambda^k J_s - U_s(\vec{x}, t, \lambda), \]

\[ \mathcal{K}(\vec{x}, t, \lambda) \equiv \left( \lambda^k \xi^\pm(\vec{x}, t, \lambda) K \xi^\pm(\vec{x}, t, \lambda) \right)_+ = \lambda^k K - V(\vec{x}, t, \lambda). \]
Express $U_s(x) \in g$ in terms of $Q_s(x)$:

$$J_s(\vec{x}, t, \lambda) = J_s + \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}^k Q J_s, \quad K(\vec{x}, t, \lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}^k K,$$

and therefore for $U_{s;l}$ we get:

$$U_{s;1}(\vec{x}, t) = -\text{ad}_{Q_1} J_s, \quad U_{s;2}(\vec{x}, t) = -\text{ad}_{Q_2} J_s - \frac{1}{2} \text{ad}^2_{Q_1} J_s$$

$$U_{s;3}(\vec{x}, t) = -\text{ad}_{Q_3} J_s - \frac{1}{2} (\text{ad}_{Q_2} \text{ad}_{Q_1} + \text{ad}_{Q_1} \text{ad}_{Q_2}) J_s - \frac{1}{6} \text{ad}^3_{Q_1} J_s$$

$$\vdots$$

$$U_{s;k}(\vec{x}, t) = -\text{ad}_{Q_k} J_s - \frac{1}{2} \sum_{s+p=k} \text{ad}_{Q_s} \text{ad}_{Q_p} J_s$$

$$- \frac{1}{6} \sum_{s+p+r=k} \text{ad}_{Q_s} \text{ad}_{Q_p} \text{ad}_{Q_r} J_s - \cdots - \frac{1}{k!} \text{ad}^k_{Q_1} J_s,$$

and similar expressions for $V_l(\vec{x}, t)$ with $J_s$ replaced by $K$. 

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5 Reductions of polynomial bundles

a) \( A\xi^+(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^-(x,t,\lambda) \), \( AQ^+(x,t,\epsilon\lambda^*)\hat{A} = -Q(x,t,\lambda) \),
b) \( B\xi^+,(x,t,\epsilon\lambda^*)\hat{B} = \hat{\xi}^-(x,t,\lambda) \), \( BQ^+(x,t,\epsilon\lambda^*)\hat{B} = Q(x,t,\lambda) \),
c) \( C\xi^+,T(x,t,-\lambda)\hat{C} = \hat{\xi}^-(x,t,\lambda) \), \( CQ^+(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda) \),

where \( \epsilon^2 = 1 \) and \( A, B \) and \( C \) are elements of the group \( \mathfrak{g} \) such that \( A^2 = B^2 = C^2 = 1 \). As for the \( \mathbb{Z}_N \)-reductions we may have:

\[ D\xi^\pm(x,t,\omega\lambda)\hat{D} = \xi^\pm(x,t,\lambda), \quad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda), \]

where \( \omega^N = 1 \) and \( D^N = 1 \).
6 On $N$-wave equations $(k = 1)$ in 2 and more dimensions

Lax representation involves two Lax operators linear in $\lambda$:

\[ L\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial x} + [J, Q(x, t)]\xi^\pm(x, t, \lambda) - \lambda [J, \xi^\pm(x, t, \lambda)] = 0, \]
\[ M\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial t} + [K, Q(x, t)]\xi^\pm(x, t, \lambda) - \lambda [K, \xi^\pm(x, t, \lambda)] = 0. \]

The corresponding equations take the form:

\[ i \left[ J, \frac{\partial Q}{\partial t} \right] - i \left[ K, \frac{\partial Q}{\partial x} \right] - [[J, Q], [K, Q(x, t)]] = 0 \]

\[ Q(x, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \quad J = \text{diag} (a_1, a_2, a_3), \quad K = \text{diag} (b_1, b_2, b_3), \]
Then the 3-wave equations take the form:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2} \frac{\partial u_1}{\partial x} + \kappa \epsilon_1 \epsilon_2 u_2^* u_3 &= 0, \\
\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3} \frac{\partial u_2}{\partial x} + \kappa \epsilon_1 u_1^* u_3 &= 0, \\
\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3} \frac{\partial u_3}{\partial x} + \kappa \epsilon_2 u_1^* u_2^* &= 0,
\end{align*}
\]

where

\[
\kappa = a_1 (b_2 - b_3) - a_2 (b_1 - b_3) + a_3 (b_1 - b_2).
\]

For 3-dimensional space-time we consider \( Q \) as above, but now let \( u_j \) and \( v_j \) be functions of \( x_1 = x, x_2 = y \) and \( t \). Let also \( J_1 = J \) and \( J_2 = I = \text{diag} (c_1, c_2, c_3) \). Now the corresponding solution of the RHP \( \xi^\pm (x, y, t, \lambda) \) will be FAS not only of \( L \) and \( M \) above, but also of

\[
P \xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial y} + [I, Q(x, t)] \xi^+(\bar{x}, t, \lambda) - \lambda [I, \xi^+(\bar{x}, t, \lambda)] = 0,
\]
and all these three operators will mutually commute, i.e. along with 
\([L, M] = 0\) we will have also \([L, P] = 0\) and \([P, M] = 0\). As a result 
\(Q(x, y, t)\) will satisfy two more 3-wave NLEE

\[
2 \frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2} \frac{\partial u_1}{\partial x} - \frac{a_1 - a_2}{c_1 - c_2} \frac{\partial u_1}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_1\epsilon_2 u_2^* u_3 = 0, \\
2 \frac{\partial u_2}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3} \frac{\partial u_2}{\partial x} - \frac{a_1 - a_3}{c_1 - c_3} \frac{\partial u_2}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_1 u_1^* u_3 = 0, \\
2 \frac{\partial u_3}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3} \frac{\partial u_3}{\partial x} - \frac{a_2 - a_3}{c_2 - c_3} \frac{\partial u_3}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_2 u_1^* u_2^* = 0.
\]

\[\kappa_1 = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2),\]
\[\kappa_2 = a_1(c_2 - c_3) - a_2(c_1 - c_3) + a_3(c_1 - c_2).\]

For \(N\)-wave equations related to Lie algebras \(\mathfrak{g}\) of higher rank \(r\) we can add up to \(r\) auxiliary variables:

\[
r \frac{\partial Q}{\partial t} - \sum_{s=1}^{r} (\text{ad}_{J_s}^{-1}\text{ad}_J) \frac{\partial Q}{\partial x_s} - i \sum_{s=1}^{r} \text{ad}_{J_s}^{-1} [[J, Q], [J_s, Q(\vec{x}, t)]] = 0
\]
where $Q$ is an $n \times n$ off-diagonal matrix depending on $r + 1$ variables. We remind that if $J = \text{diag} (a_1, \ldots, a_n)$ then

$$(\text{ad}_J Q)_{jk} \equiv ([J, Q])_{jk} = (a_j - a_k)Q_{jk},$$

$$(\text{ad}^{-1}_J Q)_{jk} = \frac{1}{a_j - a_k}Q_{jk},$$

and similarly for the other $J_s$.

7 New $N$-wave equations ($k = 2$) in 2 and more dimensions

Let $\mathfrak{g} = sl(3)$ and

$$Q_1(\vec{x}, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \quad Q_2(\vec{x}, t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix},$$
Fix up $k = 2$. Then the Lax pair becomes

$$L\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial x} + U(x, t, \lambda)\xi^\pm(x, t, \lambda) - \lambda^2]J, \xi^\pm(x, t, \lambda)\right] = 0,$$

$$M\xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial t} + V(x, t, \lambda)\xi^\pm(x, t, \lambda) - \lambda^2]K, \xi^\pm(x, t, \lambda)\right] = 0,$$

where

$$U \equiv U_2 + \lambda U_1 = \left([[J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)]\right) + \lambda[J, Q_1],$$

$$V \equiv V_2 + \lambda V_1 = \left([[K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)]\right) + \lambda[K, Q_1].$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \text{diag}(1, \epsilon, 1), \epsilon^2 = 1$. Thus $Q_1$ and $Q_2$ get reduced into:

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix}.$$
New type of integrable 3-wave equations:

\[ i(a_1 - a_2) \frac{\partial u_1}{\partial t} - i(b_1 - b_2) \frac{\partial u_1}{\partial x} + \epsilon \kappa u_2^* u_3 + \epsilon \frac{\kappa(a_1 - a_2)}{(a_1 - a_3)} u_1 |u_2|^2 = 0, \]

\[ i(a_2 - a_3) \frac{\partial u_2}{\partial t} - i(b_2 - b_3) \frac{\partial u_2}{\partial x} + \epsilon \kappa u_1^* u_3 - \epsilon \frac{\kappa(a_2 - a_3)}{(a_1 - a_3)} |u_1|^2 u_2 = 0, \]

\[ i(a_1 - a_3) \frac{\partial u_3}{\partial t} - i(b_1 - b_3) \frac{\partial u_3}{\partial x} - \frac{i \kappa}{a_1 - a_3} \frac{\partial (u_1 u_2)}{\partial x} + \epsilon \kappa \left( \frac{a_1 - a_2}{a_1 - a_3} |u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3} |u_2|^2 \right) u_1 u_2 + \epsilon \kappa u_3 (|u_1|^2 - |u_2|^2) = 0, \]

where:

\[ u_3 = w_3 + \frac{2a_2 - a_1 - a_3}{2(a_1 - a_3)} u_1 u_2. \]

The diagonal terms in the Lax representation are \( \lambda \)-independent.
Two of them read:

\[
\begin{align*}
    i(a_1 - a_2) \frac{\partial |u_1|^2}{\partial t} & - i(b_1 - b_2) \frac{\partial |u_1|^2}{\partial x} - \epsilon \kappa (u_1 u_2 u^*_3 - u_1^* u_2^* u_3) = 0, \\
    i(a_2 - a_3) \frac{\partial |u_2|^2}{\partial t} & - i(b_2 - b_3) \frac{\partial |u_2|^2}{\partial x} - \epsilon \kappa (u_1 u_2 u^*_3 - u_1^* u_2^* u_3) = 0,
\end{align*}
\]

These relations are satisfied identically as a consequence of the NLEE.

Let the sewing function \( G \) of the RHP depends on 3 variables: \( t, x_1 = x \) and \( x_2 = y \) with \( J_1 = J \) and \( J_2 = I = \text{diag}(c_1, c_2, c_3) \). For \( k = 2 \) we obtain: \( L, M \) and

\[
P \xi^\pm \equiv i \frac{\partial \xi^\pm}{\partial y} + W(x, y, t, \lambda) \xi^\pm(x, y, t, \lambda) - \lambda^2 I, \xi^\pm(x, y, t, \lambda)] = 0,
\]

\[
W \equiv W_2 + \lambda W_1
\]

\[
= \left( [I, Q_2(x, y, t)] - \frac{1}{2} [[I, Q_1], Q_1(x, y, t)] \right) + \lambda [I, Q_1(x, y, t)],
\]
commuting identically with respect to $\lambda$. It is obvious that $[L, P] = 0$ if

$$i(a_1 - a_2) \frac{\partial u_1}{\partial t} - i(c_1 - c_2) \frac{\partial u_1}{\partial y} + \epsilon \kappa_2 u_2^* u_3 + \epsilon \frac{\kappa_2 (a_1 - a_2)}{(a_1 - a_3)} u_1 |u_2|^2 = 0,$$

$$i(a_2 - a_3) \frac{\partial u_2}{\partial t} - i(c_2 - c_3) \frac{\partial u_2}{\partial y} + \epsilon \kappa_2 u_1^* u_3 - \epsilon \frac{\kappa_2 (a_2 - a_3)}{(a_1 - a_3)} |u_1|^2 u_2 = 0,$$

$$i(a_1 - a_3) \frac{\partial u_3}{\partial t} - i(c_1 - c_3) \frac{\partial u_3}{\partial y} - \frac{i\kappa_2}{a_1 - a_3} \frac{\partial (u_1 u_2)}{\partial y}$$

$$+ \epsilon \kappa_2 \left( \frac{a_1 - a_2}{a_1 - a_3} |u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3} |u_2|^2 \right) u_1 u_2 + \epsilon \kappa_2 u_3 (|u_1|^2 - |u_2|^2) = 0,$$
\[
2i \frac{\partial u_1}{\partial t} - i(\vec{v}_{(1)} \cdot \nabla)u_1 + \epsilon(\kappa_1 + \kappa_2) \left( \frac{u_2^* u_3}{a_1 - a_2} + \frac{u_1 |u_2|^2}{(a_1 - a_3)} \right) = 0,
\]
\[
2i \frac{\partial u_2}{\partial t} - i(\vec{v}_{(2)} \cdot \nabla)u_2 + \epsilon(\kappa_1 + \kappa_2) \left( \frac{u_1^* u_3}{a_1 - a_3} - \frac{u_2 |u_1|^2}{(a_1 - a_3)} \right) = 0,
\]
\[
2i \frac{\partial u_3}{\partial t} - i(\vec{v}_{(3)} \cdot \nabla)u_3 - i(\vec{\kappa} \cdot \nabla)(u_1 u_2) + \frac{\epsilon(\kappa_1 + \kappa_2)}{a_1 - a_3} (|u_1|^2 - |u_2|^2) u_3
\]
\[+ \frac{\epsilon(\kappa_1 + \kappa_2)}{(a_1 - a_3)^2} ((a_1 - a_2)|u_1|^2 + (a_2 - a_3)|u_2|^2) u_1 u_2 = 0.
\]

Here \( \nabla = (\partial_x, \partial_y)^T \), the characteristic velocities \( \vec{v}_{(j)}, j = 1, 2, 3 \) and \( \vec{\kappa} \) are two-component vectors given by:

\[
\vec{v}_{(1)} = \frac{1}{a_1 - a_2} \begin{pmatrix} b_1 - b_2 \\ c_1 - c_2 \end{pmatrix}, \quad \vec{v}_{(2)} = \frac{1}{a_2 - a_3} \begin{pmatrix} b_2 - b_3 \\ c_2 - c_3 \end{pmatrix},
\]

\[
\vec{v}_{(3)} = \frac{1}{a_1 - a_3} \begin{pmatrix} b_1 - b_3 \\ c_1 - c_3 \end{pmatrix}, \quad \vec{\kappa} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix},
\]

and \( \kappa_1 = \kappa \).
8 Conclusions and open questions

• We constructed new integrable spinor models and new integrable 3- and \( N \)-wave equations.

• These new NLEE must be Hamiltonian. The Poisson brackets for polynomial bundles – see Kulish, Reyman and Semenov-Tyan-Shanskii (1981-1983);

• The method allows one also to apply Zakharov-Shabat dressing method for constructing their explicit (\( N \)-soliton) solutions – T. Valchev.

• This approach improves Gel’fand-Dickey approach.