Recursion Operators and expansions over adjoint solutions for the Caudrey-Beals-Coifman system with $\mathbb{Z}_p$ reductions of Mikhailov type

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June 5, 2012

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Introduction

Nonlinear evolution equations (NLEEs) of soliton type

$$(q_\alpha)_t = F_\alpha(q, q_x, ...), \quad q = (q_\alpha)_{1\leq\alpha\leq s} \tag{1}$$

are equations admitting **Lax representation**

$$[L, A] = 0$$

where L, A are linear operators on ∂_x, ∂_t depending also on some functions $q_\alpha(x,t)$, $1 \leq \alpha \leq s$ (called 'potentials') and a spectral parameter λ.

Hierarchy of NLEEs related to $L\psi = 0$ (auxiliary linear problem) – the evolution equations obtaine changing A.

Integration. Most of the schemes share the property: the Lax representation permits to pass from the original evolution defined by the equations (1) to the evolution of some spectral data related to the problem $L\psi = 0$: **Faddeev, Takhtadjian 1987; Gerdjikov, Vilasi, Yanovski 2008**.

The Caudrey-Beals-Coifman system (CBC system), called the **Generalized Zakharov-Shabat system (GZS system)** in the case when the element J is real, is one of the best known auxiliary linear problems:

$$L\psi = (\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = 0 \tag{2}$$

Originally J was *fixed, real and traceless* $n \times n$ *diagonal matrix with mutually distinct diagonal elements and* $q(x)$ *is a matrix function with values in the space of the off-diagonal matrices*, **Zakharov, Manakov, Novikov, Pitaevski 1981**. The assumption that J is a real simplifies substantially both the spectral theories of L and the Recursion Operators **Gerdjikov, Kulish 1981; Gerdjikov 1986**.

Next step: the case when J *is a complex, traceless* $n \times n$ *matrix with mutually distinct diagonal elements and* $q(x)$ *is a matrix function taking values in the space of the off-diagonal matrices.* **Caudrey 1982, Beals and Coifman 1984, 1985; Beals, Sattinger 1991; Zhou 1989**

Final step: The case when $q(x)$ *and* J *belong to a fixed simple Lie algebra* $\mathfrak{g}$ *in some finite dimensional irreducible representation*, **Gerdjikov, Yanovski, 1994**. *The element* J *should be regular, that is* $\ker \mathrm{ad}_J$ *(*$\mathrm{ad}_J(X) \equiv [J, X]$, $X \in \mathfrak{g}$*) is a Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$. $q(x)$ *belongs to the orthogonal complement* $\mathfrak{h}^\perp = \bar{\mathfrak{g}}$ *of* $\mathfrak{h}$ *with respect to the Killing form:* $\langle X, Y \rangle = \mathrm{tr}\,(\mathrm{ad}_X \mathrm{ad}_Y)$; $X, Y \in \mathfrak{g}$. Thus $q(x) = \sum_{\alpha\in\Delta} q_\alpha E_\alpha$ where E_α are the root vectors; Δ is the root system of $\mathfrak{g}$. The scalar functions $q_\alpha(x)$ are defined on $\mathbb{R}$, are *complex valued, smooth and tend to zero as* $x \to \pm\infty$. *We can assume that they are Schwartz-type functions.* Classical Zakharov-Shabat system is obtained for $\mathfrak{g} = \mathrm{sl}\,(2, \mathbb{C})$, $J = \mathrm{diag}\,(1, -1)$.

AKNS approach to the soliton equations.

We construct the so-called **adjoint solutions of** L that is functions of the type $w = mXm^{-1}$ where $X = \mathrm{const}\,$, $X \in \mathfrak{g}$ and m is fundamental solution of $Lm = 0$. Indeed they satisfy the equation:

$$[L, w] = (\mathrm{i}\partial_x w + [q(x) - \lambda J, w]) = 0$$

Let $w^{\rm a} = \pi_0$, $w^{\rm d} = ({\rm id} - \pi_0)w$ where π_0 is the orthogonal projector (with respect to the Killing form) of w over $\mathfrak{h}^{\perp}$ and $\mathfrak{h}$ respectively. Then
1. If a suitable set of adjoint solutions $(w_i(x,\lambda))_i$ is taken, for λ on the spectrum of L the functions $w_i^{\rm a}(x,\lambda)$ form a complete set in the space of potentials $q(x)$.
2. If one expands the potential over $(w_i(x,\lambda))_i$ as coefficients one gets the minimal scattering data for L.

Recursion Operators

Passing from the potentials to the scattering data can be considered as Generalized Fourier Transform. For it $w_i^{\rm a}(x,\lambda)$ play the same role the exponents play in the Fourier Transform. **The Recursion Operators (Generating Operators, Λ-operators) are the operators for which the adjoint solutions $w_i^{\rm a}(x,\lambda)$ introduced above are eigenfunctions and therefore for the Generalized Fourier Transform they play the same role as the differentiation operator in the Fourier Transform method.**

For the above reason Recursion Operators play important role in the theory of soliton equations - it is a theoretical too which apart from explicit solutions can give most of the information about the NLEEs. Through them can be obtained:

i) The hierarchies of the nonlinear evolution equations solvable through L

ii) The conservation laws for these NLEEs

iii) The hierarchies of Hamiltonian structures for these NLEEs

It is not hard to get that the Recursion Operators related to L have the form

$$\Lambda_{\pm}(X(x)) = \mathrm{ad}_J^{-1}\left(\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{i}\,\mathrm{ad}_q \int_{\pm\infty}^{x} (\mathrm{id} - \pi_0)[q(y), X(y)]\mathrm{d}y.\right) \tag{3}$$

where of course $\mathrm{ad}_q(X) = [q, X]$ and X is a smooth, fast decreasing function with values in $\mathfrak{h}^{\perp}$.

Recursion Operators name origin

For NLEEs such that $[L, A] = 0$ where A is of the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \quad A_n \in \mathfrak{h}, \quad A_n = \mathrm{const}\,, \quad A_{n-1} \in \mathfrak{h}^{\perp}$$

it follows that $A_{n-1} = \mathrm{ad}_J^{-1}[q, A]$ and for $0 < k < n-1$ and the recursion relations

$$\pi_0 A_{k-1} = \Lambda_{\pm}(\pi_0 A_k), \quad (\mathrm{id} - \pi_0) A_k = \mathrm{i}(\mathrm{id} - \pi_0) \int_{\pm\infty}^{x} [q, \pi_0 A_k](y) \mathrm{d}y \tag{4}$$

Then the NLEEs related to L can be written into one of the two forms:

$$\mathrm{iad}_J^{-1} q_t + \Lambda_{\pm}^{n} \left(\mathrm{ad}_J^{-1} [A_n, q] \right) = 0 \tag{5}$$

Thus the Recursion Operators can be introduced also purely algebraically as the operators solving the above recursion relations.

Geometric Interpretation

The Recursion Operators have interesting geometric interpretation as dual objects to a Nijenhuis tensors N on the manifold of potentials on which it is defined a special geometric structure, Poisson- Nijenhuis structure. In their turn the NLEEs related to L are fundamental fields of that structure. This interpretation has been given by F Magri, **Magri 1978**. In full the geometric theory of the Recursion Operators is presented in **Gerdjikov, Vilasi, Yanovski 2008**. Summarizing, the Recursion Operators have three important aspects:

- **They appear naturally considering recursion relations arising from the Lax representations of the NLEEs related with L.**
- **In the Generalized Fourier expansions they play the role similar of the role of differentiation in the Fourier expansions.**
- **Their adjoint operatos are Nijenhuis tensors for some special geometric structure on the manifold of potentials - Poisson-Nijenhuis structures.**

We shall discuss here the implications of the Mikhailov-type reductions on the theory of Recursion Operators. It has been considered recently in several papers, for example **Gerdjikov, Mikhailov, Valchev 2010; Valchev 2011, Gerdjikov, Grahovski, Mikhailov, Valchev, 2011; Yanovski 2011**. In these papers the case of the CBC system in pole gauge is treated. The CBC system in canonical gauge (the one we discuss) subject to reductions has been considered earlier. For example, in **Grahhovski 2002, Grahovski 2003** were investigated the implications to the scattering data. In **Gerdjikov, Kostov, Valchev 2009** the Recursion Operators has been considered from spectral theory viewpoint. A general result about the geometry of the Recursion Operators for L is presented in **Yanovski 2012**. From the other side, though there are number of papers treating what happens with the spectral expansions related with the Recursion Operators in concrete situations with $\mathbb{Z}_p$ reductions, there has been no general treatment and in this article we shall try to fill this gap.

Fundamental analytical solutions for the CBC system

If $q(x) = \sum_{\alpha\in\Delta} q_\alpha(x) E_\alpha$ we define: $\|q\|_1 = \sum\limits_{\alpha\in\Delta} \int\limits_{-\infty}^{+\infty} |q_\alpha(x)| \mathrm{d}x$. Potentials for which $\|q\|_1 < \infty$ form a Banach space $L^1(\bar{\mathfrak{g}}, \mathbb{R})$. Main facts related to the spectral properties of the solutions of the (2) with $q \in L^1(\bar{\mathfrak{g}})$ were CBC system is considered in some irreducible matrix representation defined on a space V are obtained in **Gerdjikov,Yanovski 1994**. Let $m(x,\lambda) = \psi(x,\lambda) \exp \mathrm{i}\lambda J x$ where ψ satisfies CBC system. Then:

$$\mathrm{i}\partial_x m + q(x)m - \lambda J m + \lambda m J = 0 \quad \lim_{x\to-\infty} m = \mathbf{1}_V \tag{6}$$

Theorem 0.1 *Suppose that for fixed λ the bounded fundamental solution $m(x,\lambda)$, satisfying the equation (2) exists. Suppose that λ does not belong to the bunch of straight lines $\Sigma = \cup_{\alpha\in\Delta} l_\alpha$ where*

$$l_\alpha = \{\lambda : \mathrm{Im}(\lambda\alpha(J)) = 0\} \tag{7}$$

Then the solution $m(x,\lambda)$ is unique. (In the above Im *denotes the imaginary part).*

Next, suppose Γ is the system of weights in the representation of $\mathfrak{g}$ for which we are considering the CBC system. We then have the following system of integral equations which as readily checked is equivalent to the differential equation (6):

$$\langle\gamma_1|m|\gamma_2\rangle = \langle\gamma_1|\gamma_2\rangle + \mathrm{i}\int\limits_{-\infty}^{x}\langle\gamma_1|q(y)m(y)|\gamma_2\rangle e^{-\mathrm{i}\lambda(\gamma_1-\gamma_2)(J)(x-y)}\mathrm{d}y \tag{8}$$

$$\text{for}\quad \mathrm{Im}(\lambda(\gamma_1-\gamma_2)(J)) \leq 0, \quad \gamma_1,\gamma_2\in\Gamma$$

$$\langle\gamma_1|m|\gamma_2\rangle = \mathrm{i}\int\limits_{+\infty}^{x}\langle\gamma_1|q(y)m(y)|\gamma_2\rangle e^{-\mathrm{i}\lambda(\gamma_1-\gamma_2)(J)(x-y)}dy \tag{9}$$

$$\text{for}\quad \mathrm{Im}(\lambda(\gamma_1-\gamma_2)(J)) > 0, \quad \gamma_1,\gamma_2\in\Gamma$$

For $\gamma_1,\gamma_2\in\Gamma$, consider the lines:

$$l_{\gamma_1,\gamma_2} = \{\lambda : \mathrm{Im}\lambda(\gamma_1-\gamma_2)(J) = 0\},\ \ (\gamma_1-\gamma_2)(J)\neq 0 \tag{10}$$

The set of these lines coincides with the set of lines $\Sigma = \cup_{\alpha\in\Delta} l_\alpha$ **introduced earlier in (7). The connected components of the set** $\mathbb{C}\setminus\Sigma$ **are open sectors in the** λ**-plain. In every such sector either** $\mathrm{Im}[\lambda(\gamma_1-\gamma_2)(J)]$**,** $\gamma_1,\gamma_2\in\Gamma$ **is identically zero or it has the same sign. We denote these sectors by** Ω_ν **and order them anti-clockwise.** Clearly ν takes values from 1 to some even number $2M$. Thus:

$$\mathbb{C}\setminus\Sigma = \bigcup_{\nu=1}^{2M}\Omega_\nu, \;\; \Omega_\nu\bigcap\Omega_\mu = \emptyset, \quad \nu\neq\mu \tag{11}$$

In the ν-th sector we introduce the ordering :

$$\begin{aligned} \alpha \geq_\nu \beta \quad &\text{iff} \quad \mathrm{Im}\lambda(\alpha-\beta)(J)\geq 0 \\ \alpha >_\nu \beta \quad &\text{iff} \quad \mathrm{Im}\lambda(\alpha-\beta)(J) > 0 \end{aligned} \tag{12}$$

Then the system of integral equations can be written in every sector Ω_ν :

$$\begin{aligned}
&\langle\alpha|m|\beta\rangle = \langle\alpha|\beta\rangle + \mathrm{i}\int\limits_{-\infty}^{x} \langle\alpha|q(y)m(y)|\beta\rangle e^{-\mathrm{i}\lambda(\alpha-\beta)(J)(x-y)}\mathrm{d}y \\
&\text{for}\quad \alpha-\beta \leq_\nu 0, \quad \alpha,\beta\in\Gamma \\
&\langle\alpha|m\beta|\rangle = \mathrm{i}\int\limits_{+\infty}^{x} \langle\alpha|q(y)m(y)|\beta\rangle e^{-\mathrm{i}\lambda(\alpha-\beta)(J)(x-y)}\mathrm{d}y \\
&\text{for}\quad \alpha-\beta >_\nu 0, \quad \alpha,\beta\in\Gamma
\end{aligned} \tag{13}$$

Thus there is system of integral equations in every Ω_ν, $\nu = 1, 2, \ldots, 2M$. We count the sectors anticlockwise and then the boundary of the sector Ω_ν consists of two rays - $L_{\nu-1}$ and L_ν ($L_{\nu-1}$ comes before L_ν when we turn anti-clockwise) so that $\bar{\Omega}_\nu \cap \bar{\Omega}_{\nu+1} = L_\nu$. Of course, we understand the number ν modulo $2M$.

For small potentials there is no discrete spectrum, more precisely one has the following Theorem:

Theorem 0.2 *If the potential* $q(x) \in L^1(\bar{\mathfrak{g}}, \mathbb{R})$ *is such that* $q_1 < 1$ *then for* $\lambda \in \Omega_\nu$ *there exists unique analytical solution* $m(x, \lambda)$ *with the following properties:*

1. *If* q *has integrable derivatives up to the* n*-th order then* $m(x, \lambda) = \mathbf{1}_V + \sum_{i=1}^{n} a_i(x)\lambda^{-i} + o(\lambda^{-(n+1)})$ *when* $|\lambda| \to \infty$, *uniformly in* $x \in \mathbb{R}$, *where the coefficients* $a_i(x)$ *are calculated through* q *and its* x*-derivatives. In particular, for absolutely integrable* q *we have* $\lim_{\lambda \to \infty} m(x, \lambda) = \mathbf{1}_V$

2. *The solution* $m(x, \lambda)$ *allows continuous extension to the closure* $\overline{\Omega}_\nu$ *of the sector* Ω_ν.

3. *The solution* $m(x, \lambda)$ *and its inverse obey the estimates* $m_\infty < (1 - q_1)^{-1}$, $m^{-1} < (1 - q_1)^{-1}$.

For potentials that are not small the typical approach is to consider potentials on compact support and then to pass to Lebesgue integrable potentials. The situation is complicated, there is discrete spectrum etc.

Expansions over adjoint solutions

In order to introduce them we first define in each Ω_ν analytic solutions $\chi_\nu(x,\lambda)$ of (2)

$$m_\nu(x,\lambda) = \chi_\nu(x,\lambda)e^{i\lambda Jx} \tag{14}$$

and then we set

$$e_\alpha^\nu(x,\lambda) = \pi_0(\chi_\nu(x,\lambda)E_\alpha\chi_\nu^{-1}(x,\lambda)), \quad \lambda\in\bar{\Omega}_\nu \tag{15}$$

This notation is better to be changed a little because for $\lambda\in L_\nu$ it will be good to retain the index ν to refer to the ray L_ν. Then it becomes necessary to distinguish from what sector the solution is extended. So for $\lambda\in L_\nu$ we shall write $e_\alpha^{(+;\nu)}(x,\lambda)$ if the solution is extended from the sector $\Omega_{\nu-1}$ and $e_\alpha^{(-;\nu)}(x,\lambda)$ if the solution is extended from the sector Ω_ν. In other words, for $\lambda\in L_\nu$

$$\begin{aligned} e_\alpha^{\nu;+}(x,\lambda) &= \pi_0(\chi_\nu(x,\lambda)E_\alpha\chi_\nu^{-1}(x,\lambda)) \\ e_\alpha^{\nu;-}(x,\lambda) &= \pi_0(\chi_{\nu-1}(x,\lambda)E_\alpha\chi_{\nu-1}^{-1}(x,\lambda)) \end{aligned} \tag{16}$$

Then the completeness relations (no discrete spectrum) run:

$$\Pi_0\delta(x-y) = \frac{1}{2\pi}\sum_{\nu=1}^{M}\int_{L_\nu} \mathrm{d}\lambda\{\sum_{\alpha\in\Delta_\nu^+} e_\alpha^{(-;\nu)}(x)\otimes e_{-\alpha}^{(-;\nu)}(y) - \sum_{\alpha\in\Delta_{\nu-1}^+} e_\alpha^{(+;\nu)}(x)\otimes e_{-\alpha}^{(+;\nu)}(y)\} \tag{17}$$

where $\Pi_0 = \sum_{\gamma\in\Delta}\frac{|\gamma\rangle\langle\gamma|}{\gamma(J)}$. Here we assumed that the rays are oriented from 0 to ∞ and we have omitted the dependence on λ in order to be able to write nicely the formula.

The formula itself must be understood in the following way. First, it it assumed that $\mathfrak{g}^*$ is identified with $\mathfrak{g}$, assuming that the pairing is given by the Killing form. So for example, for $X, Y, Z \in \mathfrak{g}$ making a contraction of $X\otimes Y$ with Z on the right we obtain $X\langle Y, Z\rangle$ and making contraction from the left we get $\langle Z, X\rangle Y$. Next, the formula for Π_0 implies that making a contraction with Π_0 the right we get $\Pi_0 X = \mathrm{ad}_J^{-1}\pi_0 X$ and similarly from the left $X\Pi_0 = \mathrm{ad}_J^{-1}\pi_0 X$. (On the space $\bar{\mathfrak{g}}$ the operator ad_J is invertible).

Suppose that we have a L^1-integrable function $h : \mathbb{R} \mapsto \bar{\mathfrak{g}}$. Making a contraction of $\mathrm{ad}\,_J h = [J, h]$ with (17) from the right (left) and integrating over y from $-\infty$ to $+\infty$ we get:

$$h(x) =$$

$$\frac{1}{2\pi}\sum_{\nu=1}^{M}\int_{L_\nu}\{\sum_{\alpha\in\Delta_\nu^+} e_\alpha^{(-;\nu)}(x)\langle\langle e_{-\alpha}^{(-;\nu)}, [J,h]\rangle\rangle - \sum_{\alpha\in\Delta_{\nu-1}^+} e_\alpha^{(+;\nu)}(x)\langle\langle e_{-\alpha}^{(+;\nu)}, [J,h]\rangle\rangle\}\mathrm{d}\lambda \tag{18}$$

$$= \frac{1}{2\pi}\sum_{\nu=1}^{M}\int_{L_\nu}\{\sum_{\alpha\in\Delta_\nu^+} e_{-\alpha}^{(-;\nu)}(y)\langle\langle e_{\alpha}^{(-;\nu)}, [J,h]\rangle\rangle - \sum_{\alpha\in\Delta_{\nu-1}^+} e_{-\alpha}^{(+;\nu)}(y)\langle\langle e_{\alpha}^{(+;\nu)}, [J,h]\rangle\rangle\}\mathrm{d}\lambda \tag{19}$$

In the above

$$\langle\langle e_{-\alpha}^{(-;\nu)}, [J,h]\rangle\rangle = \int_{-\infty}^{+\infty}\langle e_{-\alpha}^{(-;\nu)}(x), [J, h(x)]\rangle\rangle\mathrm{d}x \tag{20}$$

$$\langle\langle e_{-\alpha}^{(+;\nu)}, [J,h]\rangle\rangle = \int_{-\infty}^{+\infty}\langle e_{-\alpha}^{(+;\nu)}(x), [J, h(x)]\rangle\mathrm{d}x \tag{21}$$

1. It can be shown that the expansion (18) converges in the same sense as the Fourier expansions for $h(x)$. These are the so-called Generalized Fourier Expansions and the functions $e_\alpha^{\pm;\nu}(x,\lambda)$ are the Generalized Exponents. When one expands over the Generalized Exponents the potential $q(x)$ one gets as coefficients the minimal scattering data.

2. One can prove that

$$(\Lambda_- - \lambda)e_\alpha^{(-;\nu)} = 0, \quad \alpha \in \Delta_\nu^+, \qquad (\Lambda_- - \lambda)e_\alpha^{(+;\nu)} = 0, \quad \alpha \in \Delta_{\nu-1}^+ \quad (22)$$

$$(\Lambda_+ - \lambda)e_{-\alpha}^{(-;\nu)} = 0, \quad \alpha \in \Delta_\nu^+, \qquad (\Lambda_+ - \lambda)e_{-\alpha}^{(+;\nu)} = 0, \quad \alpha \in \Delta_{\nu-1}^+ \quad (23)$$

and therefore **the expansions (18) and (19) are in fact the spectral decompositions for the operators Λ_- and Λ_+, that is they play for these expansions the role that $\mathrm{i}\partial_x$ plays for the Fourier expansion.**

$\mathbb{Z}_p$ reductions in the CBC system defined by an automorphism

We shall consider now special type of linear problems of the type (2) in which the potential function $q(x)$ and the element J obey some special requirements resulting from Mikhailov-type reductions. We shall consider Mikhailov reduction group G_0 is generated by one element, which we denote by H.

$$H(\psi(x,\lambda)) = \mathcal{K}(\psi(x,\omega^{-1}\lambda)) \qquad (24)$$

where $\omega = \exp \frac{2\pi \mathrm{i}}{p}$ and $\mathcal{K}$ is automorphism of order p of the Lie group corresponding to the algebra $\mathfrak{g}$. $\mathcal{K}$ generates an automorphism of $\mathfrak{g}$ which we shall denote by the same letter $\mathcal{K}$. We shall require in the above situation that the automorphism leave invariant the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to which the element J in the CBC system belongs.

General remarks

- Suppose $\mathcal{K}$ is an automorphism of $\mathfrak{g}$ and $\mathcal{K}^p = \mathrm{id}\,$, $\mathcal{K}\mathfrak{h} \subset \mathfrak{h}$. (In case of Coxeter automorphisms p is called the Coxeter number). The Coxeter automorphisms are internal that is each $\mathcal{K}$ is internal and can be represented as $\mathcal{K} = Ad(K)$, K belonging to the corresponding group G with algebra $\mathfrak{g}$.

- The automorphisms leave the Killing form invariant, a fact that we shall use constantly.

- The algebra $\mathfrak{g}$ splits into a direct sum of eigenspaces of $\mathcal{K}$, that is:

$$\mathfrak{g} = \oplus_{s=0}^{p-1} \mathfrak{g}^{[s]} \tag{25}$$

  where for each $X \in \mathfrak{g}^{[s]}$ we have $\mathcal{K}X = \omega^s X$ and the spaces $\mathfrak{g}^{[s]}$, $\mathfrak{g}^{[k]}$ for $k \neq s$ are orthogonal with respect to the Killing form.

- Because $\mathcal{K}$ is an automorphism of $\mathfrak{g}$ leaving $\mathfrak{h}$ invariant, it leaves invariant also the orthogonal complement $\bar{\mathfrak{g}}$ of $\mathfrak{h}$. Thus each $\mathfrak{g}^{[s]}$ splits into $\bar{\mathfrak{g}}^{[s]} \oplus \mathfrak{h}^{[s]}$ and

$$\bar{\mathfrak{g}} = \oplus_{s=0}^{p-1} \bar{\mathfrak{g}}^{[s]}, \qquad \mathfrak{h} = \oplus_{s=0}^{p-1} \mathfrak{h}^{[s]} \tag{26}$$

For different k and s the spaces $\mathfrak{g}^{[k]}$ and $\mathfrak{g}^{[s]}$ are orthogonal with respect to the Killing form and the spaces $\bar{\mathfrak{g}}^{[k]}$ and $\mathfrak{h}^{[s]}$ are orthogonal for arbitrary k and s. Further, if we denote the orthogonal projections on $\mathfrak{g}^{[k]}$ by $\mathbf{1}^{[k]}$ we shall have that $\zeta^{[k]} = \mathbf{1}^{[k]}(\mathbf{1} - \pi_0)$ are the projections on $\mathfrak{h}^{[k]}$ and $\mathbf{1}^{[k]}\pi_0 = \pi_0^{[k]}$ are the orthogonal projector on $\bar{\mathfrak{g}}^{[k]}$.

- If as before the orthogonal projector $\mathfrak{g} \mapsto \bar{\mathfrak{g}}$ is denoted by π_0 we shall have:

$$\pi_0 = \sum_{k=0}^{p-1} \pi_0^{[k]}, \quad \pi_0^{[l]}\pi_0^{[s]} - \pi_0^{[s]}\pi_0^{[l]} = 0 \tag{27}$$

$$\mathbf{1} - \pi_0 = \sum_{k=0}^{p-1} \zeta^{[k]}, \quad \zeta^{[l]}\zeta^{[s]} - \zeta^{[s]}\zeta^{[l]} = 0 \tag{28}$$

$$\pi_0^{[k]} + \zeta^{[k]} = \mathbf{1}^{[k]}, \quad \zeta^{[l]}\pi_0^{[s]} = \pi_0^{[s]}\zeta^{[l]} = 0 \tag{29}$$

Let us assume that the set of fundamental solutions for the spectral problem (2) are invariant under G_0. Then as it is easy to see that we must have

$$\mathcal{K}(J) = \omega J, \quad \mathcal{K}q = q \tag{30}$$

that is, $J \in \mathfrak{g}^{[1]}$, $q(x) \in \mathfrak{g}^{[0]}$. In fact, suppose we have a Lax representation $[L, A] = 0$ where A has the form:

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \quad A_n \in \mathfrak{h}, \quad A_n = \mathrm{const}\,, \quad A_{n-1} \in \bar{\mathfrak{g}}$$

If the common fundamental solutions for $L\psi = 0, A\psi = 0$ are invariant under G_0 then we also have:

$$\mathcal{K}(A_s) = \omega^s A_s \quad s = 0, 1, 2, \ldots n \tag{31}$$

The above reductions are compatible with the evolution in the sense that if at the moment $t = 0$ we have (30, 31) we have the same relations at arbitrary moment t.

The invariance of the set of the fundamental solutions can be additionally specified if we take the fundamental analytic solutions $m_\nu(x, \lambda)$ **defined in the sectors** Ω_ν**,** $\nu = 1, 2, \ldots h$ **defined by the straight lines** $l_\alpha = \{\lambda : \mathrm{Im}(\lambda\alpha(\mathrm{J})) = 0\}, \quad \alpha \in \Delta$. (Of course, one obtains the same line for α and $-\alpha$ but it can happen that $\alpha \neq \beta$ and $l_\alpha = l_\beta$).

Taking into account the uniqueness of the solutions $m(x,\lambda)$ we get that $\mathcal{K}(m(x,\lambda))$ is equal to $m(x,\omega\lambda)$. **Consequently, we obtain that**

$$\mathcal{K}(\chi(x,\lambda)) = \mathcal{K}(m(x,\lambda)e^{-iJx\lambda}) = m(x,\omega\lambda)e^{-iJx\omega\lambda} = \chi(x,\omega\lambda) \qquad (32)$$

is analytic in $\omega\Omega_\nu$. If l_α, l_β form the boundary of Ω_ν then $\omega l_\alpha, \omega l_\beta$ are the straight lines defining the boundary of $\omega\Omega_\nu$.

Let us define $\hat{\mathcal{K}} : \mathfrak{h} \mapsto \mathfrak{h}$ by $\hat{\mathcal{K}} = (\mathcal{K}^*)^{-1}$. The map $\hat{\mathcal{K}}$ defines the coadjoint action of $\mathcal{K}$ on $\mathfrak{h}^*$. Naturally $\hat{\mathcal{K}}^p = \mathrm{id}$ and

$$\langle \hat{\mathcal{K}}\xi, \mathcal{K}H\rangle = \langle \xi, H\rangle, \quad \xi \in \mathfrak{h}^*, H \in \mathfrak{h} \qquad (33)$$

It is a general fact from the theory of the automorphisms is that for all roots we have $\mathcal{K}E_\alpha = q(\alpha)E_{\hat{\mathcal{K}}\alpha}$, where $q(\alpha) = \pm 1$, $q(\alpha)q(-\alpha) = 1$, $q(\alpha)q(\beta) = q(\alpha+\beta)$ if $\alpha+\beta \in \Delta$. **One easily gets that $\omega l_\alpha = l_{\hat{\mathcal{K}}^{-1}\alpha}$. Thus we have an action of the automorphism $\mathcal{K}$ (the group $\mathbb{Z}_p$) on the bunch of lines $\{l_\alpha\}_{\alpha\in\Delta}$ defined by $\hat{\mathcal{K}}^{-1}$ and similarly the action on the set of sectors Ω_ν, $\nu = 1, 2, \ldots, h$.** We have

Proposition 0.1 *The representatives from the different orbits of the* $\mathbb{Z}_p$ *on the set of sectors* Ω_ν, $\nu = 1, 2 \ldots, a$ *can be taken to be adjacent,* **which we shall always assume.**

Reductions defined by Coxeter automorphisms

Coxeter automorphisms are the automorphisms for which

$$\hat{\mathcal{K}} = S_{\alpha_1} S_{\alpha_2} \ldots S_{\alpha_r}$$

where S_{α_i} are the Weyl reflections corresponding to the simple roots $\alpha_1, \alpha_2, \alpha_r$ of $\mathfrak{g}$. We are able to prove the following:

Theorem 0.3 *Assume we have the CBC problem for the classical series of simple Lie algebras and the* $\mathbb{Z}_p$ *reduction is defined as in the above using the Coxeter automorphism* $\mathcal{K}$. *Then we have two adjacent fundamental sectors of analyticity for the fundamental analytic solutions* $m_\nu(x, \lambda)$ *and they can be chosen to be*

$$\begin{aligned} \Omega_0 &= \{\lambda : \frac{\pi}{2} < \arg(\lambda) < \frac{\pi}{2} + \frac{\pi}{p}\} \\ \Omega_1 &= \{\lambda : \frac{\pi}{2} + \frac{\pi}{p} < \arg(\lambda) < \frac{\pi}{2} + \frac{2\pi}{p}\} \end{aligned} \qquad (34)$$

Expansions in presence of reductions defined by automorphisms

$\mathbb{Z}_p$ reductions of general type

Consider the general case of automorphism $\mathcal{K}$ of order p, let Ω_1, Ω_2 ... Ω_a be the fundamental sectors (moving anticlockwise when we go from Ω_1 to Ω_a) and let us label the rays that form the boundaries of the sectors in such a way that Ω_ν is locked between the rays L_ν and $L_{\nu+1}$ that are oriented from zero to infinity. Since multiplication by ω^p is identity (turning by angle 2π) the number of sectors is $M = pa$ and M is even number. Multiplying by ω we get from the sector Ω_ν the sector $\Omega_{a+\nu}$ and multiplying by ω^M we get again Ω_1 so we shall understand the labels modulo M. Naturally, $L_{a+\nu} = \omega L_\nu$ and $\Omega_{a+\nu} = \omega\Omega_\nu$. For each $\alpha \in \Delta$ we have $\mathcal{K}(E_\alpha) = q(\alpha)E_{\hat{\mathcal{K}}\alpha}$, where $q(\alpha)$ are numbers, such that $q(\alpha) = \pm 1$, $q(\alpha)q(-\alpha) = 1$ and $q(\alpha)q(\beta) = q(\alpha+\beta)$ if $\alpha + \beta \in \Delta$.

It is not hard to obtain that

$$[\mathcal{K}\circ\pi_0](\chi_\nu(x,\lambda)E_\alpha\chi_\nu^{-1}(x,\lambda)) = \pi_0(\chi_{\nu+a}(x,\omega\lambda)\mathcal{K}(E_\alpha)\chi_{\nu+a}^{-1}(x,\omega\lambda)) =$$
$$q(\alpha)\pi_0(\chi_{\nu+a}(x,\omega\lambda)E_{\hat{\mathcal{K}}\alpha}\chi_{\nu+a}^{-1}(x,\omega\lambda))$$

and as a consequence :

$$\mathcal{K}(e_\alpha^\nu(x,\lambda)) = q(\alpha)e_{\hat{\mathcal{K}}\alpha}^{\nu+a}(x,\omega\lambda) \tag{35}$$

Changing variables for the integrals over the rays that do not belong to the closures of the fundamental sectors and taking into account (35) we transform expansion (17) into

$$\Pi_0\ \delta(x-y) =$$
$$\frac{1}{2\pi}\sum_{\nu=1}^{a}\sum_{k=1}^{p}\int_{L_\nu}\{\sum_{\alpha\in\Delta_\nu^+}\omega^k\mathcal{K}^k\otimes\mathcal{K}^k(e_\alpha^{(-;\nu)}(x)\otimes e_{-\alpha}^{(-;\nu)}(y)) - \tag{36}$$
$$\sum_{\alpha\in\Delta_{\nu-1}^+}\omega^k\mathcal{K}^k\otimes\mathcal{K}^k(e_\alpha^{(+;\nu)}(x)\otimes e_{-\alpha}^{(+;\nu)}(y))\}\mathrm{d}\lambda$$

where

$$(\mathcal{K} \otimes \mathcal{K})(X \otimes Y) = \mathcal{K}(X) \otimes \mathcal{K}(Y) \tag{37}$$

Note that the numbers $q(\alpha)$ don't appear any more, this occurs because we apply $\mathcal{K}$ always on products of the type $E_\alpha \otimes E_{-\alpha}$. The rays L_ν are orientated from 0 to ∞.

The expansions of a function $h(x)$ over the adjoint solutions can be simplified further, if for arbitrary x the value $h(x) \in \mathfrak{g}^{[s]}$, where $\mathfrak{g}^{[s]}$ is the eigenspace corresponding to the eigenvalue ω^s.

As the Killing form is invariant with respect to the action of the automorphism, we get

$$\begin{aligned}
&\langle \mathcal{K}^k(e^\nu_\alpha(x,\lambda)), [J, h(x)]\rangle = \langle e^\nu_\alpha(x,\lambda, \mathcal{K}^{-k}([J, h(x)])\rangle = \\
&= \omega^{-k(s+1)} \langle e^\nu_\alpha(x,\lambda), [J, h(x)]\rangle
\end{aligned}$$

The expansions over the adjoint solutions run as follows:

$$h(x) =$$

$$\begin{gathered}\frac{\epsilon}{2\pi}\sum_{\nu=1}^{a}\int_{L_\nu}\{\sum_{\alpha\in\Delta_\nu^+}\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_{\epsilon\alpha}^{(-;\nu)}(x,\lambda))\langle\langle e_{-\epsilon\alpha}^{(-;\nu)},[J,h]\rangle\rangle- \\ -\sum_{\alpha\in\Delta_{\nu-1}^+}\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_{\epsilon\alpha}^{(+;\nu)}(x,\lambda))\langle\langle e_{-\epsilon\alpha}^{(+;\nu)},[J,h]\rangle\rangle\}\mathrm{d}\lambda\end{gathered} \tag{38}$$

In the above are written two expansions, one for $\epsilon = +1$ and the other for $\epsilon = -1$.

Thus we see that $h(x)$ is actually expanded over the functions:

$$e_\alpha^{(\pm;\nu;s)}(x,\lambda) = \sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e^{(\pm;\nu)}(x,\lambda)) \in \mathfrak{g}^{[s]}, \quad \nu = 1,2,\ldots,a \tag{39}$$

since for arbitrary $X \in \mathfrak{g}$ we have $\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(X) \in \mathfrak{g}^{[s]}$.

We shall denote by $e_\alpha^{(\nu;s)}(x,\lambda)$ the expressions:

$$e_\alpha^{(\nu;s)}(x,\lambda) = \sum_{k=1}^{p} \omega^{-ks} \mathcal{K}^k(e_\alpha^\nu(x,\lambda)), \qquad \lambda \in \Omega_\nu \tag{40}$$

Clearly, $e_\alpha^{(\pm;\nu;s)}(x,\lambda)$ are just the limits of $e_\alpha^{(\nu-1;s)}(x,\lambda)$ and $e_\alpha^{(\nu;s)}(x,\lambda)$ when λ approaches one of the rays L_ν from one or the other side.

If as before $h(x) \in \mathfrak{g}^{[s]}$, we get

$$\langle e_\alpha^{(\nu;s)}(x,\lambda), [J, h(x)]\rangle = p\langle e_\alpha^\nu, [J, h(x)]\rangle$$

and the expansions (38) can be cast into the form

$$\begin{aligned} h(x) = \frac{\epsilon}{2\pi p} \sum_{\nu=1}^{a} \int_{L_\nu} \{ \sum_{\alpha\in\Delta_\nu^+} e_{\epsilon\alpha}^{(-;\nu;s)}(x,\lambda) \langle\langle e_{-\epsilon\alpha}^{(-;\nu;s)}, [J,h]\rangle\rangle - \\ - \sum_{\alpha\in\Delta_{\nu-1}^+} e_{\epsilon\alpha}^{(+;\nu;s)}(x,\lambda)) \langle\langle e_{-\epsilon\alpha}^{(+;\nu;s)}, [J,h]\rangle\rangle \} \mathrm{d}\lambda \end{aligned} \tag{41}$$

(We have two expansions, for $\epsilon = +1$ and for $\epsilon = -1$).

Coxeter automorphisms reductions

When $\mathbb{Z}_p$ reduction defined by a Coxeter automorphism of degree p on some of the simple Lie algebras from the classical series the above expansion specify further. Note that in this case the number p is equal to the dimension of the Cartan subalgebra. For the sake of symmetry we label the fundamental sectors by 0 and 1, that is they are Ω_0 and Ω_1. Their boundaries are formed by the rays L_{-1}, L_0, L_1. Next, if $\alpha \in \Delta_\nu^+$ then

- $\nu = 2k$ leads to $\hat{\mathcal{K}}^{-k}\alpha \in \Delta_0^+ = \Delta_{2p}^+$
- $\nu = 2k + 1$ leads to $\hat{\mathcal{K}}^{-k}\alpha \in \Delta_1^+$.

Using the same type of notation as in the general case, the completeness relation (in case we do not write the discrete sector terms) can be cast into the form:

$$
\begin{aligned}
&\Pi_0\ \delta(x-y) = \\
&\frac{1}{2\pi}\sum_{\nu=-1}^{+1}\int_{L_\nu} \mathrm{d}\lambda\{\sum_{\alpha\in\Delta_\nu^+}\sum_{k=1}^{p}\omega^k\mathcal{K}^k\otimes\mathcal{K}^k(e_\alpha^{(-;\nu)}(x,\lambda)\otimes e_{-\alpha}^{(-;\nu)}(y,\lambda))- \\
&-\sum_{\alpha\in\Delta_{\nu-1}^+}\sum_{k=1}^{p}\omega^k\mathcal{K}^k\otimes\mathcal{K}^k(e_\alpha^{(+;\nu)}(x,\lambda)\otimes e_{-\alpha}^{(+;\nu)}(y,\lambda))\}
\end{aligned}
\tag{42}
$$

(The rays $L_0, L_{\pm 1}$ are orientated from 0 to ∞.)

If the function $h(x)$ is such that for arbitrary x the value $h(x)\in\mathfrak{g}^{[s]}$, where $\mathfrak{g}^{[s]}$ is the eigenspace for the Coxeter automorphism, we get $\langle\mathcal{K}^k(e_\alpha^\nu(x,\lambda)),[J,h(x)]\rangle = \omega^{-k(s+1)}\langle e_\alpha^\nu(x,\lambda),[J,h(x)]\rangle$.

The corresponding expansions over the adjoint solutions run as follows:

$$h(x) =$$

$$\frac{\epsilon}{2\pi}\sum_{\nu=-1}^{+1}\int_{L_\nu} \mathrm{d}\lambda\{\sum_{\alpha\in\Delta_\nu^+}\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_{\epsilon\alpha}^{(-;\nu)}(x,\lambda))\langle\langle e_{-\epsilon\alpha}^{(-;\nu)},[J,h]\rangle\rangle - \tag{43}$$
$$-\sum_{\alpha\in\Delta_{\nu-1}^+}\sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e_{\epsilon\alpha}^{(+;\nu)}(x,\lambda))\langle\langle e_{-\epsilon\alpha}^{(+;\nu)},[J,h]\rangle\rangle\}$$

In the above are written two expansions, for $\epsilon = +1$ and $\epsilon = -1$.

As before we see that $h(x)$ is actually expanded over the functions:

$$e_{\alpha}^{(\pm;\nu;s)}(x,\lambda) = \sum_{k=1}^{p}\omega^{-ks}\mathcal{K}^k(e^{(\pm;\nu)}(x,\lambda)), \qquad \nu = 0,1,-1 \tag{44}$$

which are the 'stratifications' of the usual adjoint solutions under the endomorphism $\mathcal{K}$.

In complete analogy with the general case, denoting by $e_\alpha^{(\nu;s)}(x,\lambda)$ the expressions:

$$e_\alpha^{(\nu;s)}(x,\lambda) = \sum_{k=1}^{p} \omega^{-ks}\mathcal{K}^k(e_\alpha^\nu(x,\lambda)), \qquad \lambda \in \Omega_\nu \tag{45}$$

we see that $e_\alpha^{(\pm;\nu;s)}(x,\lambda)$ are the limits of $e_\alpha^{(\nu;s)}(x,\lambda)$ when λ approaches one of the rays $L_0, L_{\pm 1}$ from one or the other side.

If $h(x) \in \mathfrak{g}^{[s]}$, we get $\langle e_\alpha^{(\nu;s)}(x,\lambda), [J,h(x)]\rangle = p\langle e_\alpha^\nu(x), [J,h(x)]\rangle$ As a consequence, the expansions (43) can be cast into the form

$$\begin{aligned} h(x) = \frac{\epsilon}{2\pi p} \sum_{\nu=-1}^{+1} \int_{L_\nu} \{ \sum_{\alpha\in\Delta_\nu^+} e_{\epsilon\alpha}^{(-;\nu;s)}(x,\lambda)\langle\langle e_{-\epsilon\alpha}^{(-;\nu;s)}, [J,h]\rangle\rangle - \\ - \sum_{\alpha\in\Delta_{\nu-1}^+} e_{\epsilon\alpha}^{(+;\nu;s)}(x,\lambda))\langle\langle e_{-\epsilon\alpha}^{(+;\nu;s)}, [J,h]\rangle\rangle\} \mathrm{d}\lambda \end{aligned} \tag{46}$$

(We have two expansions, for $\epsilon = +1$ and for $\epsilon = -1$.)

Recursion Operators in the presence of $\mathbb{Z}_p$ reductions defined by automorphism

Algebraic aspects

Let us see now what happens with the Recursion Operator:

$$\Lambda_\pm X = \mathrm{ad}\,_J^{-1}\left\{ \mathrm{i}\partial_x X + \pi_0[q,X] + \mathrm{i}\mathrm{ad}\,_q(\mathbf{1}-\pi_0)\partial_x^{-1}[q,X]\rangle \right\} \qquad (47)$$

when $\mathbb{Z}_p$ reductions are present. Then the algebra splits in a direct sum, see (25) and $q \in \mathfrak{g}^{[0]}$ while $J \in \mathfrak{h}^{[1]}$. In particular, this means that

$$\mathrm{ad}\,_J(\bar{\mathfrak{g}}^{[s]}) \subset \bar{\mathfrak{g}}^{[s+1]}, \qquad \mathrm{ad}\,_J^{-1}(\bar{\mathfrak{g}}^{[s]}) \subset \bar{\mathfrak{g}}^{[s-1]} \qquad (48)$$

(the superscripts are understood modulo p). Also, if $X \in \bar{\mathfrak{g}}^{[s]}$ then $\partial_x X \in \bar{\mathfrak{g}}^{[s]}$, $\partial_x^{-1} X \in \bar{\mathfrak{g}}^{[s]}$, $[q,X] \in \bar{\mathfrak{g}}^{[s]}$ and

$$\Lambda_\pm X = \mathrm{ad}\,_J^{-1}\{\mathrm{i}\partial_x X + \pi_0[q,X] + \mathrm{ad}\,_q\partial_x^{-1}(\mathbf{1}-\pi_0)[q,X]\} \in \bar{\mathfrak{g}}^{[s-1]} \qquad (49)$$

If we use the notation introduced in (27) the above expression can also be written as

$$\Lambda_\pm X = \mathrm{ad}\,_J^{-1}\{\mathrm{i}\partial_x + \pi_0\mathrm{ad}\,_q + \mathrm{ad}\,_q\partial_x^{-1}(\mathbf{1}-\pi_0)\mathrm{ad}\,_q\}\pi_0^{[s]} X \qquad (50)$$

Denote

- *By* $\mathfrak{F}(\bar{\mathfrak{g}})$ *the space of smooth, rapidly decreasing functions with values in* $\bar{\mathfrak{g}}$
- *By* $\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})$ *the space of smooth, rapidly decreasing functions with values in* $\bar{\mathfrak{g}}^{[s]}$
- *By* $\Lambda_{\pm;s}X$ *the value* $\Lambda_{\pm}X$ *if* $X \in \mathfrak{F}(\bar{\mathfrak{g}})$

As one can see $\Lambda_{\pm;s}X$ **is an operator** $\Lambda_{\pm;s}$ **acting on the space** $\mathfrak{F}(\bar{\mathfrak{g}})$ **with values in** $\bar{\mathfrak{g}}^{[s-1]}$**. The spaces** $\bar{\mathfrak{g}}^{[s]}$ **are moved one into another by** $\Lambda_{\pm}$ **and are invariant under the action of** $\Lambda_{\pm}^{p}$**. Naturally,**

$$\Lambda_{\pm}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})} = \Lambda_{\pm;s}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})}, \quad \Lambda_{\pm;s}\mathfrak{F}(\bar{\mathfrak{g}}^{[s]}) \subset \mathfrak{F}(\bar{\mathfrak{g}}^{[s-1]}) \tag{51}$$

Also,

$$\Lambda_{\pm}^{p}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[s]})} = \Lambda_{\pm;s-p+1}\dots\Lambda_{\pm;s-1}\Lambda_{\pm;s} \tag{52}$$

(the indexes $s-k$ **are understood modulo** p**). In particular,**

$$\Lambda_{\pm}^{p}|_{\mathfrak{F}(\bar{\mathfrak{g}}^{[0]})} = \Lambda_{\pm;1}\dots\Lambda_{\pm;p-2}\Lambda_{\pm;p-1}\Lambda_{\pm;p} \tag{53}$$

Recall that the Recursion Operators arise naturally when looking for the NLEEs that have Lax representation $[L, A] = 0$ with L being the CBC system operator and A is the form

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{n} \lambda^k A_k, \qquad A_n \in \mathfrak{h}, \qquad A_n = \mathrm{const}\,, \qquad A_{n-1} \in \bar{\mathfrak{g}} \quad (54)$$

Then from the condition $[L, A] = 0$ we first obtain $A_{n-1} = \mathrm{ad}\,_J^{-1}[q, A]$ and next for $0 < k < n-1$ the recursion relations $\pi_0 A_{k-1} = \Lambda_\pm(\pi_0 A_k)$ and the NLEEs (5).

Assume that we have $\mathbb{Z}_p$ reduction. Then we have $q \in \bar{\mathfrak{g}}^{[0]}$, $J \in \mathfrak{h}^{[0]}$ and we must have $\mathcal{K}(A_s) = \omega^s A_s$. Assume that $A_n \in \mathfrak{h}^{[n]}$. Then $A_{n-1} \in \bar{\mathfrak{g}}^{[n-1]}$ and we see that $A_s \in \mathfrak{g}^{[s]}$. Therefore the reduction requirements will be satisfied automatically when we choose $A_n \in \mathfrak{h}^{[n]}$. Since n is a natural number let us write it into the form $n = kp + m$ where k, p, m are natural numbers and $0 \leq m < p$. Then

$$\Lambda_\pm^n \mathrm{ad}\,_J^{-1}[A_n, q] = \Lambda_\pm^{kp} \Lambda_\pm^m \mathrm{ad}\,_J^{-1}[A_n, q] =$$

$$(\Lambda_{\pm;0} \ldots \Lambda_{\pm;p-2} \Lambda_{\pm;p-1})^k \Lambda_{\pm;0} \ldots \Lambda_{\pm;m-2} \Lambda_{\pm;m-1} \mathrm{ad}\,_J^{-1}[A_n, q]$$

Starting from the works Fordy, Gibbons 1980;1981 it is frequently said that when reductions are present the Recursion Operator becomes of higher order in the derivative ∂_x and factorizes into a product of first order operators with respect to ∂_x. The above has been used by some authors to justify the claim that the Recursion Operators $R_\pm$ in the presence of $\mathbb{Z}_p$ reduction factorize to become

$$R_\pm = \Lambda_{\pm;0} \dots \Lambda_{\pm;p-2}\Lambda_{\pm;p-1} \tag{55}$$

To our opinion more accurate would be simply to say that they are restrictions of the Recursion Operator in general position on some subspaces:

$$\mathfrak{F}(\bar{\mathfrak{g}}^{[p]}) = \mathfrak{F}(\bar{\mathfrak{g}}^{[0]}) \quad \overset{\Lambda_{\pm;0}}{\rightarrow} \quad \mathfrak{F}(\bar{\mathfrak{g}}^{[p-1]}) \quad \overset{\Lambda_{\pm;p-1}}{\rightarrow} \quad \dots \quad \overset{\Lambda_{\pm;1}}{\rightarrow} \quad \mathfrak{F}(\bar{\mathfrak{g}}^{[0]}) = \mathfrak{F}(\bar{\mathfrak{g}}^{[p]}) \tag{56}$$

The above suggests that the role of the Recursion Operators $\Lambda_\pm$ in case of $\mathbb{Z}_p$ reductions is taken now by $\Lambda_\pm^p$. It is also supported by the geometric picture, Yanovski 2012.

Expansions over adjoint solutions

Let us see how this operators act on the set of functions (39), (40) over which the expansions (38) are written. Using the properties of the automorphism $\mathcal{K}$ (the fact that it commutes with the projection π_0 on $\mathfrak{h}$) and the facts that $\mathcal{K}q = q$ and $\mathcal{K}J = \omega J$ we easily get

Lemma 0.1 ***If*** $\mathcal{K}$ ***is an automorphism of order*** p ***defining the*** $\mathbb{Z}_p$ ***reduction then***

$$\Lambda_\pm \circ \mathcal{K} = \omega \mathcal{K} \circ \Lambda_\pm \tag{57}$$

As a consequence,

$$\Lambda_\pm^p \circ \mathcal{K} = \mathcal{K} \circ \Lambda_\pm^p \tag{58}$$

Then for $\lambda \in \Omega_\nu$ we immediately obtain:

$$\Lambda_\pm e_\alpha^{(\nu;s)}(x,\lambda) = \lambda \sum_{k=1}^{p} \omega^{-k(s-1)} \mathcal{K}^k \Lambda_\pm(e_\alpha^\nu(x,\lambda)) = \lambda e_\alpha^{(\nu;s-1)}(x,\lambda), \tag{59}$$

After some calculations we get that

$$\Lambda_- e_\alpha^{(-;\nu;s)} = \lambda e_\alpha^{(-;\nu,s-1)}, \ \ \alpha \in \Delta_\nu^+ \tag{60}$$
$$\Lambda_- e_\alpha^{(+;\nu,s)} = \lambda e_\alpha^{(+;\nu.s-1)}, \ \ \alpha \in \Delta_{\nu-1}^+$$

$$\Lambda_+ e_{-\alpha}^{(-;\nu,s)} = \lambda e_{-\alpha}^{(-;\nu,s-1)}, \ \ \alpha \in \Delta_\nu^+ \tag{61}$$
$$\Lambda_- e_{-\alpha}^{(+;\nu,s)} = \lambda e_{-\alpha}^{(+;\nu,s-1)}, \ \ \alpha \in \Delta_{\nu-1}^+$$

As a corollary

$$\Lambda_-^p e_\alpha^{(-;\nu;s)} = \lambda^p e_\alpha^{(-;\nu,s)}, \ \ \alpha \in \Delta_\nu^+ \tag{62}$$
$$\Lambda_-^p e_\alpha^{(+;\nu,s)} = \lambda^p e_\alpha^{(+;\nu.s)}, \ \ \alpha \in \Delta_{\nu-1}^+$$

$$\Lambda_+^p e_{-\alpha}^{(-;\nu,s)} = \lambda^p e_{-\alpha}^{(-;\nu,s)}, \ \ \alpha \in \Delta_\nu^+ \tag{63}$$
$$\Lambda_+^p e_{-\alpha}^{(+;\nu,s)} = \lambda^p e_{-\alpha}^{(+;\nu,s)}, \ \ \alpha \in \Delta_{\nu-1}^+$$

and we have:

Theorem 0.4 ***For the expansions (38) the role of the Recursion Operators are played by the p-th powers of the operators $\Lambda_\pm$.***

Conclusions

The above considerations show that both from recursion relations viewpoint and expansion over adjoint solutions viewpoint the role of the Recursion Operators in case of $\mathbb{Z}_p$ reductions are played by the operators $\Lambda_{\pm}^{p}$. Since the same conclusion is drawn from the geometric considerations, Yanovski 2012, the theory now is complete in all aspects - algebraic, spectral and geometric.

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