A relation between fluid membranes and motions of planar curves

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June 09\textsuperscript{th} 2012

\textit{XIV\textsuperscript{th} International Conference “Geometry, Integrability and Quantization”, Varna, Bulgaria}
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This is joint work with I.M. Mladenov (Institute of biophysics, BAS).
Equilibrium shapes of fluid membranes

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- $\Delta_S$ - Surface Laplacian
If one puts certain symmetry to the equation and focuses on cylindrical membranes it becomes the ordinary differential equation

\[ 2 \frac{d^2 \kappa(s)}{ds^2} + \kappa^3(s) - \mu \kappa(s) - \sigma = 0. \]
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- \( \kappa(s) \) is a curvature of the directrix of the cylindrical fluid membrane.
- \( \sigma \) and \( \mu \) are physical parameters, more precisely

\[ \mu = h^2 + \frac{2 \lambda}{k_c}, \quad \sigma = -\frac{2p}{k_c}. \]
Cylindrical equilibrium shapes of fluid membranes

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where \( P(\kappa) \) is a fourth degree polynomial in \( \kappa \) with zero cubic term. Obviously, the roots add up to zero. This equation was solved for all cases of interest depending on the roots of \( P(\kappa) \).
Motions of planar curves

The general evolution of a curve in the plane is given by

\[ \frac{d\bar{r}(s, t)}{ds} = U\bar{t} + W\bar{n} \]
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$$\frac{d\bar{r}(s, t)}{ds} = U\tilde{t} + W\tilde{n}$$

where $\bar{r}$ is the position vector in the plane, $\tilde{n}$, $\tilde{t}$ are the unit normal and the unit tangent to the curve at given time $t$ and $U$, $W$ are certain velocities that are determined by the curvature of the curve.
The evolution of the curvature is given by

\[ \frac{\partial \kappa}{\partial t} = \frac{\partial^2 W}{\partial s^2} + \kappa^2 W + \frac{\partial \kappa}{\partial s} \int kW \, ds \equiv RW \]
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$$\frac{\partial \kappa}{\partial t} - \frac{\partial^3 \kappa}{\partial s^3} - \frac{3}{2} \kappa^2 \frac{\partial \kappa}{\partial s} = 0$$
The evolution of the curvature

mKdV equation:

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one gets an ODE which after one integration becomes

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which is the same equation derived in the membranes study. Therefore one can apply results from elastic membrane theory to the current topic.
Overview

We solve the equation

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depending on the roots of \( P(\kappa) \). There are three relevant cases.
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- **Case 1** Two real roots \( \alpha < \beta \), pair of complex roots \( \gamma, \bar{\gamma} \) with

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- **Case 2** Two real roots \( \alpha < \beta \), pair of complex roots \( \gamma, \bar{\gamma} \) with
  \[
  (3\alpha + \beta)(\alpha + 3\beta) = 0
  \]
- **Case 3** Four distinct real roots
Case 1

\[
\kappa_1(s) = \frac{A\beta + B\alpha - (A\beta - B\alpha) \text{cn}(us, k)}{A + B - (A - B) \text{cn}(us, k)}
\]
Case 1

\[ \kappa_1(s) = \frac{A\beta + B\alpha - (A\beta - B\alpha) \text{cn}(us, k)}{A + B - (A - B) \text{cn}(us, k)} \]

\[ \theta_1(s) = \frac{(A\beta - B\alpha) s}{A - B} + \frac{(A + B)(-\beta + \alpha)}{2u(A - B)} \prod \left( \text{sn}(us, k), -\frac{(A - B)^2}{4BA}, k \right) \]

\[ + \frac{\alpha - \beta}{u\sqrt{4k^2 + \frac{(A-B)^2}{BA}}} \arctan \left( \sqrt{k^2 + \frac{(A - B)^2}{4BA}} \frac{\text{sn}(us, k)}{\text{dn}(us, k)} \right) \]
Case 1

- \( \text{cn}(x, k) \), \( \text{dn}(x, k) \), \( \text{sn}(x, k) \) and \( \Pi(\text{sn}(x, k), n, k) \) are Jacobi elliptic functions with elliptic modulus \( k \).
Case 1

- \( cn(x, k), \ dn(x, k), \ sn(x, k) \) and \( \Pi(sn(x, k), n, k) \) are Jacobi elliptic functions with elliptic modulus \( k \)
- \( A = \sqrt{4\eta^2 + (3\alpha + \beta)^2} \) and \( B = \sqrt{4\eta^2 + (\alpha + 3\beta)^2} \) with \( \eta \) being the imaginary part of \( \gamma \)
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- $u = 1/4\sqrt{AB}$.
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- \( u = 1/4 \sqrt{AB} \)
- \( k = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{4 \eta^2 + (3 \alpha + \beta)(\alpha + 3 \beta)}{(4 \eta^2 + (3 \alpha + \beta)(\alpha + 3 \beta))^2 + 16 \eta^2 (\beta - \alpha)^2}} \)
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Now one can write the formulae for the solution curve. Let us set...
Case 1

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Now one can write the formulae for the solution curve. Let us set
Case 1

The solution curve is given if we plug the quantities from the previous page in

\[ x(s) = \frac{2}{\sigma} \frac{d\kappa(s)}{ds} \cos \theta(s) + \frac{1}{\sigma} (\kappa^2(s) - \mu) \sin \theta(s) \]

\[ z(s) = \frac{2}{\sigma} \frac{d\kappa(s)}{ds} \sin \theta(s) - \frac{1}{\sigma} (\kappa^2(s) - \mu) \cos \theta(s) \]
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\end{align*}
\]

That is for the case \( \sigma \neq 0 \). One can get the solution curves in the zero case too.
Case 1

Figure: Solution curve (left) and phase portrait (right) for $\alpha = 0$, $\beta = 2$, $\gamma = -1 - i$. 
Here the polynomial $P(\kappa)$ has two real roots $\alpha < \beta$ and a pair of complex roots $\gamma, \bar{\gamma}$ with $(3\alpha + \beta)(\alpha + 3\beta) = 0$. Let $\xi = \alpha$ if $3\alpha + \beta = 0$ and $\xi = \beta$ otherwise. Again we need the roots to sum up to zero. These two conditions actually imply that $\sigma \neq 0$. 
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\[
\kappa_2(s) = \xi - 4 \frac{\xi}{1 + \xi^2 s^2}
\]

\[
\theta_2(s) = \xi s - 4 \arctan(\xi s)
\]
Case 2

Equations for the solution curve:

\[
\begin{align*}
\sigma (1 + \xi^2 s^2)^2 + 1 & - 1 \sigma \left( (\xi - 4 \xi^2 s^2)^2 - \mu \right) \\
\cdot & \cos (\xi s - 4 \arctan (\xi s))
\end{align*}
\]

\[
\begin{align*}
\sigma (1 + \xi^2 s^2)^2 & - 1 \sigma \left( (\xi - 4 \xi^2 s^2)^2 - \mu \right) \\
\cdot & \sin (\xi s - 4 \arctan (\xi s))
\end{align*}
\]
Case 2

Equations for the solution curve:

\[
x_2(s) = 16 \frac{\xi^3 s \cos (\xi s - 4 \arctan (\xi s))}{\sigma (1 + \xi^2 s^2)^2} \\
+ \frac{1}{\sigma} \left( \left( \xi - 4 \frac{\xi}{1 + \xi^2 s^2} \right)^2 - \mu \right) \sin (\xi s - 4 \arctan (\xi s))
\]

\[
z_2(s) = 16 \frac{\xi^3 s \sin (\xi s - 4 \arctan (\xi s))}{\sigma (1 + \xi^2 s^2)^2} \\
- \frac{1}{\sigma} \left( \left( \xi - 4 \frac{\xi}{1 + \xi^2 s^2} \right)^2 - \mu \right) \cos (\xi s - 4 \arctan (\xi s))
\]
Case 2

**Figure:** Solution curve (left) and phase portrait (right) for $\alpha = \beta = \gamma = -1, \delta = 3$
Case 3

In the last case we will consider the polynomial $P(\kappa)$ with four real roots $\alpha < \beta < \gamma < \delta$. One possible solution (i.e. the curvature, etc.) is given below. Let
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\kappa_3(s) = \delta - (\delta - \alpha)(\delta - \beta)(\delta - \beta + (\beta - \alpha)\hat{s}^2(s))^{-1}
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In the last case we will consider the polynomial $P(\kappa)$ with four real roots $\alpha < \beta < \gamma < \delta$. One possible solution (i.e. the curvature, etc.) is given below. Let

\[ p = \frac{(\gamma - \alpha)(\delta - \beta)}{4}, \quad q = \sqrt{\frac{(\beta - \alpha)(\delta - \gamma)}{(\gamma - \alpha)(\delta - \beta)}}, \quad \text{s}_n^\wedge(s) = \text{sn}(ps, q) \]

\[ \kappa_3(s) = \delta - (\delta - \alpha)(\delta - \beta)(\delta - \beta + (\beta - \alpha)s_n^2(s))^{-1} \]

\[ \theta_3(s) = \delta s - 4\pi \left( \text{s}_n(s), \frac{\beta - \alpha}{\beta - \delta}, q \right) (\delta - \alpha)(\gamma - \alpha)^{-1/2}(\delta - \beta)^{-1/2} \]
Case 3

Figure: Solution curve (left) and phase portrait (right) for
\[ \alpha = -4, \beta = -2, \gamma = 0, \delta = 6 \]
Summary

We use results from the theory of fluid membranes to solve the mKdV equation which arises from the evolution of planar curves.
Thank you for your patience!