A relation between fluid membranes and motions of planar curves

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This is joint work with I.M. Mladenov (Institute of biophysics, BAS).

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- Δ_S - Surface Laplacian

Cylindrical equilibrium shapes of fluid membranes

If one puts certain symmetry to the equation and focuses on cylindrical membranes it becomes the ordinary differential equation

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- σ and μ are physical parameters, more precisely

$$\mu = \mathbf{h}^2 + \frac{2\lambda}{k_c}, \qquad \sigma = -\frac{2p}{k_c}.$$

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$$\left(\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\right)^2=P(\kappa)$$

where $P(\kappa)$ is a fourth degree polynomial in κ with zero cubic term. Obviously, the roots add up to zero.
This equation was solved for all cases of interest depending on the roots of $P(\kappa)$.

Motions of planar curves

The general evolution of a curve in the plane is given by

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where $\bar{r}$ is the position vector in the plane, $\bar{n}, \bar{t}$ are the unit normal and the unit tangent to the curve at given time t and U, W are certain velocities that are determined by the curvature of the curve.

The evolution of the curvature

The evolution of the curvature is given by

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 W}{\partial s^2} + \kappa^2 W + \frac{\partial \kappa}{\partial s} \int k W \mathrm{d}s \equiv RW$$

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$$\frac{\partial \kappa}{\partial t} - \frac{\partial^3 \kappa}{\partial s^3} - \frac{3}{2}\kappa^2 \frac{\partial \kappa}{\partial s} = 0$$

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which is the same equation derived in the membranes study. Therefore one can apply results from elastic membrane theory to the current topic.

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- **Case 3** Four distinct real roots

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$$\kappa_1(s) = \frac{A\beta + B\alpha - (A\beta - B\alpha)\,\mathrm{cn}\,(us, k)}{A + B - (A - B)\,\mathrm{cn}\,(us, k)}$$

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$$\theta_1(s) = \frac{(A\beta - B\alpha)\,s}{A - B} + \frac{(A+B)(-\beta+\alpha)}{2u\,(A-B)}\Pi\left(\mathrm{sn}\,(us,k)\,, -\frac{(A-B)^2}{4BA}, k\right)$$
$$+ \frac{\alpha - \beta}{u\sqrt{4\,k^2 + \frac{(A-B)^2}{BA}}}\,\mathrm{arctan}\left(\sqrt{k^2 + \frac{(A-B)^2}{4BA}}\,\frac{\mathrm{sn}\,(us,k)}{\mathrm{dn}\,(us,k)}\right)$$

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The solution curve is given if we plug the quantities from the previous page in

$$x(s) = \frac{2}{\sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\cos\theta(s) + \frac{1}{\sigma}(\kappa^2(s) - \mu)\sin\theta(s)$$

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That is for the case $\sigma \neq 0$. One can get the solution curves in the zero case too.

Case 1

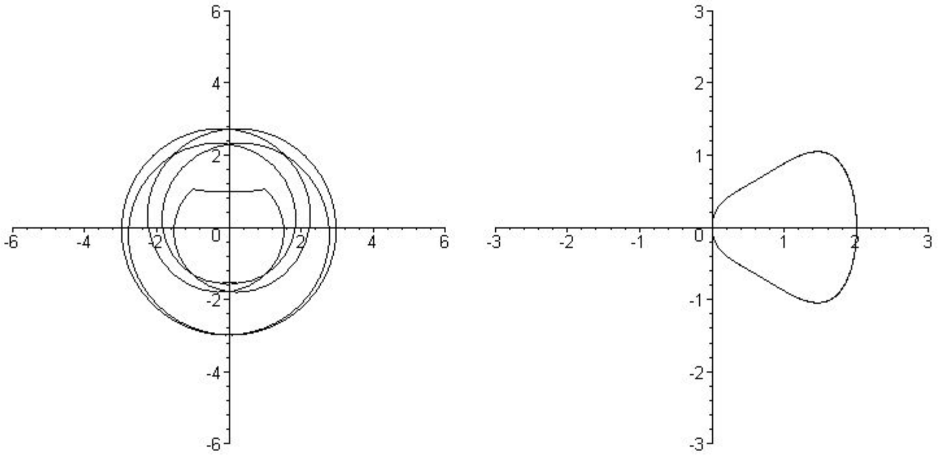

Figure: Solution curve (left) and phase portrait (right) for $\alpha = 0$, $\beta = 2$, $\gamma = -1 - \mathrm{i}$.

Case 2

Here the polynomial $P(\kappa)$ has two real roots $\alpha < \beta$ and a pair of complex roots $\gamma, \bar{\gamma}$ with $(3\alpha + \beta)(\alpha + 3\beta) = 0$. Let $\xi = \alpha$ if $3\alpha + \beta = 0$ and $\xi = \beta$ otherwise. Again we need the roots to sum up to zero. These two conditions actually imply that $\sigma \neq 0$.

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$$\kappa_2(s) = \xi - 4\,\frac{\xi}{1 + \xi^2 s^2}$$

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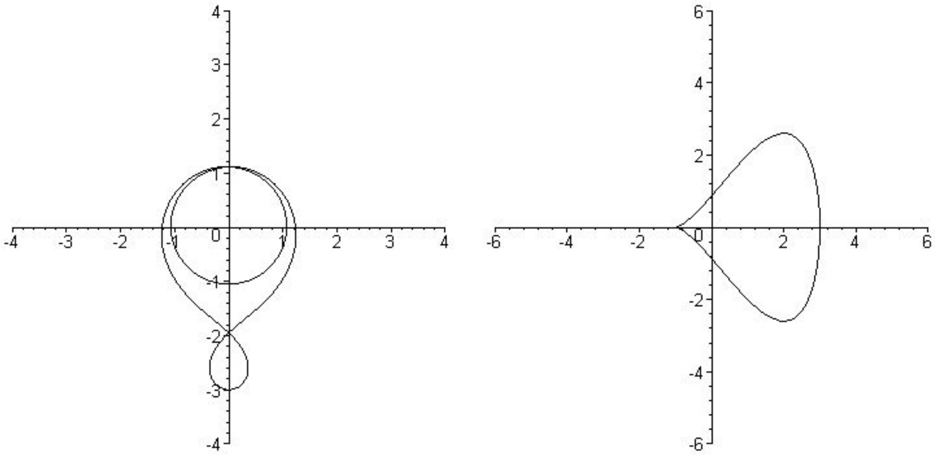

Figure: Solution curve (left) and phase portrait (right) for $\alpha = \beta = \gamma = -1, \delta = 3$

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In the last case we will consider the polynomial $P(\kappa)$ with four real roots $\alpha < \beta < \gamma < \delta$. One possible solution (i.e. the curvature, etc.) is given below. Let

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$$\theta_3(s) = \delta s - 4\Pi\left(\hat{\mathrm{sn}}(s), \frac{\beta-\alpha}{\beta-\delta}, q\right)(\delta-\alpha)(\gamma-\alpha)^{-1/2}(\delta-\beta)^{-1/2}$$

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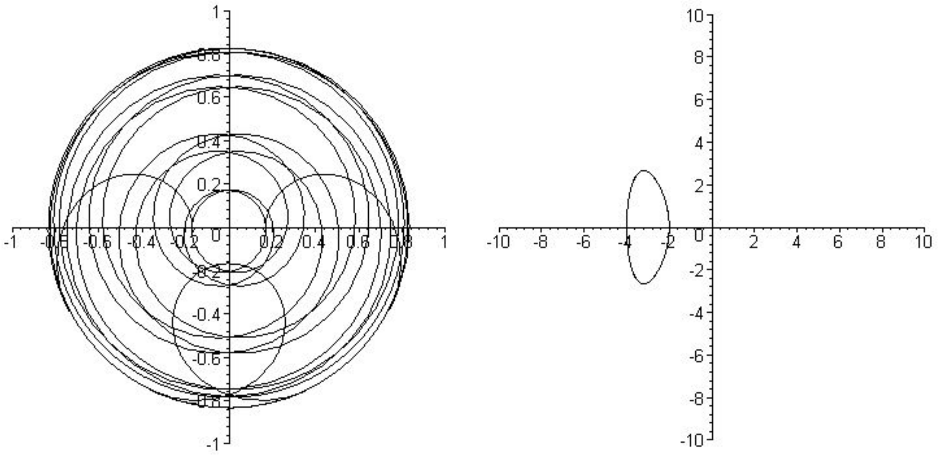

Figure: Solution curve (left) and phase portrait (right) for $\alpha = -4, \beta = -2, \gamma = 0, \delta = 6$

Summary

We use results from the theory of fluid membranes to solve the mKdV equation which arises from the evolution of planar curves.

Thank you for your patience!