XIV^{th} International Conference on Geometry, Integrability and Quantization, 8-13 June, Varna, Bulgaria

Some Remarks on the Exponential Map on the Groups $\mathrm{SO}(n)$ and $\mathrm{SE}(n)$

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12th June 2012

1 Introduction. Lie groups and the exponential map

Let G be a Lie group with its Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma_X(1)$, where $X \in \mathfrak{g}$ and γ_X is the one-parameter subgroup of G induced by X. Recall the following general properties of the exponential map.

1. For every $t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(tX) = \gamma_X(t)$;
2. For every $s, t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(sX)\exp(tX) = \exp(s+t)X$;
3. For every $t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(-tX) = \exp(tX)^{-1}$;
4. $\exp : \mathfrak{g} \to G$ is a smooth mapping, it is a local diffeomorphism at $0 \in \mathfrak{g}$ and $\exp(0) = e$, where e is the unity element of the group G;
5. The image $\exp(\mathfrak{g})$ of the exponential map generates the connected component G_e of the unity $e \in G$;
6. If $f : G_1 \to G_2$ is a morphism of Lie groups and $f_* : \mathfrak{g}_1 \to \mathfrak{g}_1$ is the induced morphism of Lie algebras, then $f \circ \exp_1 = \exp_2 \circ f$.

As we can note from the previous *property 5*, the following problems are of special importance :
Problem 1. *Find the conditions on the group G such that the exponential is surjective.*
Problem 2. *Determine the image $E(G)$ of the exponential map.*

Concerning *Problem 1*, only in a few special situations we have $G = E(G)$, i.e. the surjectivity of the exponential map. A Lie group satisfying this property is called *exponential*. Every compact and connected Lie group is exponential , but there are exponential Lie groups which are not compact.

Even if we know that the exponential map is surjective, to get closed formulas for the exponential map for different Lie groups is an interesting problem. Such formulas are well-known for the special orthogonal group $SO(n)$ and for the special Euclidean group $SE(n)$, when $n = 2, 3$, as Rodrigues' formulas. One of the main goal of this presentation is to discuss the possibility to extend the Rodrigues' formulas for these two Lie groups in dimensions $n \geq 4$.

2 The Rodrigues formula for $\mathrm{SO}(n)$, $n = 2$ and $n = 3$

It is well-known that the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ consists in all skew-symmetric matrices in $M_n(\mathbb{R})$ and the Lie bracket is the standard commutator $[A, B] = AB - BA$.

The exponential map $\exp : \mathfrak{so}\,(n) \to SO(n)$ is defined by

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

When $n = 2$, a skew-symmetric matrix B can be written as $B = \theta J$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and from the series expansion of $\sin\theta$ and $\cos\theta$ it is easy to show that:

$$e^B = e^{\theta J} = (\cos\theta)I_2 + (\sin\theta)J = (\cos\theta)I_2 + \frac{\sin\theta}{\theta}B.$$

Given the matrix $R \in \mathbf{SO}(2)$, we can find $\cos\theta$ because we have $tr(R) = 2\cos\theta$ (where $tr(R)$ denotes the trace of R). Thus, the formula is completely proved.

Proposition 1 (Rodrigues) The exponential map $\exp : \mathfrak{so}(3) \to \mathbf{SO}(3)$ *is given by the following formula:*

$$\exp(\widehat{v}) = I_3 + \frac{\sin||v||}{||v||}\widehat{v} + \frac{1}{2}\left(\frac{\sin\frac{||v||}{2}}{\frac{||v||}{2}}\right)^2\widehat{v}^2.$$

Proof. Indeed, we obtain successively:

$$\widehat{v}^3 = -||v||^2\widehat{v}$$

$$\widehat{v}^4 = -||v||^2\widehat{v}^2$$

$$\widehat{v}^5 = -||v||^4\widehat{v}$$

$$\widehat{v}^6 = -||v||^4\widehat{v}^2$$

$$\vdots$$

Consequently,

$$\exp(\widehat{v}) = \sum_{n=0}^{\infty}\frac{\widehat{v}^n}{n!}$$

$$= I_3 + \frac{\widehat{v}}{1!} + \frac{\widehat{v}^2}{2!} + \frac{\widehat{v}^3}{3!} + \frac{\widehat{v}^4}{4!} + \dots$$

$$= I_3 + \frac{\widehat{v}}{1!} + \frac{\widehat{v}^2}{2!} - \frac{||v||^2}{3!}\widehat{v} - \frac{||v||^2}{4!}\widehat{v}^2 + \dots$$

$$= I_3 + \left[I_3 - \frac{||v||^2}{3!} + \frac{||v||^4}{5!} + \dots\right]\widehat{v} + \left[\frac{1}{2!}I_3 - \frac{||v||^2}{4!} + \dots\right]\widehat{v}^2$$

$$= I_3 + \frac{\sin||v||}{||v||}\widehat{v} + \frac{1-\cos||v||}{||v||^2}\widehat{v}^2$$

$$= I_3 + \frac{\sin||v||}{||v||}\widehat{v} + \frac{1}{2}\left(\frac{\sin\frac{||v||}{2}}{\frac{||v||}{2}}\right)^2\widehat{v}^2.$$

■

Even if the following result is clear because for every $n \geq 1$, the group $SO(n)$ is compact, we prefer to present the alternative proof because it gives an explicit way to find solutions to the equation $\exp(X) = R$.

Proposition 2 The exponential map

$$\exp : \mathfrak{so}(3) \to \mathbf{SO}(3)$$

is surjective.

Proof. We show that for any rotation matrix $R \in \mathbf{SO}(3)$,

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

there is $\widehat{\omega} \in \mathfrak{so}(3)$ so that

$$\exp(\widehat{\omega}) = R,$$

or equivalent to

$$I_3 + \frac{\sin||\omega||}{||\omega||}\widehat{\omega} + \frac{1-\cos||\omega||}{||\omega||^2}\widehat{\omega}^2 = R.$$

From the above relation we obtain that:

$$1 + 2\cos||\omega|| = Trace(R).$$

Because

$$-1 \leq Trace(R) \leq 3$$

we can conclude that:

$$||\omega|| = arc\cos\frac{Trace(R)-1}{2}.$$

On the other hand we obtain

$$r_{32} - r_{23} = 2\omega_1\frac{\sin||\omega||}{||\omega||}$$

$$r_{13} - r_{31} = 2\omega_2\frac{\sin||\omega||}{||\omega||}$$

$$r_{21} - r_{12} = 2\omega_3 \frac{\sin||\omega||}{||\omega||}.$$

So, we can consider

$$\omega = \frac{||\omega||}{2\sin||\omega||}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

and we obtain

$$\exp(\widehat{\omega}) = R.$$

■

3 A Rodrigues-like formula for $\mathrm{SO}(n)$, $n \geq 4$

When $n = 3$, a real skew-symmetric matrix B is of the form:

$$B = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

and letting $\theta = \sqrt{a^2 + b^2 + c^2}$, we have the well-known formula due to Rodrigues:

$$e^B = I_3 + \frac{\sin\theta}{\theta} B + \frac{1 - \cos\theta}{\theta^2} B^2$$

with $e^B = I_3$ when $B = 0$.

It turns out that it is more convenient to normalize B, that is, to write $B = \theta B_1$ (where $B_1 = B/\theta$, assuming that $\theta \neq 0$), in which case the formula becomes:

$$e^{\theta B_1} = I_3 + \sin\theta B_1 + (1 - \cos\theta) B_1^2.$$

Also, given the matrix $R \in \mathbf{SO}(3)$, we can find $\cos\theta$ because $tr(R) = 1 + 2\cos\theta$, and we can find B_1 by observing that:

$$\frac{1}{2}(R - R^\top) = \sin\theta B_1.$$

Actually, the above formula cannot be used when $\theta = 0$ or $\theta = \pi$, as $\sin\theta = 0$ in these cases. When $\theta = 0$, we have $R = I_3$ and $B_1 = 0$, and when $\theta = \pi$, we need to find B_1 such that:

$$B_1^2 = \frac{1}{2}(R - I_3).$$

As B_1 is a skew-symmetric 3×3 matrix, this amounts to solving some simple equations with three unknowns. Again, the problem is completely solved.

In this presentation, it is shown that there is a generalization of Rodrigues' formula for computing the exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$, when $n \geq 4$. The key to the solution is that, given a skew-symmetric $n \times n$ matrix B, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that B can be expressed as:

$$B = \theta_1 B_1 + \ldots + \theta_p B_p$$

where $\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$

is the set of distinct eigenvalues of B, with $\theta_i > 0$ and where:

$$B_i B_j = B_j B_i = 0_n \quad (i \neq j)$$

$$B_i^3 = -B_i.$$

This reduces the problem to the case of 3×3 matrices.

Lemma 1 Given any skew-symmetric $n \times n$ matrix B ($n \geq 2$), there is some orthogonal matrix P and some block diagonal matrix E such that:$B = PEP^{\top}$,
with E of the form:

$$E = \begin{pmatrix} E_1 & \cdots & & \\ \vdots & \ddots & \vdots & \\ & \cdots & E_m & \\ \cdots & & & 0_{n-2m} \end{pmatrix}$$

where each block E_i is a real two-dimensional matrix of the form:

$$E_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix} = \theta_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ with } \theta_i > 0.$$

Observe that the eigenvalues of B are $\pm i\theta_j$, or 0, reconfirming the well-known fact that the eigenvalues of a skew-symmetric matrix are purely imaginary, or null. We now prove that the existence and uniqueness of the B_j's as well as the generalized Rodrigues' formula.

Theorem 1 *Given any non-null skew-symmetric $n \times n$ matrix B, where $n \geq 3$, if:*

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, where $\theta_j > 0$ and each $i\theta_j$ (and $-i\theta_j$) has multiplicity $k_j \geq 1$, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that the following relations hold:

$$B = \theta_1 B_1 + \ldots + \theta_p B_p \quad (1)$$

$$B_i B_j = B_j B_i = O_n \quad (i \neq j) \quad (2)$$

$$B_i^3 = -B_i \quad (3)$$

for all i, j with $1 \leq i, j \leq p$, and $2p \leq n$. Furthermore, we have

$$e^B = e^{\theta_1 B_1 + \ldots + \theta_p B_p} = I_n + \sum_{i=1}^{p} [(\sin\theta_i) B_i + (1 - \cos\theta_i) B_i^2]$$

and $\{\theta_1, \ldots, \theta_p\}$ is the set of the distinct positive square roots of the $2m$ positive eigenvalues of the symmetric matrix $-1/4(B - B^\top)^2$, where $m = k_1 + \ldots + k_p$.

Proof. By *Lemma 1*, the matrix B can be written as:

$$B = PEP^\top,$$

where E is a block diagonal matrix consisting of m non-zero blocks of the form:

$$E_i = \theta_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ with } \theta_i > 0.$$

If:

$$\{i\theta_1, -i\theta_1, \dots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, where $\theta_j > 0$, for every j, there is a non-empty set:

$$S_j = \{i_1, \dots, i_{k_j}\}$$

of indices (in the set $\{1, \dots, m\}$) corresponding to all blocks E_j in which θ_j occurs. Let F_j be the matrix obtained by zeroing from E the blocks E_k, where $k \notin S_j$.
By factorizing θ_j in F_j, we have

$$F_j = \theta_j G_j$$

and we let

$$B_j = P G_j P^\top.$$

It is obvious by construction that the three equations $(1) - (3)$ hold.

As B_i and B_j comute for all i, j, we have:

$$e^B = e^{\theta_1 B_1 + \ldots + \theta_p B_p} = e^{\theta_1 B_1} \ldots e^{\theta_p B_p}.$$

However, using:

$$B_i^3 = -B_i$$

as in the 3×3 case, we can show that:

$$e^{\theta_i B_i} = I_n + (\sin\theta_i)B_i + (1 - \cos\theta_i)B_i^2.$$

Indeed, $B_i^3 = -B_i$ implies that:

$$B_i^{4k+j} = B_i^j \text{ and } B_i^{4k+2+j} = -B_i^j \text{ for } j = 1, 2 \text{ and all } k \geq 0.$$

Thus, we get

$$e^{\theta_i B_i} = I_n + \sum_{k \geq 1} \frac{\theta_i^k B_i^k}{k!}$$

$$= I_n + \left(\frac{\theta_i}{1!} - \frac{\theta_i^3}{3!} + \frac{\theta_i^5}{5!} + \ldots\right) B_i + \left(\frac{\theta_i^2}{2!} - \frac{\theta_i^4}{4!} + \frac{\theta_i^6}{6!} + \ldots\right) B_i^2$$

$$= I_n + (\sin\theta_i)B_i + (1 - \cos\theta_i)B_i^2.$$

Since

$$B_iB_j = B_jB_i = O_n \quad (i \neq j)$$

we obtain:

$$e^B = \prod_{i=1}^{p} e^{\theta_i B_i} = \prod_{i=1}^{m} [I_n + (\sin\theta_i)B_i + (1 - \cos\theta_i)B_i^2]$$

$$= I_n + \sum_{i=1}^{p} [(\sin\theta_i)B_i + (1 - \cos\theta_i)B_i^2].$$

The matrix $1/4(B - B^\top)^2$ is of the form $PE^2P^\top$, where

$$E_i^2 = \begin{pmatrix} -\theta_i^2 & 0 \\ 0 & -\theta_i^2 \end{pmatrix}.$$

Thus, the eigenvalues of $-1/4(B - B^\top)$ are

$$(\theta_1^2, \theta_1^2, \ldots, \theta_m^2, \theta_m^2, \underbrace{0, \ldots, 0}_{n-2m})$$

and thus $(\theta_1, \ldots, \theta_m)$ are the positive square roots of the eigenvalues of the symmetric matrix $-1/4(B - B^\top)^2$.
We now prove the uniqueness of the B_j's. If we assume that matrices B_j's, we get the following system:

$$B = \sum_{i=1}^{p} \theta_i B_i$$

$$B^3 = -\sum_{i=1}^{p} \theta_i^3 B_i$$

$$B^5 = \sum_{i=1}^{p} \theta_i^5 B_i$$

$$\vdots$$

$$B^{2p-1} = (-1)^{p-1} \sum_{i=1}^{p} \theta_i^{2p-1} B_i.$$

The determinant of the system is

$$\delta_n = \begin{vmatrix} \theta_1 & \theta_2 & \cdots & \theta_p \\ -\theta_1^3 & -\theta_2^3 & \cdots & -\theta_p^3 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{p-1}\theta_1^{2p-1} & (-1)^{p-1}\theta_2^{2p-1} & \cdots & (-1)^{p-1}\theta_p^{2p-1} \end{vmatrix}.$$

Observe that the matrix defining δ_n is the product of the diagonal matrix

$$diag(1, -1, 1, -1, \ldots, 1, (-1)^{p-1})$$

by the matrix $\left(\prod_{i=1}^{p}\theta_i\right) V(\theta_1^2, \ldots, \theta_p^2)$

where $V(\theta_1^2, \ldots, \theta_p^2)$ is a Vandermonde matrix. Therefore, the determinant δ_n can be immediately computed, and we get

$$\delta_n = (-1)^{p(p-1)/2}\prod_{i=1}^{p}\theta_i \prod_{1\leq i,j\leq p}(\theta_j^2 - \theta_i^2).$$

Since the θ_i's are positive and all distinct, we have $\delta_n \neq 0$. Thus $B_1, \ldots, B_p$ are uniquely determined from B and its non-null eigenvalues. ■

Given a skew-symmetric $n \times n$ matrix B, we can compute $\theta_1, \dots, \theta_p$ and $B_1, \dots, B_p$ as follows. By *Thorem 1*, $\theta_1^2, \dots, \theta_p^2$ are the distinct non-null eigenvalues of the symmetric matrix $-1/4(B - B^\top)^2$. There are several numerical methods for computing eigenvalues of symmetric matrices. Then, we find $B_1, \dots, B_p$ by solving the linear system used in the proof of *Theorem 1*.
Note that B_j has the eigenvalues $i, -i$, each with multiplicity k_j, and 0 with multiplicity $n - 2k_j$. Now recall the following structure lemma for rotations in $\mathbf{SO}(n)$.

Lemma 2 For every rotation matrix $R \in \mathbf{SO}(n)$, there is a block diagonal matrix D and an orthogonal matrix P such that:

$$R = PDP^\top,$$

with D a block diagonal matrix of the form

$$D = \begin{pmatrix} D_1 & \cdots & & \\ \vdots & \ddots & \vdots & \\ & \cdots & D_m & \\ \cdots & & & I_{n-2m} \end{pmatrix}$$

where the first m blocks D_i are of the form:

$$D_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \text{ with } 0 < \theta_i \leq \pi.$$

Using the surjectivity of the exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$, which esaily follows from *Lemma 1*, *Lemma 2* and the fact that if

$$E_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}$$

then $e^{E_i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$

we obtain the following characterization of rotations in $\mathbf{SO}(n)$, where $n \geq 3$:

Lemma 3 Given any rotation matrix $R \in \mathbf{SO}(n)$, where $n \geq 3$, if:

$$\left\{e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, -e^{-i\theta_p}\right\}$$

is the set of distinct eigenvalues of R different from 1, where $0 < \theta_i \leq \pi$, there are p skew-symmetric matrices $B_1, \ldots, B_p$ such that:

$$B_i B_j = B_j B_i = O_n \ (i \neq j) \text{ and } B_i^3 = -B_i,$$

for all i, j with $1 < i, j \leq p$, and $2p \leq n$, and furthermore:

$$R = e^{\theta_1 B_1 + \ldots + \theta_p B_p} = I_n + \sum_{i=1}^{p} [(\sin\theta_i) B_i + (1 - \cos\theta_i) B_i^2].$$

Lemma 3 implies that

$$\{\cos\theta_1, \ldots, \cos\theta_p\}$$

is the set of eigenvalues of the symmetric matrix $1/2(R + R^\top)$ that are different from 1. However, the matrices $B_1, \ldots, B_p$ are not necessarily unique. This has to do with the fact that we may have $\sin\theta_i = 0$ when $\theta_i = \pi$. Nevertheless, it is possible to find $B_1, \ldots, B_p$ from R.

4 The special Euclidean group $\mathrm{SE}(n)$

The Euclidean group $E(n)$ is the group of all isometries of the Euclidean space $\mathbb{R}^n$. When $n = 2$, $E(n)$ consists in all plane translations, rotations and reflections. This group of isometries can be represented by the matrix group denoted also by $\mathbf{E}(n)$,

$$\mathbf{E}(n) := \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} \in \mathbf{GL}_{n+1}(\mathbb{R}) | \mathbf{v} \in \mathbb{R}^n \right\} \text{ in terms of } (n+1) \times (n+1) \text{ matrices.}$$

The set of affine maps ρ of $\mathbb{R}^n$ defined such that:

$$\rho(X) = RX + U$$

where R is a rotation matrix $(R \in \mathbf{SO}\,(n))$ and U is some vector in $\mathbb{R}^n$, is a group under composition called the group of direct affine isometries, or rigid motions, denotes as $\mathbf{SE}\,(n)$.

Every rigid motion can be represented by the $(n+1) \times (n+1)$ matrix:

$$\begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix}$$

in the sense that:

$$\begin{pmatrix} \rho\,(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

if and only if

$$\rho(X) = RX + U.$$

The vector space of real $(n+1) \times (n+1)$ matrices of the form

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where B is a skew-symmetric matrix and U is a vector in $\mathbb{R}^n$ is denoted by $\mathfrak{se}(n)$. The group $\mathbf{SE}(n)$ is a Lie group, and $\mathfrak{se}(n)$ is its Lie algebra. In what follows we will concentrate on the properties of the group $\mathbf{SE}(2)$.

It turns out that the group $\mathbf{E}(2)$ is not a connected Lie group. We restrict ourselves to the Lie subgroup $\mathbf{SE}(2)$, the connected component of the identity. The Lie subgroup $\mathbf{SE}(2)$ corresponds to the group of all orientation-preserving isometries, where $\det R_\theta = 1$.

So

$$\mathbf{SE}(2) := \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \in GL(3,\mathbb{R}) | \mathbf{v} \in \mathbb{R}^{2\times 1} \text{ and } R_\theta \in \mathbf{SO}(2) \right\}$$

$$\text{where } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

4.1 Topological properties of $\mathrm{SE}(2)$

Proposition 3 The group $\mathbf{SE}(2)$ is closed in $GL(3,\mathbb{R})$.

Proposition 4 The group $\mathbf{SE}(2)$ is not bounded, hence it is not compact.

The Lie algebra has the following form:

$$\mathfrak{se}(2) = \left\{ A \in \mathbb{R}^{3\times 3} | A = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} ; x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The standard basis of $\mathfrak{se}(2)$ is given by

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The Lie brackets of these elements are

$$[E_1, E_2] = 0,\ [E_2, E_3] = E_1,\ [E_3, E_1] = E_2.$$

We have $\exp A \in \mathbf{SE}(2)$ for all $A \in \mathfrak{se}(2)$. Indeed for all $x_3 \neq 0$

$$\exp A = I_3 + \frac{\sin(x_3)}{x_3} A + \frac{(1 - \cos x_3)}{x_3^2} A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{x_1 \sin x_3 + x_2 \cos x_3 - x_2}{x_3} & \cos x_3 & -\sin x_3 \\ \frac{x_2 \sin x_3 - x_1 \cos x_3 + x_1}{x_3} & \sin x_3 & \cos x_3 \end{bmatrix}.$$

For $x_3 = 0$ we take the limiting case of the above, as $x_3 \to 0$, and we obtain that:

$$\exp A = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix}$$

Proposition 5 The map $\exp : \mathfrak{se}(2) \to \mathbf{SE}(2)$ *is surjective and it is not injective.*

Proof. Let

$$(v, R_\theta) = \begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \in \mathbf{SE}(2).$$

Then for

$$x_1 = \frac{v_1\theta\sin\theta}{2(1-\cos\theta)} + \frac{v_2\theta}{2},\; x_2 = \frac{v_2\theta\sin\theta}{2(1-\cos\theta)} - \frac{v_1\theta}{2}$$

we have that

$$\exp\begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -\theta \\ x_2 & \theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix}.$$

For $\cos\theta = 1$

$$\exp\begin{bmatrix} 0 & 0 & 0 \\ v_1 & 0 & 0 \\ v_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{bmatrix}.$$

Consider the following two elements

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\pi \\ 0 & \pi & 0 \end{bmatrix},\; \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -\pi \\ 1 & \pi & 0 \end{bmatrix} \in \mathfrak{se}(2).$$

Then

$$\exp\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -\pi\\ 0 & \pi & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

and

$$\exp\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -\pi\\ 1 & \pi & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ \frac{1(0)+1(1)-1}{\pi} & \cos\pi & -\sin\pi\\ \frac{1(0)-1(1)+1}{\pi} & \sin\pi & \cos\pi\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Therefore it follows that $\exp : \mathfrak{se}(2) \to \mathbf{SE}(2)$ is not injective. ■

4.2 The Rodrigues formula for $\mathrm{SE}(n)$, $n \geq 3$

In this subsection, we give a Rodrigues-like formula showing how to compute the exponential e^{Ω} of an element Ω of the Lie algebra $\mathfrak{se}(n)$, where $n \geq$. In order to give a Rodrigues-like formula for computing the exponential map $\exp : \mathfrak{se}(n) \to \mathbf{SE}(n)$, we need the following key lemma.

Lemma 4 Given any $(n+1) \times (n+1)$ *matrix of the form:*

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

$$\textit{then } e^{\Omega} = \begin{pmatrix} e^{B} & VU \\ 0 & 1 \end{pmatrix}$$

$$\textit{where } V = I_n + \sum_{k \geq 1} \frac{B^k}{(k+1)!}.$$

Observing that:

$$V = I_n + \sum_{k \geq 1} \frac{B^k}{(1+k)!} = \int_0^1 e^{Bt} dt$$

we can now prove our main result:

Theorem 2 Given any $(n+1)\times(n+1)$ *matrix of the form:*

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where B *is a non-null skew-symmetric matrix and* $U \in \mathbb{R}^n$*, with* $n \geq 3$*, if:*

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B*, where* $\theta_i > 0$*, there are* p *unique skew-symmetric matrices* $B_1, \ldots, B_p$ *such that the three equations* $(1)-(3)$ *hold. Furthermore:*

$$e^\Omega = \begin{pmatrix} e^B & VU \\ 0 & 1 \end{pmatrix}$$

where $e^B = I_n + \sum_{i=1}^{p} \left(\sin\theta_i B_i + (1-\cos\theta_i)B_i^2\right)$

and $V = I_n + \sum_{i=1}^{p} \left(\frac{1-\cos\theta_i}{\theta_i} B_i + \frac{\theta_i - \sin\theta_i}{\theta_i^2} B_i^2\right)$*.*

Proof. The existence and uniqueness of $B_1, \ldots, B_p$ and the formula for e^B come from *Theorem 1.* Since:

$$V = I_n + \sum_{k \geq 1} \frac{B^k}{(k+1)!} = \int_0^1 e^{Bt} dt$$

we have:

$$V = \int_0^1 \left[I_n + \sum_{i=1}^{p} \left(\sin t\theta_i B_i + (1 - \cos t\theta_i) B_i^2 \right) \right] dt$$

$$= \left[tI_n + \sum_{i=1}^{p} \left(-\frac{\cos t\theta_i}{\theta_i} B_i + \left(t - \frac{\sin t\theta_i}{\theta_i} \right) B_i^2 \right) \right]_0^1$$

$$= I_n + \sum_{i=1}^{p} \left(\frac{1 - \cos \theta_i}{\theta_i} B_i + \frac{\theta_i - \sin \theta_i}{\theta_i} B_i^2 \right)$$

■

Remark 1 Given:

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where $B = \theta_1 B_1 + \ldots + \theta_p B_p$, if we let:

$$\Omega_i = \begin{pmatrix} B_i & U/\theta_i \\ 0 & 0 \end{pmatrix}$$

using the fact that $B_i^3 = -B_i$ and:

$$\Omega_i^k = \begin{pmatrix} B_i^k & B_i^{k-1} U/\theta_i \\ 0 & 0 \end{pmatrix}$$

it is easily verified that:

$$e^{\Omega} = I_{n+1} + \Omega + \sum_{i=1}^{p} \left((1 - \cos\theta_i)\, \Omega_i^2 + (\theta_i - \sin\theta_i)\, \Omega_i^3 \right).$$

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Thank you!