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**The Dynamics of the Field of Linear Frames and Gauge Gravitation**

_Some Comments-2_

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\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\rho\sigma} g^{\nu\lambda} = \frac{1}{2} (\overline{E}^2 - \overline{B}^2) \]

\[ P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{8} \varepsilon^{\mu\nu\lambda\xi} F_{\mu\nu} F_{\lambda\xi} = \overline{E} \cdot \overline{B} \]

\[ L[F] = l(S, P) \]

**Maxwell:** \[ L = S \]

**Born-Infeld:** \[ L = b^2 - b^2 \sqrt{1 - \frac{2}{b^2}} S - \frac{1}{b^4} P^2 = \]

\[ = b^2 \sqrt{| \text{det} [g_{\mu\nu}] |} - \sqrt{| \text{det} [bg_{\mu\nu} + F_{\mu\nu}] |} \]

**Born original:** \[ b^2 \left( \sqrt{1 + \frac{1}{b^2} (\overline{B}^2 - \overline{E}^2)} - 1 \right) \]

\[ g_{\mu\nu} \text{ - Minkowskian} \]
Motivation: finite electromagnetic self-energy of the electron (maximal electrostatic field)

Field equations $\Rightarrow$ equations of motion (like in GR)

$\varphi = A_0$

$\varphi, \bar{E} - \text{finite}$

$\bar{D} - \text{infinite at } r=0$

$\omega = T_{00} - \text{infinite at } r=0$, but $E = \int \omega d\varphi - \text{finite}$

$$\bar{E}(\bar{r}) = \frac{e}{\sqrt{r_0^4 + r^4}} \frac{\bar{r}}{r}$$

$$r_0 = \sqrt{\frac{e}{b}}$$

$$\varphi(r) = \int_{r}^{\infty} \frac{e dx}{\sqrt{r_0^4 + x^4}}$$
Exceptionality of the Born-Infeld model:

~ gauge-invariant
~ energy positively definite
~ finite electromagnetic mass of point sources
~ energy current - non-spacelike
~ no birefringence
~ plane waves on the background of the constant electromagnetic field, solitary waves

( J. Plebański, Z. Białynicka-Birula )
Most interesting among nonlinear models, nevertheless, disappointing(?):

~ no convincing results in: field equations $\Rightarrow$ equations of motion
   (success of G.R.: Bianchi identities following from the general covariance
    nonlinearity relevant, but indirectly, as implied by general covariance)

~ the spectra of superheavy atoms do not seem to support B.-I.

~ quantization difficulties (non-polynomial structure)

~ QED is not afraid of infinities (renormalization). The electron mass
   is not purely electromagnetic (cancellation of infinities)

~ B.-I. does not suit well the external charged matter (other than
   the internal one, described by singularities of $\bar{D}$).

   E.g., for the quantum coherent matter:

   \[ L = b^2 \sqrt{1g} - \sqrt{1b}g + F^1 + g^{I\nu} \overleftrightarrow{D}_\mu \overleftrightarrow{D}_\nu 4 \sqrt{1g} - m^2 \overleftrightarrow{D}_I 4 \sqrt{1g} ; \quad D_\mu = \partial_\mu + ieA_\mu \]

(Non-rational structure of field equations).
B-I: useful as a classical model of some QEM-effects like the light-light-scattering:

Nonlinearity of the electromagnetic field: an effective description (Ersatz-Modell) of the nonlinear interaction between the linear Maxwell field and matter. (Nonlinearity replaces matter).
Nevertheless, the peculiarity of B-I seems to suggest that it was motivated by some good intuitions. One should discuss it and reformulate in more geometric terms, including the general covariance.

Linear models: \( L(y^a; y^a, \mu) \) - quadratic in \( y^a, \mu, y^a \) with \( y \)-independent coefficients; typically:

\[
L(y, \delta y) = l(y, \delta y) \sqrt{g^1},
\]
\[
l(y, \delta y) = a_{KL}(x) y^k, \mu y^L, \nu g^{\mu \nu} + b_{KL}(x) y^k y^L.
\]

Quasilinear: coefficients at \( \delta y \) depend on \( y \):

\[
l(y, \delta y) = a_{KL}(x,y) y^k, \mu y^L, \nu g^{\mu \nu} + b(x,y)
\]

GR - quasilinear, but generally-covariant theories must be nonlinear.
Simple nonlinearities introduced „by hand“: adding to $L$ some terms of degree higher than 2 in $\delta y$, $y$. It does not work even in quasilinear GR.

**Density philosophy:**

Primary dynamical quantities are Lagrangian tensors $L_{\mu\nu}(y, \delta y)$.

$$L(y, \delta y) := \sqrt{|\det [L_{\mu\nu}(y, \delta y)]|}$$

**GR - artificial in this language:**

$$L = \text{sign} \, R \sqrt{|\det [\sqrt{|R|} g_{\mu\nu}]|} = \text{sign} \, R \sqrt{|\det [\sqrt{|R|} g_{\mu\nu}]|}$$

if $n = 4$

Locally:

$$L_{\mu\nu} = |R|^{n/2} g_{\mu\nu} = \sqrt{|R|} g_{\mu\nu}$$

if $n = 4$
The simplest models, opposite to linear theories: $L_{\mu\nu}$ is a low-order polynomial of derivatives. B-I - first-order polynomial.

Scalar field; the B-I counterpart of the scalar theory of light:

$$L = b^2 \sqrt{|\det [g_{\mu\nu}]|} - \sqrt{|\det [bg_{\mu\nu} + \phi_{,\mu} \phi_{,\nu}]|}$$

instead the linear model $g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \sqrt{|\det [g_{\mu\nu}]|}$.

For the spherically-symmetric stationary field (the point particle) the result identical with that for the usual B-I, although $L_{\mu\nu}$ -quadratic in $\mathcal{D}\Phi$. Barbashov- scalar Born-Infeld theory.
If so, perhaps the second-order B-I electrodynamics could also work:

\[ \sqrt{\det [\alpha g_{\mu\nu} + \beta F_{\mu\nu} + \sigma g^{\kappa\lambda} F_{\mu\kappa} F_{\nu\lambda} + \delta g^{\kappa\eta} g^{\lambda\sigma} F_{\mu\kappa} F_{\nu\sigma} + g_{\mu\nu}]} \]

additional terms: familiar from the Maxwell energy–momentum tensor.

This modification: necessary in the Born-Infeld theory of gauge fields ruled by semisimple groups:

\[ \sqrt{\det [\alpha g_{\mu\nu} + \sigma g^{\kappa\lambda} F_{\mu\kappa} F_{\nu\lambda} h_{KL}]} \]

If so, it is perhaps possible to overcome the problem of incompatibility between B-I and "external" matter, e.g., in this way:

\[ \sqrt{\det [\alpha g_{\mu\nu} + b F_{\mu\nu} + c \overline{D_{\mu} 41} D_{\nu} 41]} \]

\[ \hat{L} + \text{perhaps the terms quadratic in } F. \]

The resulting field equations are rational, although Lagrangian is not.
Natural models for scalar multiplets.

\((M, g)\) - space-time, coordinates \(x^\mu\)

\((W, \eta)\) - target space, coordinates \(y^A\)

\(\eta\) - pseudo-Euclidean or hermitian

\(\Phi : M \rightarrow W\)

\(y^A = \Phi^A(x^\mu)\)

\[ L = \frac{1}{2} \eta_{AB} \left( \Phi(x^\mu) \right) \frac{\partial \Phi^A}{\partial x^\mu} \frac{\partial \Phi^B}{\partial x^\nu} g^{\nu\mu} \sqrt{\det[g_{\mu\nu}]} \] - "d'Alembert"

(but \(\eta\) may depend on \(\Phi\)).

If \(W\) - linear and \(\eta\) - constant, algebraic (mass) term possible:

\[ L = \frac{1}{2} \eta_{AB} \frac{\partial \Phi^A}{\partial x^\mu} \frac{\partial \Phi^B}{\partial x^\nu} g^{\nu\mu} \sqrt{\det[g_{\mu\nu}]} - \frac{m^2}{2} \eta_{AB} \Phi^A \Phi^B \sqrt{\det[g_{\mu\nu}]} \]

+ possibly nonlinear terms \(f(\eta_{AB} \Phi^A \Phi^B)\), e.g., \(c (\eta_{AB} \Phi^A \Phi^B - \lambda)^2\)
Born–Infeld version:

\[ L = \sqrt{\left| \det \left[ \alpha g_{\mu\nu} + \beta \eta_{AB} \frac{\partial \Phi^A}{\partial x^r} \frac{\partial \Phi^B}{\partial x^y} + \gamma \eta_{AB} \Phi^A \Phi^B g_{\mu\nu} \right] } \]

fixed \( g_{\mu\nu} \) - nongeometric feature; violation of the general covariance.

From now on: generally covariant B–I schemes:

\( M \) - amorphous; no fixed metric \( g \)

\( (W, \eta) \) - target space endowed with the geometry

\[ g[\Phi] := \Phi^* \eta \, , \quad g[\Phi]_{\mu\nu} := \eta_{AB} \frac{\partial \Phi^A}{\partial x^r} \frac{\partial \Phi^B}{\partial x^y} \]
\[ L[\Phi] = \sqrt{|\det[g(\Phi)_{\mu\nu}]|} = \sqrt{|\det[\eta_{AB} \Phi^A_{,\mu} \Phi^B_{,\nu}]|} \]

\( L_{\mu\nu} \) - quadratic in derivatives.

Minimizing the \( n \)-dimensional \( \eta \)-volume of \( \Phi(\Omega) \), keeping the \( (n-1) \)-dimensional boundary \( \Phi(\partial\Omega) = \partial \Phi(\Omega) \) fixed.
Minimal surfaces, soap films. If $N \leq n$ - trivial.

Invariant under $\text{Diff} M \times \text{Diff}(W, \eta)$.

Without $\text{Diff}(W, \eta)$-invariance - a wider class:

\[ L[\Phi] = f(\Phi) \sqrt{|\det[g[\Phi]_{\mu\nu}]|} = \sqrt{|k(\Phi) \det[g[\Phi]_{\mu\nu}]|} \]

$f, k$ - a "potential" term.

If $W$-complex-linear, and $\eta$-hermitian, then:

\[ L[\Phi] = \sqrt{|\det[g[\Phi]_{\mu\nu}]|} = \sqrt{|\det[\eta_{ab} \Phi^a_\mu \Phi^b_\nu]|} \]

$g[\Phi]_{\mu\nu}$ - $\Phi$-dependent hermitian metric on $M$. 
If \( W \)-linear, \( \eta \)-constant, \( (W, \eta) \)-pseudo-Euclidean, the natural class of scalars is given by:

\[
 f[\Phi] = F(\|\Phi\|^2) = F(\eta_{AB}\Phi^A\Phi^B).
\]

\( L \)-invariant under \( \text{Diff} M \times O(W, \eta) \). Other representation:

\[
 L[\Phi] = \sqrt{|\det[T_{\mu\nu}]|} = \sqrt{|\det[\omega \eta_{AB}\Phi^A,\mu\Phi^B,\nu + \kappa \lambda_\mu \lambda_\nu]|},
\]

\( \omega, \kappa \)-functions of \( \|\Phi\|^2 \), \( \lambda_\mu = \frac{1}{2}(\|\Phi\|^2)_{,\mu} = \eta_{AB}\Phi^A\Phi^B,\mu \).
No absolute conceptual gap between Lagrangians:

\[ L = \frac{1}{2} \eta_{\alpha\beta} \Phi^A_{\alpha\beta} \Phi^B_{\alpha\beta} \sqrt{|g|} \quad \text{and} \quad L_{\text{BI}} = \sqrt{\det[\eta_{\alpha\beta} \Phi^A_{\alpha\beta} \Phi^B_{\alpha\beta}]} \]

(fixed \( g \))

Namely, let us take:

\[ \mathcal{L}[\Phi, G] = \frac{1}{2} \eta_{\alpha\beta} \Phi^A_{\alpha\beta} \Phi^B_{\alpha\beta} G^{\mu\nu} \sqrt{|G|} + C \sqrt{|G|} \quad \text{, C - constant,} \]

\( \Phi, G \) - dynamical on the equal footing. Generally-covariant. (Polyakov approach).

\(~Jf \quad n > 2, \quad G_{\mu\nu} = \frac{2-n}{2C} g[\Phi]_{\mu\nu}, \text{ and } \Phi \text{ satisfies } L_{\text{BI}} \text{-equations.} \)

\( Jf \quad C = \frac{2-n}{2}, \quad G_{\mu\nu} = g[\Phi]_{\mu\nu} \)

\(~Jf \quad n = 2, \quad C = 0, \quad G_{\mu\nu} = \lambda g[\Phi]_{\mu\nu}, \lambda \text{-arbitrary function,} \)

and \( \Phi \) again satisfies \( L_{\text{BI}} \text{-equations.} \)
Minimal surfaces, general covariance, and scalar B-I-models

\[ L = \sqrt{\det[\eta_{AB} \Phi^A,_{\mu} \Phi^B,_{\nu}]} \quad , \quad \eta_{AB} = \text{const}. \]

\[ g^{\mu} \nabla_{\mu} \nabla_{\nu} \Phi^a = 0 \quad , \quad a = 1, \ldots, N \quad \text{``} \text{d'Alembert} \text{''} \]

\[ g^{[\phi]} \nabla - \text{covariant differentiation with respect to the } g^{[\phi]}-\text{Levi-Civita} \]

Written down:

\[ g^{\mu \nu} \Phi^a,_{\mu \nu} + \Phi^a,_{\nu} \left( \frac{1}{2} g^{\mu \nu} g^{\alpha \beta} - g^{\mu \alpha} g^{\nu \beta} \right) g_{\alpha \beta, \mu} = 0 \]

The mean curvature of \( \Phi(M) \) in \((W, \eta)\) vanishes.
General covariance, one needs coordinate conditions, e.g.,

\[
\left(\frac{1}{2} g^{\mu\nu} g_{\alpha\beta} - g^\mu g_{\nu}\right) g_{\alpha\beta,\mu} = 0
\]

\[(\ast)\]

thus:

\[
g^{\mu\nu} \Phi^\alpha,_{\mu\nu} = 0 \quad , \quad \alpha = 1, \ldots, N
\]

The simplest gauge for eliminating superfluous variables:

\[
\Phi^g = x^g, \quad g = 1, \ldots, m \]

\[(\ast)\)-holds automatically

\[
\Phi^\sigma, \quad \sigma = m+1, \ldots, N \quad - \text{true degrees of freedom.}
\]

If \(\eta_{\mu\sigma} = 0\), \(\mu = 1, \ldots, m\), \(\sigma = m+1, \ldots, N\) - block structure, then:

\[
g_{\mu\nu} = \eta_{\mu\nu} + \eta_{\nu s} \Phi^s,_{\mu} \Phi^s,_{\nu}
\]

The true dynamics:

\[
g^{\mu\nu} \Phi^\sigma,_{\mu\nu} = 0 \quad , \quad \sigma = m+1, \ldots, N
\]

Exactly equivalent to the vanishing of the mean curvature.
Effective Lagrangian for $\Phi^r$, $r = m+1, \ldots, N$:

$$
L_{\text{eff}} = \sqrt{\left| \det \left[ \eta_{\mu\nu} + \eta_{rs} \Phi^r, \mu \Phi^s, \nu \right] \right|}
$$

"Traditional" B-I form with the fixed spatio-temporal metric $\eta_{\mu\nu}$

Without the block structure:

$$
L_{\text{eff}} = \sqrt{\left| \det \left[ \eta_{\mu\nu} + 2 \eta_{\tau(\mu} \Phi^{r, \nu)} + \eta_{rs} \Phi^r, \mu \Phi^s, \nu \right] \right|}
$$

"Vacuum" solutions - affine injections:

$$
\Phi^r = C^r_\mu x^\mu + C^r
$$

$$
g_{\mu\nu} = \eta_{\mu\nu} + \eta_{rs} C^r_\mu C^s_\nu
$$

$C^r_\mu$, $C^r$ - constants. $C^r_\mu = 0$ if well-behaving at infinity.

$$
\Phi^r = C^r
$$

$g_{\mu\nu} = \eta_{\mu\nu}$
Small corrections $\varphi^r$ to the vacuum background, Jacobi fields:

$$\Phi^r = C^r + \varphi^r, \quad \eta^{\mu\nu} \varphi^r_{,\mu\nu} = 0.$$ 

Ruled by the quadratic background Lagrangian:

$$\frac{1}{2} \eta_{rs} \Phi^r_{,s} \Phi^s_{,r} \eta^{\mu\nu} \sqrt{\text{det} [\eta_{\mu\nu}]}.$$ 

Special example: scalar Born-Infeld electrodynamics:

$N=5, \ m=4, \ [\eta_{\alpha\beta}] = \text{diag} (\eta, 1, -1, -1, -1), \ \eta > 0.$

$$\text{Leff} = \sqrt{\text{det} [\eta_{\mu\nu} + \eta \Phi_{,\mu} \Phi_{,\nu}]}.$$ 

Spherically-symmetric, stationary solutions:

$$\Phi (\sigma) = \pm \sqrt{\frac{C}{\eta}} \int_{0}^{\sigma} \frac{dx}{\sqrt{C + x^4}}, \quad C > 0 - \text{integration constant}.$$
Conclusion: traditional Born-Infeld schemes are fixed-gauge-descriptions of generally-covariant models with higher-dimensional target spaces.

Special examples:
- $N=3$, $m=2$ - soap films, rubber films
- $N$-arbitrary, $m=1$ - geodesics
- $N=4$, $\eta$-Minkowskian, $m=1$, $L \approx \sqrt{1 - \frac{v^2}{c^2}}$ - relativistic material point
- $N=4$, $\eta$-Minkowskian, $m=2$ - strings
- $N$-arbitrary, $\eta$-Riemannian, $m=1$, $f = \sqrt{2(E-V)}$ - Jacobi-Maupertuis variational principle with the potential $V$.
Is it possible to remove a fixed geometry also from the linear target space? For scalar field multiplets - NO!

Let us consider multiplets of covectors. Scalars $\Phi^a$ gave rise to local injections $E^K \mu = \Phi^K_\mu$. One can start from non-holonomic fields of such injections, i.e., from $W$-valued differential one-forms $E$.

\[
\begin{align*}
\text{dim } W &= N \\
\text{dim } M &= m
\end{align*}
\]

\[
g(E)_{\mu\nu} = \eta_{KL} E^K_\mu E^L_\nu \\
g(E)_x = E^*_x - \eta
\]

\[
e_x \in L(T_x M, W)
\]
\[ E'_{K} := \gamma_{KL} E_{L} \ , \ g^{\mu \nu} = \delta_{\mu \nu} \]

\[ E'_{K} E^{K}_{\nu} = \delta_{\nu}^{\nu} - \text{left \ inverse} \]

\[ F^{K} = d E^{K}, \quad F^{K}_{\mu \nu} = \partial_{\mu} E^{K}_{\nu} - \partial_{\nu} E^{K}_{\mu} \]

\[ S^{a}_{\mu \nu} := E^{a}_{K} F^{K}_{\mu \nu} \]

*The simplest Lagrangian:*

\[ L = \gamma_{KL} F^{K}_{\mu \nu} F^{L}_{\chi \alpha} g^{\mu \chi} g^{\nu \alpha} \sqrt{|g|} + c \sqrt{|g|} \]

*Quadratic in derivatives, invariant under \text{Diff} M \times O(W, \gamma)*

*Non-quadratic in \text{E itself} (g_{\alpha \beta}, \sqrt{|g|})*

*Quasilinear field equations.*
Another possibility:

\[ A g_{\alpha\gamma} g^{\beta\gamma} S^\alpha_{\beta\gamma} S^\alpha_{\mu\nu} \sqrt{|g|} + B g^{\mu\nu} S^\alpha_{\beta\gamma} S^\beta_{\alpha\nu} \sqrt{|g|} + C g^{\mu\nu} S^\alpha_{\beta\mu} S^\beta_{\beta\nu} \sqrt{|g|} \]

The same invariance under \( \text{Diff} M \times O(W,\gamma) \), although capitals hidden.

Born-Infeld schemes:

\[ \sqrt{|\det [C g_{\mu\nu} + \gamma_{KL} F_{\mu\kappa} F_{\nu\lambda} g^{\kappa\lambda}]|} \quad \text{, etc.} \]

invariant under \( \text{Diff} M \times O(W,\gamma) \)

\[ \sqrt{|\det [S^\alpha_{\beta\mu} S^\beta_{\alpha\nu}]|} \]

Killing structure of \( S^\alpha_{\beta\mu} S^\beta_{\alpha\nu} \).

\( O(W,\gamma) \) still coded in this Lagrangian (although capital indices and \( \gamma \) are hidden).
The special case: \( N = m \); E-field of linear frames (if \( \det[E^\mu_\nu] \neq 0 \)). Then the left inverse is just the \( \eta \)-independent contravariant inverse frame. More generally:

\[
L = \sqrt{\det \left[ A S^\alpha_\beta S^\beta_\alpha + B S^\alpha_\alpha S^\beta_\nu + C S^\alpha_\alpha S^\beta_\mu \right]}
\]

\[
\mathcal{L}_{\mu} = A S^\alpha_\beta S^\beta_\alpha + B S^\alpha_\alpha S^\beta_\nu + C S^\alpha_\alpha S^\beta_\mu
\]

\( GL(W) \approx GL(m, \mathbb{R}) \)-invariant.

\( S \)-torsion of the E-teleparallelism connection
- special solutions - Lie-group-spaces (E_k's span a Lie algebra under the Lie bracket operation), or appropriately deformed Lie-group-spaces
- correspondence with GR (with cosmological constant)

With matter, e.g.,

\[
L = \sqrt{\det \left[ S^\alpha_{\beta \kappa} S^\beta_{\alpha \nu} + \lambda \bar{\psi},_r \psi,^r,_{\nu} \right]} \]

Special reference solutions: E- (deformed) Lie group space, \(W\) - combined of matrix elements of a given irreducible representation of the mentioned group.
Heisenberg ideas about fundamental spinors and $B$-$T$-models

$\Psi: \mathcal{M} \to \mathbb{C}^4$

$(\mathbb{C}^4, \mathcal{G})$, \hspace{1cm} $\mathcal{G} = \text{diag}(++--)$ – hermitian form of the neutral signature

$$L = \int \sqrt{\left| \det \left[ \mathcal{G}_{\pi s} \bar{\Psi}^\mu \Psi^\nu \right] \right|}$$

$f$ – “potential” depending on $\mathcal{G}_{\pi s} \bar{\Psi}^\mu \Psi^\nu$

(e.g. const $- \mathcal{G}_{\pi s} \bar{\Psi}^\mu \Psi^\nu$)

$g[\Psi]_{\mu \nu} = \mathcal{G}_{\pi s} \bar{\Psi}^\mu \Psi^\nu$, assumed normal-hyperbolic.

Generally covariant and $U(2,2)$-globally invariant.

Small vibrations about affine background – ruled by Klein-Gordon Lagrangian.
Local $U(2,2)$-invariance - gauge field $A_{\mu}^{\nu}$ with values in the Lie algebra $U(2,2)$.

$$g_{\nu}^{\mu} = G_{\alpha\beta} D_\mu \bar{\psi}^\alpha D_\nu \psi^\beta$$

$$D_\mu \psi = \partial_\mu \psi + g A_\mu \psi + \frac{\mu - \gamma}{4} Tr A_\mu \psi$$

$$L = (f + \pi \text{Tr}(F_{\mu\nu} F_{\alpha\beta}) g^{\mu\alpha} g^{\nu\beta}) \sqrt{\det [g_{\nu}^{\mu}]}$$

where:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu].$$

Another possibility:

$$L = \sqrt{\det [k g_{\nu}^{\mu} + \pi \text{Tr}(F_{\mu\nu} F_{\alpha\beta}) g^{\mu\alpha}]},$$

$k$ - function of $G_{\alpha\beta} \bar{\psi}^\alpha \psi^\beta$. 