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On Multicomponent Derivative Nonlinear Schrödinger Equation Related to Symmetric Spaces

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1. Introduction

Derivative nonlinear Schrödinger equation (DNLS) has the form:

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0,$$

where $q(x,t)$ is a smooth complex-valued function. DNLS describes the propagation of circular polarized nonlinear Alfvén waves in plasma.

DNLS is S-integrable [Kaup-Newell, 1977], i.e. it possesses a quadratic bundle Lax pair:

$$\begin{aligned} L(\lambda) &:= \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2\sigma_3, \\ A(\lambda) &:= \mathrm{i}\partial_t + \sum_{k=1}^{3} A_k(x,t)\lambda^k - 2\lambda^4\sigma_3, \end{aligned}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ q^*(x,t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Purpose of the talk: Study of certain examples of multicomponent generazitations of DNLS related to Hermitian symmetric spaces.

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2. Preliminaries

- Multicomponent DNLS equation related to **A.III** symmetric space

  Our main object of study is:

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + \frac{2\mathrm{i}}{n+1}\left(\left(\mathbf{q}^T\mathbf{q}^*\right)\mathbf{q}\right)_x = 0,$$

  where $\mathbf{q} : \mathbb{R}^2 \to \mathbb{C}^n$ is an infinitely smooth function. It is also assumed that $\mathbf{q}$ obeys zero boundary conditions, i.e.

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{0}.$$

- Lax representation and connection with Hermitian symmetric spaces

$$
\begin{aligned}
L(\lambda) &:= \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J, \\
A(\lambda) &:= \mathrm{i}\partial_t + \sum_{k=1}^{4} \lambda^k A_k(x,t).
\end{aligned}
$$

All coefficients above are Hermitian traceless $(n+1)\times(n+1)$ matrices. Moreover, the following $\mathbb{Z}_2$ reduction is imposed on the Lax pair:

$$
\begin{gathered}
\mathbf{C}L(-\lambda)\mathbf{C} = L(\lambda), \\
\mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda),
\end{gathered}
$$

where $\mathbf{C} = \mathrm{diag}\,(1, -1\ldots, -1)$. Due to the form of $\mathbf{C}$ the potential Q has the block structure:

$$
Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^T(x,t) \\ \mathbf{q}^*(x,t) & 0 \end{pmatrix}.
$$

while J is block diagonal. More particularly, we pick it up in the form $J = \mathrm{diag}\,(n, -1, \ldots, -1)$.

The matrix $\mathbf{C}$ represents action of Cartan's involutive automorphism to define $SU(n+1)/S(U(1)\times U(n))$ symmetric space of the type **A.III**. It induces a $\mathbb{Z}_2$ grading in $\mathfrak{sl}(n+1)$ as follows

$$\mathfrak{sl}(n+1) = \mathfrak{sl}^0(n+1) + \mathfrak{sl}^1(n+1), \quad [\mathfrak{sl}^{\sigma}(n+1), \mathfrak{sl}^{\sigma'}(n+1)] = \mathfrak{sl}^{\sigma+\sigma'}(n+1),$$

where

$$\mathfrak{sl}^{\sigma}(n+1) := \{X \in \mathfrak{sl}(n+1) | \mathbf{C}X\mathbf{C}^{-1} = (-1)^{\sigma}X\}.$$

It is easy to see that Q as well as A_1 and A_3 belong to $\mathfrak{sl}^1(n+1)$ while J, A_2 and A_4 belong to $\mathfrak{sl}^0(n+1)$. The subspace $\mathfrak{sl}^0(n+1)$ coincides with the centralizer of J.

- Direct scattering problem

  In order to formulate the direct scattering theory one introduces auxilary linear problem:

$$L(\lambda)\psi(x,t,\lambda) = \mathrm{i}\partial_x \psi(x,t,\lambda) + \lambda(Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0.$$

  It is evident that $\det \psi = 1$. Since $[L, A] = 0$ any fundamental solution satisfies as well

$$A(\lambda)\psi = \mathrm{i}\partial_t \psi + \sum_{k=1}^{4} \lambda^k A_k \psi = \psi f(\lambda),$$

  where

$$f(\lambda) = \lim_{x\to\pm\infty} \sum_{k=1}^{4} \lambda^k A_k(x,t) = -(n+1)\lambda^4 J.$$

  is called dispersion law. It is a fundamental property of any soliton equation.

A special case of solutions are Jost solutions defined as follows:

$$\lim_{x \to \pm\infty} \psi_{\pm}(x, t, \lambda) \mathrm{e}^{\mathrm{i}\lambda^2 J x} = \mathbb{1}.$$

The Jost solutions are defined only on the real and imaginery axes in the λ-plane (continuous spectrum of $L(\lambda)$). The transition matrix

$$\psi_{-}(x, t, \lambda) = \psi_{+}(x, t, \lambda) T(t, \lambda)$$

is called scattering matrix. Its time evolution is given by:

$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0 \quad \Rightarrow \quad T(t, \lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t} T(0, \lambda) \mathrm{e}^{-\mathrm{i}f(\lambda)t}.$$

- Fundamental analytic solutions There exist two fundamental solutions $\chi^{+}(x, \lambda)$ and $\chi^{-}(x, \lambda)$ to be analytic in the upper and lower half-plane of the λ^2-plane respectively. They can be constructed from Jost solutions through the formulae:

$$\chi^{\pm}(x, \lambda) = \psi_{-}(x, \lambda) S^{\pm}(\lambda) = \psi_{+}(x, \lambda) T^{\mp}(\lambda) D^{\pm}(\lambda).$$

The matrices $S^{\pm}(\lambda)$, $T^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ are involved in the generalized Gauss decomposition

$$T(\lambda) = T^{\mp}(\lambda)D^{\pm}(\lambda)(S^{\pm}(\lambda))^{-1}.$$

As a simple consequence of their construction we see that

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G(\lambda)$$

for some sewing function $G(\lambda) = (S^{-}(\lambda))^{-1}S^{+}(\lambda)$.

- Reduction conditions on the Jost solutions, the scattering matrix and fundamental analytic solutions

$$\begin{array}{rcl} \left[\psi_{\pm}^{\dagger}(x,\lambda^*)\right]^{-1} & = & \psi_{\pm}(x,\lambda), \qquad \left[T^{\dagger}(\lambda^*)\right]^{-1} = T(\lambda), \\ \mathbf{C}\psi_{\pm}(x,-\lambda)\mathbf{C} & = & \psi_{\pm}(x,\lambda), \qquad \mathbf{C}T(-\lambda)\mathbf{C} = T(\lambda), \\ (\chi^{+})^{\dagger}(x,\lambda^*) & = & [\chi^{-}(x,\lambda)]^{-1}, \qquad \mathbf{C}\chi^{+}(x,-\lambda)\mathbf{C} = \chi^{-}(x,\lambda). \end{array}$$

3. Dressing method and special solutions

- Dressing method

  Concept of the dressing method:

$$Q_0 \to L_0 \to \psi_0 \to \psi_1 \to Q_1.$$

Realization: let ψ_0 be a fundamental solution of

$$L_0\psi_0 = \mathrm{i}\partial_x\psi_0 + \lambda(Q_0 - \lambda J)\psi_0 = 0$$

where

$$Q_0(x) = \begin{pmatrix} 0 & \mathbf{q}_0(x) \\ \mathbf{q}_0^*(x) & 0 \end{pmatrix}, \qquad J = \mathrm{diag}\,(n, -1, \ldots, -1).$$

for some vector $\mathbf{q}_0^T = (q_0^1, \ldots, q_0^n)$ assumed to be known. Now construct another function $\psi_1(x,\lambda) := g(x,\lambda)\psi_0(x,\lambda)$ and assume it satisfies the linear system

$$L_1\psi_1 = \mathrm{i}\partial_x\psi_1 + \lambda(Q_1 - \lambda J)\psi_1 = 0$$

for some potential

$$Q_1(x) := \begin{pmatrix} 0 & \mathbf{q}_1(x) \\ \mathbf{q}_1^*(x) & 0 \end{pmatrix}.$$

to be found. Therefore the dressing factor g satisfies:

$$\mathrm{i}\partial_x g + \lambda Q_1\, g - \lambda g Q_0 - \lambda^2 [J, g] = 0.$$

The $\mathbb{Z}_2$ reductions imposed on the Lax pair implies that g is obliged to fulfill similar set of symmetry conditions:

$$\begin{aligned} \left[g^\dagger(x, \lambda^*)\right]^{-1} &= g(x, \lambda), \\ \mathbf{C} g(x, -\lambda)\mathbf{C} &= g(x, \lambda). \end{aligned}$$

We pick up the dressing factor in the form:

$$g(x, \lambda) = \mathbb{1} + \frac{\lambda B(x)}{\mu(\lambda - \mu)} + \frac{\lambda \mathbf{C} B(x) \mathbf{C}}{\mu(\lambda + \mu)}, \quad \operatorname{Re}\mu_k \neq 0,\ \operatorname{Im}\mu_k \neq 0.$$

The inverse of the dressing factor reads

$$[g(x,\lambda)]^{-1} = \mathbb{1} + \frac{\lambda B^\dagger(x)}{\mu^*(\lambda - \mu^*)} + \frac{\lambda \mathbf{C} B^\dagger(x)\mathbf{C}}{\mu^*(\lambda + \mu^*)}.$$

There exists the following connection between Q_1 and Q_0

$$\lambda Q_1 = -\mathrm{i}\partial_x g g^{-1} + \lambda g Q_0 g^{-1} + \lambda^2 [J, g] g^{-1}.$$

After dividing by λ and taking $|\lambda| \to \infty$ we obtain

$$Q_1 = A Q_0 A^\dagger + [J, B - \mathbf{C}B\mathbf{C}]A^\dagger,$$

where

$$A = \mathbb{1} + \frac{1}{\mu}(B + \mathbf{C}B\mathbf{C}).$$

From the obvious identity $gg^{-1} = \mathbb{1}$ it follows that the residue B satisfies:

$$B\left(\mathbb{1} + \frac{\mu B^\dagger}{\mu^*(\mu-\mu^*)} + \frac{\mu \mathbf{C} B^\dagger \mathbf{C}}{\mu^*(\mu+\mu^*)}\right) = 0.$$

$B(x,t)$ is a degenerate matrix. Therefore we have $B = XF^T$ for some $(n+1)\times k$ rectangular matrices $X(x)$ and $F(x)$. Then the algebraic relation obtains the form

$$F^* = \frac{\mu^*}{\mu}\left(\frac{F^T F^*}{\mu-\mu^*} - \frac{F^T \mathbf{C} F^*}{\mu+\mu^*}\mathbf{C}\right) X.$$

It can be solved easily to give

$$X = \frac{\mu}{\mu^*}\left(\frac{F^T F^*}{\mu-\mu^*} - \frac{F^T \mathbf{C} F^*}{\mu+\mu^*}\mathbf{C}\right)^{-1} F^*.$$

Thus we have expressed X through F. In order to find the latter we consider the differential equation for g. After calculating the residue at $\lambda = \mu$ we obtain

$$\mathrm{i}\partial_x F^T - \mu F^T(Q_0 - \mu J) = 0 \quad \Rightarrow \quad F^T(x) = F_0^T[\psi_0(x,\mu)]^{-1}.$$

What remains is to recover the time evolution. For this to be done one must analyse some properties of the second Lax operator $A(\lambda)$. Any fundamental solution of the bare linear problem also satisfies:

$$\mathrm{i}\partial_t\psi_0 + \sum_k \lambda^k A_k^{(0)}\psi_0 = \psi_0 f(\lambda)$$

while the dressed fundamental solution solves

$$\mathrm{i}\partial_t\psi_1 + \sum_k \lambda^k A_k^{(1)}\psi_1 = \psi_1 f(\lambda).$$

As a result the dressing factor satisfies:

$$\mathrm{i}\partial_t g + \sum_{k=1}^{2N} \lambda^k A_k^{(1)} g - g\sum_{k=1}^{2N} \lambda^k A_k^{(0)} = 0.$$

Detailed analysis shows that

$$\mathrm{i}\partial_t F^T - F^T\sum_{k=1}^{2N} \mu^k A_k = 0.$$

Therefore we have

$$\mathrm{i}\partial_t F_0^T - F_0^T f(\mu) = 0.$$

Thus we are able to propose a simple rule to derive the time dependence of potential, namely:

$$F_0^T \longrightarrow F_0^T \mathrm{e}^{-\mathrm{i}f(\mu)t}.$$

For the DNLS equation $f(\lambda) = -(n+1)\lambda^4 J$.

- Soliton solutions

  In the soliton sector $Q_0 \equiv 0$. Therefore we have:

$$\psi_0(x,t,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda^2 Jx}.$$

We shall resrict ourselves with the case when the rank of B is 1. Then the column-vector F is given by

$$F = \begin{pmatrix} \mathrm{e}^{n\mathrm{i}\mu^2 x} F_{0,1} \\ \mathrm{e}^{-\mathrm{i}\mu^2 x} F_{0,2} \\ \vdots \\ \mathrm{e}^{-\mathrm{i}\mu^2 x} F_{0,n+1} \end{pmatrix}.$$

It proves to be convenient to adopt polar parametrization of the pole, i.e. $\mu = \rho \exp(\mathrm{i}\varphi)$. Then the potential acquires the form:

$$
\begin{aligned}
q_1^{j-1}(x) &= (Q_1)_{1\,j}(x) = 2\mathrm{i}(n+1)\sum_{l=2}^{n+1}\frac{\rho\sin(2\varphi)\mathrm{e}^{-\mathrm{i}\sigma_l(x)}\mathrm{e}^{\theta_l(x)}}{\mathrm{e}^{-2\mathrm{i}\varphi}+\sum_{p=2}^{n+1}\mathrm{e}^{2\theta_p(x)}}\times \\
&\quad \left(\delta_{jl}-2\mathrm{i}\sin(2\varphi)\frac{\mathrm{e}^{\theta_j(x)+\theta_l(x)}\mathrm{e}^{\mathrm{i}(\delta_j-\delta_l-2\varphi)}}{\mathrm{e}^{-2\mathrm{i}\varphi}+\sum_{p=2}^{n+1}\mathrm{e}^{2\theta_p(x)}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_p(x) &= (n+1)\rho^2\sin(2\varphi)x-\xi_{0,p}, \\
\sigma_p(x) &= (n+1)\cos(2\varphi)x+\delta_1-\delta_p-\varphi. \\
\xi_{0,p} &= \ln|F_{0,1}/F_{0,p}|, \qquad \delta_1=\arg F_{0,1}, \qquad \delta_p=\arg F_{0,p}.
\end{aligned}
$$

In order to recover the time dependence one uses the following rule:

$$
\begin{aligned}
\xi_{0,p} &\to \xi_{0,p}-2(n+1)\rho^4\sin(4\varphi)t, \\
\delta_1 &\to \delta_1+2n\rho^4\cos(4\varphi)t, \\
\delta_p &\to \delta_p-2\rho^4\cos(4\varphi)t.
\end{aligned}
$$

Remark 1 *In the simplest case when $n = 1$ one can derive the soliton of the DNLS equation [Kaup-Newell, 1977]. Indeed, one should use the following dressing factor*

$$g(x,t,\lambda) = \mathbb{1} + \frac{\lambda B(x,t)}{\mu(\lambda-\mu)} + \frac{\lambda \sigma_3 B(x,t) \sigma_3}{\mu(\lambda+\mu)}.$$

As a result we reproduce the Kaup-Newel soliton

$$q_1 = 4\mathrm{i} \frac{\rho \sin(2\varphi) \mathrm{e}^{-2\mathrm{i}(\rho^2 \cos(2\varphi)x + \delta_0)} \mathrm{e}^{2\rho^2 \sin(2\varphi)x - \xi_0} \left[\mathrm{e}^{2\mathrm{i}\varphi} + \mathrm{e}^{2(2\rho^2 \sin(2\varphi)x - \xi_0)} \right]}{\left[\mathrm{e}^{-2\mathrm{i}\varphi} + \mathrm{e}^{2(2\rho^2 \sin(2\varphi)x - \xi_0)} \right]^2},$$

where $\mu = \rho \exp(\mathrm{i}\varphi)$ and

$$\delta_0 = \frac{\delta_1 - \delta_2 + 3\varphi}{2}, \qquad \xi_0 = \ln |F_{0,1}/F_{0,2}|$$

for DNLS equation. The time dependence is recovered by using the rule:

$$\xi_0 \to \xi_0 - 4\rho^4 \sin(4\varphi)t, \qquad \delta_0 \to \delta_0 + 2\rho^4 \cos(4\varphi)t.$$

- Multisoliton solutions
  - One can apply the dressing procedure to build a sequence of exact solutions to the system:

$$Q_0 \xrightarrow{g_0} Q_1 \xrightarrow{g_1} Q_2 \rightarrow \dots \xrightarrow{g_{m-1}} Q_m,$$

    where g_k is constructed by using the fundamental solution

$$\psi_k(x,t,\lambda) = \overleftarrow{\prod_{l=0,\dots,k-1}} g_l(x,t,\lambda)\psi_0(x,t,\lambda).$$

  - Multiple poles dressing factor
    In this case one uses the following factor:

$$g(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{N} \frac{\lambda}{\mu_k}\left(\frac{B_k(x,t)}{\lambda - \mu_k} + \frac{\mathbf{C}B_k(x,t)\mathbf{C}}{\lambda + \mu_k}\right),$$

    where $\mu \in \mathbb{C}$, $\operatorname{Re}\mu_k \neq 0$, $\operatorname{Im}\mu_k \neq 0$. In order to determine B_k one analyse the identity $gg^{-1} = \mathbb{1}$. After introducing the factorization $B_k = X_k F_k^T$ it reduces to a linear system for X_k, namely:

$$F_k^* = \sum_{l=1}^{m} \frac{\mu_k^*}{\mu_l} \left(X_l \frac{F_l^T F_k^*}{\mu_l - \mu_k^*} - \mathbf{C} X_l \frac{F_l | \mathbf{C} F_k^*}{\mu_l + \mu_k^*} \right).$$

Next one determines the vectors F_k from the p.d.e.

$$\mathrm{i}\partial_x g + \lambda Q_1 \, g - \lambda g Q_0 - \lambda^2 [J, g] = 0.$$

The result reads:

$$F_k^T(x,t) = F_{k,0}^T [\psi_0(x,t,\mu_k)]^{-1}.$$

Thus the dressing factor is determined if one knows the seed solution $\psi_0(x,t,\lambda)$. The multisoliton solution itself can be derived through the following formula

$$Q_1 = \sum_{k=1}^{m} [J, B_k - \mathbf{C} B_k \mathbf{C}] A^\dagger,$$

where

$$A = \mathbb{1} + \sum_{k=1}^{m} \frac{1}{\mu_k} (B_k + \mathbf{C} B_k \mathbf{C}).$$

In order to recover the time evolution we use the rule:

$$F_{k,0}^T \to F_{k,0}^T \mathrm{e}^{-\mathrm{i} f(\mu_k) t}.$$

4. Integrals of Motion

Let us consider the Lax pair

$$\begin{aligned} L(\lambda) \quad &:= \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J, \\ A(\lambda) \quad &:= \mathrm{i}\partial_t + \textstyle\sum_{k=1}^{2N} A_k(x,t)\lambda^k. \end{aligned}$$

In order to derive the integrals of motion we shall apply method of diagonalization of the Lax pair [Drinfel'd and Sokolov, 1985]. For this to be done one uses the following transformation:

$$\mathcal{P}(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{\infty} \frac{p_k(x,t)}{\lambda^k}.$$

To avoid umbiguities we assume that all $p_k \in \mathfrak{sl}^1(n+1)$.

The transformed Lax operators look as follows:

$$\begin{aligned} \mathcal{L} \quad &= \quad \mathcal{P}^{-1}\tilde{L}\mathcal{P} = \mathrm{i}\partial_x - \lambda^2 J + \lambda\mathcal{L}_{-1} + \mathcal{L}_0 + \frac{\mathcal{L}_1}{\lambda} + \cdots, \\ \mathcal{A} \quad &= \quad \mathcal{P}^{-1}\tilde{A}\mathcal{P} = \mathrm{i}\partial_t + \sum_{k=1}^{2N} \lambda^k \mathcal{A}_{-k} + \mathcal{A}_0 + \frac{\mathcal{A}_1}{\lambda} + \cdots, \end{aligned}$$

where all coefficients are block diagonal, i.e. elements of $\mathfrak{sl}^0(n+1)$. The zero curvature representation is written as

$$\partial_t \mathcal{L}_k - \partial_x \mathcal{A}_k + \sum_l^k [\mathcal{L}_l, \mathcal{A}_{k-l}] = 0.$$

Hence the matrix element $(\mathcal{L}_k)_{11}$ as well as the trace of the $n \times n$ block of $\mathcal{L}_k$ are (local) densities of the integrals of motion.

It is evident that equality can be rewritten in the following manner

$$\left(\mathbb{1} + \frac{p_1}{\lambda} + \frac{p_2}{\lambda^2} + \cdots\right)\left(\mathrm{i}\partial_x - \lambda^2 J + \lambda\mathcal{L}_{-1} + \mathcal{L}_0 + \frac{\mathcal{L}_1}{\lambda} + \cdots\right)$$
$$= \left(\mathrm{i}\partial_x + \lambda Q - \lambda^2 J\right)\left(\mathbb{1} + \frac{p_1}{\lambda} + \frac{p_2}{\lambda^2} + \cdots\right).$$

The latter is equivalent to the following set of recurrence relations:

$$
\begin{aligned}
\lambda \quad &: \quad \mathfrak{L}_{-1} - p_1 J = Q - J p_1, \\
\lambda^0 \quad &: \quad \mathfrak{L}_0 + p_1 \mathfrak{L}_{-1} - p_2 J = Q p_1 - J p_2, \\
\lambda^{-1} \quad &: \quad \mathfrak{L}_1 + p_1 \mathfrak{L}_0 + p_2 \mathfrak{L}_{-1} - p_3 J = \mathrm{i} p_{1,x} + Q p_2 - J p_3, \\
\lambda^{-2} \quad &: \quad \mathfrak{L}_2 + p_1 \mathfrak{L}_1 + p_2 \mathfrak{L}_0 + p_3 \mathfrak{L}_{-1} - p_4 J = \mathrm{i} p_{2,x} + Q p_3 - J p_4, \\
&\dots \\
\lambda^{-k} \quad &: \quad \mathfrak{L}_k + \sum_{l=1}^{k} p_l \mathfrak{L}_{k-l} + p_{k+1} \mathfrak{L}_{-1} - p_{k+2} J = \mathrm{i} p_{k,x} + Q p_{k+1} - J p_{k+2}, \\
&\dots
\end{aligned}
$$

After projecting the first recurrence relation into a part in $\mathfrak{sl}^0(n+1)$ and another one in $\mathfrak{sl}^1(n+1)$ we deduce that:

$$
\mathfrak{L}_{-1} = 0, \qquad p_1 = \mathrm{ad}\,_J^{-1} Q = \frac{1}{n+1} \begin{pmatrix} 0 & \mathbf{q}^T \\ -\mathbf{q}^* & 0 \end{pmatrix}.
$$

Similarly, from the second relation we get

$$\mathfrak{L}_0 = Qp_1 = \frac{1}{n+1}\begin{pmatrix} -\mathbf{q}^T\mathbf{q}^* & 0 \\ 0 & \mathbf{q}^T\mathbf{q}^T \end{pmatrix}, \qquad p_2 = 0.$$

Thus the first integral density is $I_1 = \mathbf{q}^\dagger\mathbf{q}$.

Theorem 1 *All conserved densities $\mathfrak{L}_k$ corresponding to odd indices vanish.*

Proof: By induction. It is easy to see that p_k vanish whenever k is even. Indeed, after splitting the k-th recurrence relation one is able to express p_k the following recursive formula:

$$p_k = \mathrm{ad}_J^{-1}\left(\mathrm{i}p_{k-2,x} - \sum_{l=1}^{k-2} p_l \mathfrak{L}_{k-2-l}\right).$$

Then the statement of theorem follows immediately from formula:

$$\mathfrak{L}_k = Qp_{k+1}.\square$$

Taking into account all this for the second nonzero integral we have:

$$\mathcal{L}_2 = \frac{\mathrm{i}}{(n+1)^2} \begin{pmatrix} \mathbf{q}^T \mathbf{q}_x^* & \mathbf{0}^T \\ \mathbf{0} & \mathbf{q}^* \mathbf{q}_x^T \end{pmatrix} + \frac{\mathbf{q}^\dagger \mathbf{q}}{(n+1)^3} \begin{pmatrix} \mathbf{q}^\dagger \mathbf{q} & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{q}^* \mathbf{q}^T \end{pmatrix}.$$

Hence as an integral density can be chosen

$$I_2 = H = \mathrm{i}\mathbf{q}^\dagger \mathbf{q}_x - \frac{1}{n+1} (\mathbf{q}^\dagger \mathbf{q})^2.$$

It represents the Hamiltonian H of the multicomponent DNLS equation if Poisson bracket is defined as:

$$\{F, G\} := \int_{-\infty}^{\infty} \mathrm{d}\, y \operatorname{tr} \left(\frac{\delta F}{\delta Q} \partial_x \frac{\delta G}{\delta Q^T} \right).$$

Thus DNLS equation can be written in a Hamiltonian form as follows:

$$q_{k,t} = \partial_x \frac{\delta H}{\delta q_k^*}, \qquad k = 1, \ldots, n.$$

Conclusions

- The direct scattering problem for quadratic bundle related to Hermitian symmetric space has been formulated.
- The soliton solutions have been constructed analytically. For that purpose we have used the dressing technique.
- The first two integrals of motion have been derived explicitly. The second integral represents the Hamiltonian of DNLS equation. A general recursion formula to calculate k-th integral has been obtained.