A NEW POINT OF VIEW ON GENERAL KALUZA-KLEIN THEORIES

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1. HISTORICAL COMMENTS AND MOTIVATION

• **MAXWELL** (1861): Electromagnetism = Electricity + magnetism.

• **GLASHOW, WEINBERG, SALAM** (1961): Electromagnetism + Weak Interactions

GRAVITY?

• **NORDSTRÖM** (1914): Electromagnetism + His Own Gravity Theory.

• **KALUZA** (1919): $5D$ Gravity Theory $\Longrightarrow$ $4D$ Gravity Theory and Electromagnetism (1921), "CYLINDER CONDITION"

• **KLEIN** (1926): "COMPACTIFICATION CONDITION" fifth dimension should be compact $\Longrightarrow$ KALUZA-KLEIN THEORY (KK THEORY).

• (1970): **STRING THEORY AND SUPERGRAVITY.**

• **EINSTEIN-BERGMANN** (1938).

• **WESSON** (1990) **STM THEORY.**

• **RANDALL-SUNDRUM** (1999) **BRANE WORLD THEORY.**

MOTIVATION: A New Approach on General KK Theories. (Remove Both Conditions; Use Nonholonomic Geometry).

2. GENERAL KALUZA-KLEIN SPACE

$$\bar{M} = M \times K, \quad \pi : \bar{M} \longrightarrow M$$

$$\tilde{x}^{\alpha} = \tilde{x}^{\alpha}(x^{\mu}), \quad \tilde{x}^4 = x^4 + h(x^{\mu}), \quad \alpha, \mu \in \{0, 1, 2, 3\} \quad (1)$$

$\bar{g}$ - pseudo-Riemannian metric on $\bar{M}$, given locally by

$$g_{\alpha\beta}(x^i)dx^{\alpha}dx^{\beta} + \epsilon\Phi^2(x^i)(dx^4 + A_{\alpha}dx^{\alpha})^2. \quad (2)$$

$V\bar{M}$ - vertical distribution spanned by $\partial/\partial x^4$.

$H\bar{M}$ - horizontal distribution spanned by

$$\frac{\delta}{\delta x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}} - A_{\alpha}\frac{\partial}{\partial x^4}. \quad (3)$$

$$T\bar{M} = H\bar{M} \oplus V\bar{M}.$$

$$H\bar{M} \perp V\bar{M}$$

A_{α}- electromagnetic potentials

$g_{\alpha\beta}(x^i)$ - define a Lorentz metric g on $H\bar{M}$.

$(\bar{M}, \bar{g})$ = **general Kaluza-Klein space.**

(2) is invariant with respect to (1) if and only if:

$$g_{\alpha\beta} = \widetilde{g}_{\mu\nu}\frac{\partial\widetilde{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\widetilde{x}^{\nu}}{\partial x^{\beta}}, \quad \text{and} \quad A_{\alpha} = \widetilde{A}_{\gamma}\frac{\partial\widetilde{x}^{\gamma}}{\partial x^{\alpha}} + \frac{\partial h}{\partial x^{\alpha}}.$$

3. $4D$ TENSOR FIELDS ON $(\bar{M}, \bar{g})$

$T = \left(T^{\alpha_1, \cdots \alpha_p}_{\beta_1 \cdots \beta_q}(x^i)\right)$ is a $4D$ tensor field of type (p, q) if:

$$T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} \frac{\partial \tilde{x}^{\gamma_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \tilde{x}^{\gamma_p}}{\partial x^{\alpha_p}} = \tilde{T}^{\gamma_1 \cdots \gamma_p}_{\mu_1 \cdots \mu_q} \frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \tilde{x}^{\mu_q}}{\partial x^{\beta_q}}.$$

Examples: $g_{\alpha\beta}$, $g^{\alpha\beta}$.

- $4D$ *electromagnetic tensor field*

$$F_{\alpha\beta} = \frac{\delta A_\beta}{\delta x^\alpha} - \frac{\delta A_\alpha}{\delta x^\beta}; \quad \left[\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right] = F_{\alpha\beta} \frac{\partial}{\partial x^4}.$$

- $4D$ *Obstruction Tensor Fields*:

$$D_{\alpha\beta} = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^4}, \quad B_\alpha = \frac{\partial A_\alpha}{\partial x^4}, \quad C_\alpha = \Phi^{-1} \frac{\delta \Phi}{\delta x^\alpha}.$$

$(\bar{M}, \bar{g})$ is the classical KK space $\Longleftrightarrow D = 0,\ B = 0,\ C = 0.$

4. A REMARKABLE LINEAR CONNECTION ON $H\bar{M}$

$$\nabla : \Gamma(T\bar{M}) \times \Gamma(H\bar{M}) \to \Gamma(H\bar{M}); \quad (X, hY) \Longrightarrow \nabla_X hY$$

Torsion Tensor Field: $T : \Gamma(T\bar{M}) \times \Gamma(H\bar{M}) \to \Gamma(H\bar{M})$;

$$T(X, hY) = \nabla_X hY - \nabla_{hY} hX - h[X, hY].$$

Theorem 1. *There exists a unique linear connection* ∇ *on* $H\bar{M}$ *satisfying:*

(i) ∇ *is a metric connection* $(\nabla_X g = 0)$

(ii) *The torsion tensor field of* ∇ *is given by*

$$T(hX, hY) = 0, \quad T(vX, hY) = D(vX, hY),$$

∇ = **Riemannian horizontal connection**

(a) $\nabla_{hX} hY = h\bar{\nabla}_{hX} hY$, (b) $\nabla_{vX} hY = h[vX, hY] + D(vX, hY)$.

Theorem 2. *The Levi-Civita connection* $\bar{\nabla}$ *on* $(\bar{M}, \bar{g})$ *is expressed as follows:*

$$\bar{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = \Gamma_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma} + \left(\tfrac{1}{2} F_{\alpha\beta} - \epsilon \Phi^{-2} D_{\alpha\beta}\right) \frac{\partial}{\partial x^4},$$

$$\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\delta}{\delta x^\alpha} = \left(D_\alpha{}^\gamma + \tfrac{\epsilon\Phi^2}{2} F_\alpha{}^\gamma\right) \frac{\delta}{\delta x^\gamma} + (C_\alpha - B_\alpha) \frac{\partial}{\partial x^4},$$

$$\bar{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^4} = \left(D_\alpha{}^\gamma + \tfrac{\epsilon\Phi^2}{2} F_\alpha{}^\gamma\right) \frac{\delta}{\delta x^\gamma} + C_\alpha \frac{\partial}{\partial x^4},$$

$$\bar{\nabla}_{\frac{\partial}{\partial x^4}} \frac{\partial}{\partial x^4} = \epsilon \Phi^2 (B^\gamma - C^\gamma) \frac{\delta}{\delta x^\gamma} + \Psi \frac{\partial}{\partial x^4},$$

$$\Gamma_\alpha{}^\gamma{}_\beta = \frac{1}{2} g^{\gamma\mu} \left\{ \frac{\delta g_{\mu\alpha}}{\delta x^\beta} + \frac{\delta g_{\mu\beta}}{\delta x^\alpha} - \frac{\delta g_{\alpha\beta}}{\delta x^\mu} \right\}, \quad \Psi = \frac{1}{\Phi} \frac{\partial \Phi}{\partial x^4}.$$

5. $4D$ EQUATIONS OF MOTION IN $(\bar{\mathbf{M}}, \bar{\mathbf{g}})$

$\{\delta/\delta x^\alpha,\ \partial/\partial x^4\}$ - adapted frame field

Theorem 3. *The fully general equations of motion for the STM Theory on $(\bar{M}, \bar{g})$ are expressed as follows*:

$$\begin{aligned}
(a)\quad & \tfrac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha}{}^{\gamma}{}_{\beta}\tfrac{dx^\alpha}{dt}\tfrac{dx^\beta}{dt} = \tfrac{\delta x^4}{\delta t}\Big\{\epsilon\Phi^2(C^\gamma - B^\gamma)\tfrac{\delta x^4}{\delta t} \\
& \qquad - \left(2D_{\alpha}{}^{\gamma} + \epsilon\Phi^2 F_{\alpha}{}^{\gamma}\right)\tfrac{dx^\alpha}{dt}\Big\}, \\
(b)\quad & \tfrac{d}{dt}(\tfrac{\delta x^4}{\delta t}) - \epsilon\Phi^{-2}D_{\alpha\beta}\tfrac{dx^\alpha}{dt}\tfrac{dx^\beta}{dt} \\
& \qquad = \tfrac{\delta x^4}{\delta t}\Big\{\Psi\tfrac{\delta x^4}{\delta t} + (B_\alpha - 2C_\alpha)\tfrac{dx^\alpha}{dt}\Big\}.
\end{aligned} \tag{5}$$

$\bar{C} \subset \bar{M}$, satisfying (5) is called *horizontal motion* if

$$\frac{\delta x^4}{\delta t} = \frac{dx^4}{dt} + A_\alpha\frac{dx^\alpha}{dt} = 0. \tag{6}$$

Corollary 1. *$\bar{C}$ is horizontal motion, if and only if,* (6) *and*

$$(a)\quad \frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha}{}^{\gamma}{}_{\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \quad (b)\quad D_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0. \tag{7}$$

Corollary 2. *Any horizontal motion is autoparallel of the Riemannian horizontal connection.*

$C \subset M$, *h-motion* if it is projection of a horizontal motion $\bar{C} \subset \bar{M}$

Corollary 3. *Any motion from general relativity of the 4D spacetime (M, g) is the projection of some horizontal motions in $(\bar{M}, \bar{g})$.*

$\bar{C} \subset \bar{M}$ is a *c-motion* if it satisfies

$$\frac{dx^4}{dt} + A_\alpha \frac{dx^\alpha}{dt} = c, \quad c \neq 0. \tag{8}$$

Corollary 4 *Any c-motion, in $(\bar{M}, \bar{g})$ is a solution of* (8) *and of the system*

$$\begin{aligned} (a) \quad & \tfrac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\ \beta}^{\ \gamma} \tfrac{dx^\alpha}{dt}\tfrac{dx^\beta}{dt} = c\left\{\epsilon c \Phi^2 (C^\gamma - B^\gamma) \right. \\ & \qquad \left. - \left(2D_\alpha^{\ \gamma} + \epsilon \Phi^2 F_\alpha^{\ \gamma}\right) \tfrac{dx^\alpha}{dt} \right\}, \\ (b) \quad & D_{\alpha\beta} \tfrac{dx^\alpha}{dt}\tfrac{dx^\beta}{dt} = \epsilon c \Phi^2 \left\{(2C_\alpha - B_\alpha)\tfrac{dx^\alpha}{dt} - c\Psi\right\}. \end{aligned} \tag{9}$$

Particular Case: Classical KK space: $D_{\alpha\beta} = 0$, $B_\alpha = 0, \quad \Phi = const.$
Then take $c = -q/m$, $q =$ charge, $\quad m =$ mass of particle.
Then (9) reduces to *Lorentz force equations*

$$\frac{d^2x^\gamma}{dt^2} + \Gamma_{\alpha\ \beta}^{\ \gamma} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \frac{q}{m} F_\alpha^{\ \gamma} \frac{dx^\alpha}{dt}.$$

Remark. Projections of $-q/m$-motions in $\bar{M}$ on M, bring more information that the solutions of Lorentz force equations.

6. FIFTH FORCE FROM FIFTH DIMENSION

$\bar{C}$ geodesic in $(\bar{M}, \bar{g})$ given by: $x^i = x^i(t), \quad t \in [a, b]$.

$$\frac{d}{dt} = \frac{dx^\alpha}{dt}\frac{\delta}{\delta x^\alpha} + \frac{\delta x^4}{\delta t}\frac{\partial}{\partial x^4}.$$

Then we call $U(t) = \frac{dx^\alpha}{dt}\frac{\delta}{\delta x^\alpha}$ the $4D$ *velocity* for STM Theory.

Case 1. $\bar{C}$ is geodesic with spacelike or timelike $4D$ velocity.
$s = 4D$ arc length parameter, that is, $ds^2 = \epsilon^\star g_{\alpha\beta} dx^\alpha dx^\beta$.
Then define the *fifth force* $\mathcal{F}(s)$ along $\bar{C}$ by

$$\mathcal{F}(s) = \nabla_{\frac{d}{ds}} U(s).$$

∇ = Riemannian horizontal connection on $H\bar{M}$.

Similarly, the dual fifth force $\mathcal{F}^\star(s)$ is given by

$$\mathcal{F}^\star(s) = \nabla_{\frac{d}{ds}} U^\star(s).$$

Properties: 1. $\mathcal{F}(s) \perp U(s)$, 2. $\mathcal{F}(s), \mathcal{F}^\star(s)$ are indeed dual to each other.

Theorem 4. *Let $\bar{C}$ be horizontal geodesic of $(\bar{M}, \bar{g})$. Then we have:*

(i) *The fifth force vanishes identically on $(\bar{M}, \bar{g})$.*

(ii) *The $4D$ arc length parameter is an affine parameter on $\bar{C}$.*

Case 2. $\bar{C}$ is geodesic in $(\bar{M}, \bar{g})$ with lightlike $4D$ velocity

$$\mathcal{F}(t) = \nabla_{\frac{d}{dt}} U(t).$$

Theorem 5. *$\bar{C}$ can be only in the following cases:*

(i) *It is horizontal and null geodesic in $(\bar{M}, \bar{g})$.*

(ii) *It is non-null geodesic given by* (5a) *and the equation*

$$\frac{d}{dt}\left(\frac{\delta x^4}{\delta t}\right) = \frac{\delta x^4}{\delta t}\left\{\Psi \frac{\delta x^4}{\delta t} - C_\alpha \frac{dx^\alpha}{dt}\right\}.$$

7. RELATIVISTIC GENERAL KALUZA-KLEIN SPACE

$(\bar{M}, \bar{g})$ satisfying conditions

$$(A) \quad \sum_{(\alpha,\beta,\gamma)} \left\{ F_{\alpha\beta} D_{\gamma}{}^{\mu} \right\} = 0, \quad (B) \quad \sum_{(\alpha,\beta,\gamma)} \left\{ F_{\alpha\beta} R_{\nu}{}^{\mu}{}_{\gamma 4} \right\} = 0.$$

Curvature tensor field R of the Riemannian horizontal connection ∇:

$$R\left(\frac{\delta}{\delta x^{\gamma}}, \frac{\delta}{\delta x^{\beta}}\right)\frac{\delta}{\delta x^{\alpha}} = R_{\alpha}{}^{\mu}{}_{\beta\gamma}\frac{\delta}{\delta x^{\mu}},$$

$$R\left(\frac{\partial}{\partial x^{4}}, \frac{\delta}{\delta x^{\beta}}\right)\frac{\delta}{\delta x^{\alpha}} = R_{\alpha}{}^{\mu}{}_{\beta 4}\frac{\delta}{\delta x^{\mu}},$$

$4D$ *Ricci tensor*: $Ric(hX, hY) = \sum_{\alpha=0}^{3} \epsilon_{\alpha} R(E_{\alpha}, hX, E_{\alpha}, hY)$

$4D$ *scalar curvature*: $\mathbf{R} = \sum_{\alpha=0}^{3} \epsilon_{\alpha} Ric(E_{\alpha}, E_{\alpha}).$

$R_{\alpha\beta} = Ric\left(\frac{\delta}{\delta x^{\beta}}, \frac{\delta}{\delta x^{\alpha}}\right), R_{\alpha\beta} = R_{\alpha}{}^{\mu}{}_{\beta\mu}, \quad \mathbf{R} = g^{\alpha\beta} R_{\alpha\beta}.$

Proved: $2R^{\mu}_{\alpha|\mu} = \mathbf{R}_{|\alpha}, \quad \mathbf{R}_{|\alpha} = \frac{\delta \mathbf{R}}{\delta x^{\alpha}}.$

$4D$ *Einstein gravitational tensor field*: $G_{\alpha\beta} = R_{\alpha\beta} - \frac{\mathbf{R}}{2} g_{\alpha\beta}.$

Theorem 6. *$G_{\alpha\beta}$ is $4D$ symmetric tensor field whose divergence vanishes identically on $\bar{M}$.*

$4D$ *electromagnetic energy-momentum tensor field*:
$E_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - F_{\alpha\mu} F_{\beta}{}^{\mu}.$

8. $4D$ EINSTEIN EQUATIONS IN $(\bar{M}, \bar{g})$

$$\bar{G}_{ij} = 0, \quad i, j \in \{0,1,2,3,4\} \Longleftrightarrow$$

$$(a) \quad G_{\alpha\beta} = -\tfrac{\epsilon\Phi^2}{2} E_{\alpha\beta} + \tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} g_{\alpha\beta} - \tfrac{1}{2} \Big\{ D_{\alpha\mu} F_{\beta}{}^{\mu} + F_{\alpha\mu} D_{\beta}{}^{\mu} \Big\} + \epsilon\Phi^{-2} \left\{ D_{\alpha\beta|4} + D_{\alpha\beta} D_{\mu}{}^{\mu} - \Psi D_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} \left((D_{\mu}{}^{\mu})^2 - D_{\mu\nu} D^{\mu\nu} \right) \right\} + \tfrac{1}{2} \left\{ C_{\alpha|\beta} + C_{\beta|\alpha} - B_{\alpha|\beta} - B_{\beta|\alpha} \right\} + (B_\alpha - C_\alpha)(B_\beta - C_\beta),$$

$$(b) \quad \tfrac{\epsilon\Phi^2}{2} \Big\{ F_{\alpha}{}^{\mu}{}_{|\mu} + 3 F_{\alpha}{}^{\mu} C_\mu - 2 F_{\alpha}{}^{\mu} B_\mu \Big\} = D_{\mu}{}^{\mu}{}_{|\alpha} - D_{\alpha}{}^{\mu}{}_{|\mu} + D_{\alpha}^{\mu} C_\mu - D_{\mu}{}^{\mu} C_\alpha, \tag{10}$$

$$(c) \quad \tfrac{\epsilon\Phi^2}{4} F_{\mu\nu} F^{\mu\nu} + \epsilon\Phi^{-2} \Big\{ \Psi D_{\mu}{}^{\mu} - D_{\mu\nu} D^{\mu\nu} - D_{\mu}{}^{\mu}{}_{|4} \Big\} + B^{\mu}{}_{|\mu} - C^{\mu}{}_{|\mu} - (B^\mu - C^\mu)(B_\mu - C_\mu) = 0.$$

(10a) represents the $4D$ *Einstein equations* in $(\bar{M}, \bar{g})$.

Wesson-Ponce de Leon Theory ($A_\alpha = 0$).

$$G_{\alpha\beta} = \Phi^{-1} \Phi_{\alpha|\beta} - \tfrac{\epsilon}{2} \Phi^{-2} \Big\{ \Phi^{-1} \overset{*}{\Phi} \overset{*}{g}_{\alpha\beta} - \overset{**}{g}_{\alpha\beta} + \overset{*}{g}^{\mu\nu} g_{\mu\alpha} \overset{*}{g}_{\nu\beta} - \tfrac{1}{2} \overset{*}{g}_{\alpha\beta} \, g^{\mu\nu} \overset{*}{g}_{\mu\nu} + \tfrac{1}{4} g_{\alpha\beta} \left(\overset{*}{g}^{\mu\nu} \overset{*}{g}_{\mu\nu} + \left(g^{\mu\nu} \overset{*}{g}_{\mu\nu} \right)^2 \right) \Big\}.$$

$\star$ means usual partial derivative with respect to x^4.

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