Reduction Groups and Darboux Transformations

George Berkeley

University of Leeds

Varna, June 2013

Contents

- Lax Operators
- Reduction Groups
- Darboux Transformations
- Integrable partial difference equations

What is a Lax operator?

- For my purposes I will consider a pair of Lax operators to be differential operators of the form:

$$L = \partial_x - X(\lambda, x, t)$$

$$S = \partial_t - T(\lambda, x, t)$$

where X and T are matrix functions of x ,t and λ, here λ is called the spectral parameter.

- Lax pairs are important in the study of non-linear pdes.
- Suppose we have a non-zero vector Ψ such that

$$L\Psi = \partial_x \Psi - X\Psi = 0$$

$$S\Psi = \partial_t \Psi - T\Psi = 0.$$

- This system is consistent iff $\partial_x \partial_t \Psi = \partial_t \partial_x \Psi$ which is iff

$$X_t - T_x + [X, T] = 0 \text{ (Zero Curvature Condition).}$$

Example

For Lax operators having X,T as

$$X = \begin{pmatrix} -i\lambda & u \\ -1 & i\lambda \end{pmatrix}$$

$$T = \begin{pmatrix} -4i\lambda^3 + 2iu\lambda - u_x & 4u\lambda^2 + 2iu_x\lambda - 2u^2 - u_{xx} \\ -4\lambda^2 + 2u & 4i\lambda^3 - 2iu\lambda + u_x \end{pmatrix},$$

the Zero Curvature condition becomes:

$$X_t - T_x + [X, T] = \begin{pmatrix} 0 & u_t + 6uu_x + u_{xxx} \\ 0 & 0 \end{pmatrix} = 0$$

so holds iff u satisfies the KdV equation.

AKNS Scheme

In fact the previous example is a specialisation of the more general Lax pair with:

$$X = \begin{pmatrix} -i\lambda & u \\ v & i\lambda \end{pmatrix}$$

$$T = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}.$$

- Take $A = \sum_{i=0}^{n} a_i \lambda^i$, $B = \sum_{i=0}^{n} b_i \lambda^i$, $C = \sum_{i=0}^{n} c_i \lambda^i$.
- Set the coefficients of λ to zero in the Zero Curvature condition
- Solve the resulting system to produce Integrable non-linear pdes for different integers n.
- For $n = 2$ we can find the non-linear Schrodinger equation.
- For $n = 3$ we may recover the KdV equation.
- For $n = -1$ the Sin-Gordan equation.
- This scheme is named after Ablowitz, Kaup, Newell and Segur.

Reduction Problem

In theory we could consider Lax operators of the form:

$$L = \partial_x + X_0 + \sum_{i=1}^{n} \frac{1}{(\lambda - \gamma_i)^{\alpha_i}} X_i$$

$$S = \partial_t + T_0 + \sum_{i=1}^{n} \frac{1}{(\lambda - \mu_i)^{\beta_i}} T_i$$

where the entries of the X_i , T_i are functions of x and t.

- Too general to be useful.
- We need a way to reduce the generality of X and T.
- We do this by assuming that X_i and T_i are invariant with respect to the action of a finite group of transformations, the Reduction Group.

Automorphic Lie Algebras

- $X, T \in \mathfrak{U}_\lambda(\Gamma) = R_\lambda(\Gamma) \otimes_{\mathbb{C}} \mathfrak{U}$ where $R_\lambda(\Gamma)$ is the ring of rational functions in λ with poles in Γ, $\mathfrak{U}$ is a finite dimensional semi simple Lie Algebra.
  - $\Gamma = \{\infty\}$ gives that $R_\lambda(\Gamma) = \mathbb{C}[\lambda]$
  - $\Gamma = \{0, \infty\}$ gives that $R_\lambda(\Gamma) = \mathbb{C}[\lambda, \lambda^{-1}]$
- Letting $G \subset Aut(\mathfrak{U}_\lambda(\Gamma))$ consider the set of elements $\mathfrak{U}_\lambda^G = \{U \in \mathfrak{U}_\lambda(\Gamma) : g(U) = U, \forall g \in G\}$.
- We call this subalgebra the Automorphic Lie algebra corresponding to the group G and the set Γ. G is called the reduction group.
- This terminology was introduced by S. Lombardo and A.V. Mikhailov in "Reduction Groups and Automorphic Lie Algebras" Commun. Math. Phys. 258, 179-202 (2005).

Types of reduction group

- We consider simultaneous automorphisms of $R_\lambda(\Gamma)$ and $\mathfrak{U}$.
- Automorphisms of $R_\lambda(\Gamma)$ are fractional linear transformations. Finite subgroups of the group of fractional linear transformations have been classified by Felix Klein. The complete list is given by:
  - $\mathbb{Z}_N$ Cyclic groups
  - $\mathbb{D}_N$ Dihedral groups
  - $\mathbb{T}$ Tetrahedral group
  - $\mathbb{O}$ Octahedral group
  - $\mathbb{I}$ Icosahedral group.
- For $\mathfrak{U} = sl(N, \mathbb{C})$ automorphisms consist of:
  - Inner automorphisms $a \mapsto QaQ^{-1}$ for $N = 2$
  - Inner and outer automorphisms $a \mapsto -a^{tr}$ for $N \geq 3$.
- We find irreducible projective representations. These consist of 2x2 representations for all reduction groups, 3x3 representations for $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$, 4x4 representations for $\mathbb{O}$ and $\mathbb{I}$, 5x5 and 6x6 for $\mathbb{I}$.

Structure of Automorphic Lie algebras

- We will consider $\Gamma = G(\gamma)$, so a single orbit. This means we have two situations, Γ is either a generic orbit or a degenerated orbit.
- Automorphic Lie algebras are constructed by taking the group averages $< \frac{\mathbf{e}_i}{(\lambda-\gamma)^{k_i}} >_G = \frac{1}{|G|}\sum_{g\in G} g(\frac{\mathbf{e}_i}{(\lambda-\gamma)^{k_i}})$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ are a basis of $\mathfrak{U}$, $\gamma \in \Gamma$ and k_i is chosen such that the sum is non-zero.
- When considering $sl(N, \mathbb{C})$ with finite reduction group and inner automorphisms the automorphic Lie algebra has a Quasi Graded structure, ie we have that $\mathfrak{U}_\lambda^G(\Gamma) = \bigoplus_{k=0}^{\infty} A^k$ where $A^k = \{J^k a_1, J^k a_2, \ldots, J^k a_m\}$ and $[A^p, A^q] \subset A^{p+q} \bigoplus A^{p+q+1}$. J is a primitive automorphic function of λ.

Example

The $\mathbb{D}_2$ reduction group can be defined by the generators:

$$g_s : \mathbf{a}(\lambda) \to \mathbf{Q}_s \mathbf{a}(\sigma_s^{-1}(\lambda))\mathbf{Q}_s^{-1}$$

$$g_r : \mathbf{a}(\lambda) \to \mathbf{Q}_r \mathbf{a}(\sigma_r^{-1}(\lambda))\mathbf{Q}_r^{-1}$$

where

$$\sigma_s(\lambda) = -\lambda \ , \sigma_r(\lambda) = 1/\lambda$$

$$\mathbf{Q}_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{Q}_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this case the group average is given by

$$< \mathbf{a}(\lambda) >_{\mathbb{D}_2} = \frac{1}{4}(\mathbf{a}(\lambda) + \mathbf{Q}_s \mathbf{a}(-\lambda)\mathbf{Q}_s^{-1} + \mathbf{Q}_r \mathbf{a}(-\lambda^{-1})\mathbf{Q}_r^{-1} + \mathbf{Q}_r \mathbf{Q}_s \mathbf{a}(-\lambda^{-1})\mathbf{Q}_s^{-1}\mathbf{Q}_r^{-1})$$

Example continued...

Taking the standard basis for $sl(2,\mathbb{C})$

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we obtain the following:

$$\hat{\mathbf{a}}_1 = < \frac{\mathbf{e}_1}{\lambda - \gamma} >_{\mathbb{D}_2} = \begin{pmatrix} 0 & \frac{\lambda}{2(\lambda^2 - \gamma^2)} \\ \frac{\lambda}{2(1-\lambda^2\gamma^2)} & 0 \end{pmatrix}$$

$$\hat{\mathbf{a}}_2 = < \frac{\mathbf{e}_2}{\lambda - \gamma} >_{\mathbb{D}_2} = \begin{pmatrix} 0 & \frac{\lambda}{2(1-\lambda^2\gamma^2)} \\ \frac{\lambda}{2(\lambda^2 - \gamma^2)} & 0 \end{pmatrix}$$

$$\hat{\mathbf{a}}_3 = < \frac{\mathbf{e}_3}{\lambda - \gamma} >_{\mathbb{D}_2} = \frac{\gamma(1-\lambda^4)}{2(\lambda^2 - \gamma^2)(1 - \lambda^2\gamma^2)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example continued...

Setting

$$\mathbf{a}_1^n = 4\hat{\mathbf{a}}_1 J^n$$

$$\mathbf{a}_2^n = 4\hat{\mathbf{a}}_2 J^n$$

$$\mathbf{a}_3^n = 4\hat{\mathbf{a}}_3 J^n$$

with

$$J = \frac{(\lambda^2 - \mu^2)(1 - \mu^2\lambda^2)}{(\lambda^2 - \gamma^2)(1 - \gamma^2\lambda^2)}, \; \mu \neq \{\pm\gamma, \pm\gamma^{-1}\}$$

it can be shown that $\mathfrak{U}_\lambda^{\mathbb{D}_2} = \bigoplus_k \{\mathbf{a}_1^k, \mathbf{a}_2^k, \mathbf{a}_3^k\}$. Moreover

$$[\mathbf{a}_1^n, \mathbf{a}_2^m] = \mathbf{a}_3^{n+m+1} + a(\gamma, \mu)\mathbf{a}_3^{n+m}$$

$$[\mathbf{a}_3^n, \mathbf{a}_1^m] = 2\mathbf{a}_1^{n+m+1} + b(\gamma, \mu)\mathbf{a}_1^{n+m} - c(\gamma, \mu)\mathbf{a}_2^{n+m}$$

$$[\mathbf{a}_3^n, \mathbf{a}_2^m] = -2\mathbf{a}_2^{n+m+1} - b(\gamma, \mu)\mathbf{a}_2^{n+m} + c(\gamma, \mu)\mathbf{a}_1^{n+m}.$$

giving that $[A^n, A^m] \subset A^{n+m} \bigoplus A^{n+m+1}$

$\mathfrak{U} = sl(2, \mathbb{C})$

- Automorphic Lie algebras corresponding to $\mathbb{Z}_N$ where Γ is a degenerated orbit are grading isomorphic for all N. For each algebra generators $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and automorphic function J can be chosen such that the commutation relations take the form:

$$[\mathbf{a}_1, \mathbf{a}_2] = J\mathbf{a}_3,\ [\mathbf{a}_1, \mathbf{a}_3] = -2\mathbf{a}_1,\ [\mathbf{a}_2, \mathbf{a}_3] = 2\mathbf{a}_2$$

- Autmorphic Lie algebras corresponding to $\mathbb{Z}_N$ where $\Gamma = \mathbb{Z}_N(\gamma)$ is a generic orbit are grading isomorphic for all N and γ. In this case the commutation relations take the form:

$$[\mathbf{a}_1, \mathbf{a}_2] = J^2\mathbf{a}_3 - J\mathbf{a}_3,\ [\mathbf{a}_1, \mathbf{a}_3] = -2\mathbf{a}_1,\ [\mathbf{a}_2, \mathbf{a}_3] = 2\mathbf{a}_2$$

- The automorphic Lie algebras corresponding to $\mathbb{D}_N, \mathbb{T}, \mathbb{O}, \mathbb{I}$ where Γ is a degenerated orbit of either group are all grading isomorphic to $\mathbb{D}_2$ with orbit $\Gamma = \{\infty, 0\}$. In this case there exist generators such that

$$[\mathbf{a}_1, \mathbf{a}_2] = \mathbf{a}_3,\ [\mathbf{a}_1, \mathbf{a}_3] = 4\mathbf{a}_2 - 2J\mathbf{a}_1,\ [\mathbf{a}_2, \mathbf{a}_3] = -4\mathbf{a}_1 + 2\mathbf{a}_2.$$

$sl(N, \mathbb{C}), N \geq 3$

For the degenerated orbit $G(\infty)$ the system arising from the automorphic Lie algebra corresponding to $\mathbb{T}$,$\mathbb{O}$ and $\mathbb{I}$ with 3x3 representation are point equivalent to the following system:

$$k_{1,t} = k_{1,xx} + k_{2,x}^2 + k_{2,x}(e^{-k_1-k_2} + e^{-\omega k_1-\omega^2 k_2} + e^{-\omega^2 k_1-\omega k_2})$$

$$k_{2,t} = -k_{2,xx} - k_{1,x}^2 + k_{1,x}(e^{-k_1-k_2} + \omega e^{-\omega k_1-\omega^2 k_2} + \omega^2 e^{-\omega^2 k_1-\omega k_2}).$$

- A Darboux transformation for a Lax operator L is a Matrix M such that $MLM^{-1} = \tilde{L}$.
- M maps solutions of $L\Psi = 0$ to solutions $\tilde{\Psi}$ of $\tilde{L}$. In fact we can see that $\tilde{L}\tilde{\Psi} = MLM^{-1}(M\Psi) = ML\Psi = 0$.
- Given $L = \partial_x + X$ we wish to find M such that $ML = \tilde{L}M$ so essentially we must solve the system:

$$M_x + \tilde{X}M - MX = 0.$$

- When solving this system for Lax operators that have a particular symmetry we assume M has the same symmetry.
- This mapping is compatible with the system $L\Psi = 0$ in the sense that $\widetilde{(\Psi_x)} = \tilde{\Psi}_x$, for

$$\widetilde{(\Psi_x)} = -\widetilde{X\Psi} = -\tilde{X}\tilde{\Psi} = -\tilde{X}M\Psi$$

$$\tilde{\Psi}_x = (M\Psi)_x = M_x\Psi + M\Psi_x = M_x\Psi - MU\Psi.$$

Darboux Transformations $sl(2,\mathbb{C})$

$\mathbb{Z}_2$ reduction group

$$g_s : L(\lambda) \to \mathbf{Q}_s L(-\lambda)\mathbf{Q}_s^{-1},$$

- Degenerated orbit:

$$L = \partial_x + \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 2\,p \\ 2\,q & 0 \end{pmatrix}$$

$$M = \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ f\,\tilde{q} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Generic orbit:

$$L = D_x + \frac{1}{\lambda - 1} S(p,q) - \frac{1}{\lambda + 1}\mathbf{Q}_s S(p,q)\mathbf{Q}_s,$$

$$S(p,q) := \frac{1}{p-q} \begin{pmatrix} p+q & -2\,p\,q \\ 2 & -p-q \end{pmatrix}.$$

Continued..

$$M = \frac{1}{\lambda - 1} f \begin{pmatrix} \tilde{q} & -p\,\tilde{q} \\ 1 & -p \end{pmatrix} - \frac{1}{\lambda + 1} \mathbf{Q}_s f \begin{pmatrix} \tilde{q} & -p\,\tilde{q} \\ 1 & -p \end{pmatrix} \mathbf{Q}_s^{-1} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

- Dihedral group $\mathbb{D}_2$ Lax operator corresponding to the degenerated orbit can be written as

$$L = D_x + \lambda^2 \mathbf{Q}_s + \lambda \begin{pmatrix} 0 & 2\,p \\ 2\,q & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & 2\,q \\ 2\,p & 0 \end{pmatrix} - \frac{1}{\lambda^2} \mathbf{Q}_s,$$

$$M = f \left(\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & p \\ \tilde{q} & 0 \end{pmatrix} + u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \tilde{q} \\ p & 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

Discrete Systems from Darboux Transformations

- Suppose we have Darboux transformations M and N such that $MLM^{-1} = \tilde{L}$ and $NLN^{-1} = \hat{L}$.
- M maps a fundamental solution Ψ to a solution $\tilde{\Psi}$.
- N maps a fundamental solution Ψ to a solution $\hat{\Psi}$.
- Imposing that $\hat{\tilde{\Psi}} = \tilde{\hat{\Psi}}$ and interpreting $\tilde{}$ and $\hat{}$ as directions on a Lattice we obtain a difference system between the entries of our Darboux matrices.
- $\hat{\tilde{\Psi}} = \tilde{\hat{\Psi}}$ is equivalent to the condition that $\hat{M}N = \tilde{N}M$.
- The differential difference equations obtained when deriving the Darboux transformations act as symmetries of our discrete system:

$$\begin{aligned}\frac{d}{dx}(\hat{M}N - \tilde{N}M) &= \hat{M}_x N + \hat{M}N_x - \tilde{N}_x M - \tilde{N}M_x \\ &= -\hat{\tilde{U}}(\hat{M}N - \tilde{N}M) + (\hat{M}N - \tilde{N}M)U \\ &= 0.\end{aligned}$$

Tetrahedral Group

For $L = \partial_x + X$ and $S = \partial_t + T$ we assume L and S are invariant with respect to the following group of transformations:

$$g_s : \mathbf{a}(\lambda) \to \mathbf{Q}_s \mathbf{a}(\sigma_s^{-1}(\lambda))\mathbf{Q}_s^{-1}$$

$$g_r : \mathbf{a}(\lambda) \to \mathbf{Q}_r \mathbf{a}(\sigma_r^{-1}(\lambda))\mathbf{Q}_r^{-1}$$

$$\sigma_s(\lambda) = \omega\lambda \ , \sigma_r(\lambda) = \frac{\lambda + 2}{\lambda - 1}.$$

Where $\omega = e^{\frac{2\pi i}{3}}$

$$\mathbf{Q}_s = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{Q}_r = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

This group is Isomorphic to the tetrahedral group. It can be readily verified that $g_s^3 = g_r^2 = (g_r g_s)^3 = id$.

Generators of Tetrahedral automorphic Lie Algebra

$$\mathbf{a}_1 = \begin{pmatrix} -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} \\ \frac{\lambda}{\lambda^3-1} & \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} \\ \frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}$$

$$\mathbf{a}_2 = \begin{pmatrix} -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} \\ \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} \\ -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} & \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}$$

$$\mathbf{a}_3 = \begin{pmatrix} \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} \\ \frac{\lambda}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda^2}{\lambda^3-1} \\ -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}.$$

Tetrahedral group example

Let $L = \partial_x + u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3$ with $u_1 u_2 u_3 = 1$. In this case we may find Darboux transformations of the form

$$M = f\mathbf{I} + \alpha(u_1 u_2 \tilde{u}_1 \mathbf{a}_1 + u_2 \tilde{u}_1 \tilde{u}_2 \mathbf{a}_2 + \mathbf{a}_3)$$

where $\tilde{u}_i$, f and α satisfy particular differential relations, ie

$$\begin{aligned} \alpha' &= A(u_i, \tilde{u}_i, f) \\ f' &= B(u_i, \tilde{u}_i, f) \\ \tilde{u}_1' &= C(u_i, \tilde{u}_i, f) \\ \tilde{u}_2' &= D(u_i, \tilde{u}_i, f). \end{aligned}$$

From direct computation we obtain that

$$\det[M] = EJ_1 + F_1 = EJ_2 + F_2$$

where

$$J_1 = \frac{\lambda^3(\lambda^3+8)^3}{4(\lambda^3-1)^3}$$

$$J_2 = \frac{\left(-8-20\lambda^3+\lambda^6\right)^2}{4\left(\lambda^3-1\right)^3}$$

$$E = \frac{1}{16}\alpha^3 u_1 u_2^2 \tilde{u}_1^2 \tilde{u}_2$$

$$F_1 = \frac{1}{27}(3f+\alpha(1+u_1u_2\tilde{u}_1-2u_2\tilde{u}_1\tilde{u}_2))(3f+\alpha(1-2u_1u_2\tilde{u}_1+u_2\tilde{u}_1\tilde{u}_2))$$
$$(3f+\alpha(-2+u_1u_2\tilde{u}_1+u_2\tilde{u}_1\tilde{u}_2))$$

$$F_2 = \frac{1}{27}(3f+\alpha(1+u_1u_2\tilde{u}_1+u_2\tilde{u}_1\tilde{u}_2))q(f,u_1,u_2,\tilde{u}_1,\tilde{u}_2),$$

Continued..

Setting $u_1 = u$ and $u_2 = v$ we obtain the Darboux transformations:

$$M_i(u, v, u_i, v_i) = \beta_i(g_i\mathbf{I} + uvu_i\mathbf{a}_1 + vu_iv_i\mathbf{a}_2 + \mathbf{a}_3).$$

for $i = 1, \ldots, 4$ where u_i, v_i are $\tilde{u}$ and $\tilde{v}$ associated to M_i $\alpha^i g_i = f_i$ are the roots of F_1 or F_2:

$$g_1 = -\frac{1}{3}(-2 + uvu_1 + vu_1v_1)$$

$$g_2 = -\frac{1}{3}(1 + uvu_2 - 2vu_2v_2)$$

$$g_3 = \frac{1}{3}(-1 + 2uvu_3 - vu_3v_3)$$

$$g_4 = -\frac{1}{3}(1 + uvu_4 + vu_4v_4).$$

Continued...

Imposing the compatibility equations,

$$M_i(u_j, v_j, u_{ij}, v_{ij})M_j(u, v, u_j, v_j) - M_j(u_i, v_i, u_{ij}, v_{ij})M_i(u, v, u_i, v_i) = 0,$$

we obtain the systems:

- For i=1, j=4

$$\begin{aligned}
&u_4 v_4 \alpha_4^1 \alpha^4 - u_1 v_1 \alpha^1 \alpha_1^4 = 0 \\
&u_{14} v_4 + uv(-1 + u_4 u_{14} v_4 - u_1 u_{14} v_4) = 0 \\
&u u_1^2 v v_4 + u v_{14} - u_1(v_4(1 + uv(u_4 - v_{14})) + u v v_1 v_{14}) = 0.
\end{aligned} \tag{1}$$

- For i=2, j=4

$$\begin{aligned}
&(-1 + u_{24} v_4 v_{24})\alpha_4^2 \alpha^4 + (1 - u_2 v v_2)\alpha^2 \alpha_2^4 = 0 \\
&u_4 v_4(1 - u_2 v v_2) + u_2 v_2(-1 + u_{24} v_4 v_{24}) = 0 \\
&- u_2 v_2 + u(-u_4 + u_2 + v_{24}) = 0.
\end{aligned} \tag{2}$$

Continued...

- For i=3, j=4

$$
\begin{aligned}
&(-1 + u_4 u_{34} v_4)\alpha_4^3 \alpha^4 + (1 - u u_3 v)\alpha^3 \alpha_3^4 = 0 \\
&- u_3 v_3 + u_4 v_4 (1 - u u_3 v + u_3 u_{34} v_3) = 0 \\
&- u_4 v_4 + (u + v_4 - v_3) v_{34} = 0.
\end{aligned} \tag{3}
$$

- For i=1, j=2

$$
\begin{aligned}
&u_1 v_1 \alpha^1 \alpha_1^2 - u_2 v_2 \alpha_2^1 \alpha^2 = 0 \\
&- u_1 v + u_{12} v_{12} = 0 \\
&- u u_1^2 v v_2 - u v_{12} + u_1 v_2 (1 - u_2 v v_2 + u v (u_2 + v_{12})) = 0.
\end{aligned} \tag{4}
$$

Continued...

- For i=2, j=3

$$
\begin{aligned}
&-u_3 v_3 \alpha_3^2 \alpha^3 + u_2 v_2 \alpha^2 \alpha_2^3 = 0 \\
&-u_2 v_2 + u v_{23} = 0 \\
&-u^2 u_3 u_2 v v_2 - u_2^2 u_{23} v_3 v_2^2 + \\
&u(u_2 v_2 + u_3 v_3(-1 + u_2(u_{23} + v) v_2)) = 0.
\end{aligned} \tag{5}
$$

- For i=1, j=3

$$
\begin{aligned}
&-u_1 v_1 \alpha^1 \alpha_1^3 + u_3 v_3 \alpha_3^1 \alpha^3 = 0 \\
&-uv + u_{13} v_3 = 0 \\
&-u u_1^2 v v_1 - u v_{13} + u_1(uv(u + v_1) v_{13} + v_3(1 - uv v_{13})) = 0.
\end{aligned} \tag{6}
$$

Corresponding differential difference relations

- For i=1

$$-u+u_1+u^2u_1v-uu_1^2v+u_1vu'+uvu_1'+uu_1v'=0$$

$$-uv+2uv_1-u_1v_1-u^2u_1vv_1+uu_1^2vv_1+$$
$$uu_1v^2v_1-uu_1vv_1^2-u_1vv_1u'+uu_1vv_1'=0$$

- For i=2

$$uv-u_2v_2+u^2u_2vv_2-uu_2^2vv_2-uu_2v^2v_2-uu_2vv_2^2+$$
$$2u_2^2vv_2^2+u_2vv_2u'+uvv_2u_2'+uu_2v_2v'=0$$
$$-u^2v_2+uu_2v_2+uv_2^2-u_2v_2^2-v_2u'+uv_2'=0$$

Continued...

- For i=3

$$uv - u^2 u_3 v^2 - u_3 v_3 + u u_3^2 v v_3 + u_3 v v_3 u' + u v v_3 u_3' + u u_3 v_3 v' = 0$$

$$-u^2 v + u^2 v_3 + u v v_3 - u v_3^2 - v_3 u' + u v_3' = 0$$

- For i=4

$$uv - u_4 v_4 + u^2 u_4 v v_4 - u u_4^2 v v_4 + u_4 v v_4 u' + u v v_4 u_4' + u u_4 v_4 v' = 0$$

$$-u^2 v_4 + u u_4 v_4 + u v v_4 - u v_4^2 - v_4 u' + u v_4' = 0$$

What have we seen?

- We can find Integrable pdes by considering reductions of general Lax pairs.
- We can impose a symmetry upon our Lax pairs to achieve this aim.
- Lax Pairs posessing such symmetries form a Lie algebra.
- For certain spectral dependence and reductions these Lie algebras are grading Isomorphic.
- Darboux transformations can be used to construct discrete systems.
- These discrete systems automatically possess a symmetry.