The motion of the fluxion in curved Josephson junction.

Josephson junction

$$\psi_T = |\psi_T| e^{\imath \varphi_T}$$

If the central dielectric layer is sufficiently narrow then one observes a phase correlation of the wave functions. The physical effect of this correlation is flow of the Cooper-pairs through the dielectric layer.

The dominating dynamical degee of freedom is gauge invaiant phase difference.

$$\psi_B = |\psi_B| e^{\imath \varphi_B}$$

$$\imath\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

$$\psi_T = |\psi_T|e^{\imath\varphi_T}$$

$$\psi_B = |\psi_B|e^{\imath\varphi_B}$$

$$|\psi_T\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi\rangle = \psi_T|\psi_T\rangle + \psi_B|\psi_B\rangle = \begin{bmatrix} \psi_T \\ \psi_B \end{bmatrix}$$

$$|\psi_B\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \langle\psi_T|\hat{H}|\psi_T\rangle, & \langle\psi_T|\hat{H}|\psi_B\rangle \\ & B\rangle \end{bmatrix} = \begin{bmatrix} E_T, & K \\ K, & E_B \end{bmatrix}$$

$$\imath\hbar\frac{\partial\psi_T}{\partial t} = E_T\psi_T + K\psi_B$$

$$\imath\hbar\frac{\partial\psi_B}{\partial t} = E_B\psi_B + K\psi_T$$

$$\boxed{\begin{gathered} |\psi_B|^2 = \rho_B \\ |\psi_T|^2 = \rho_T \end{gathered}}$$

$$\boxed{\frac{\partial}{\partial t}\phi = \frac{2\pi}{\Phi_0}V}$$

$$\boxed{I_J = I_c \sin\phi}$$

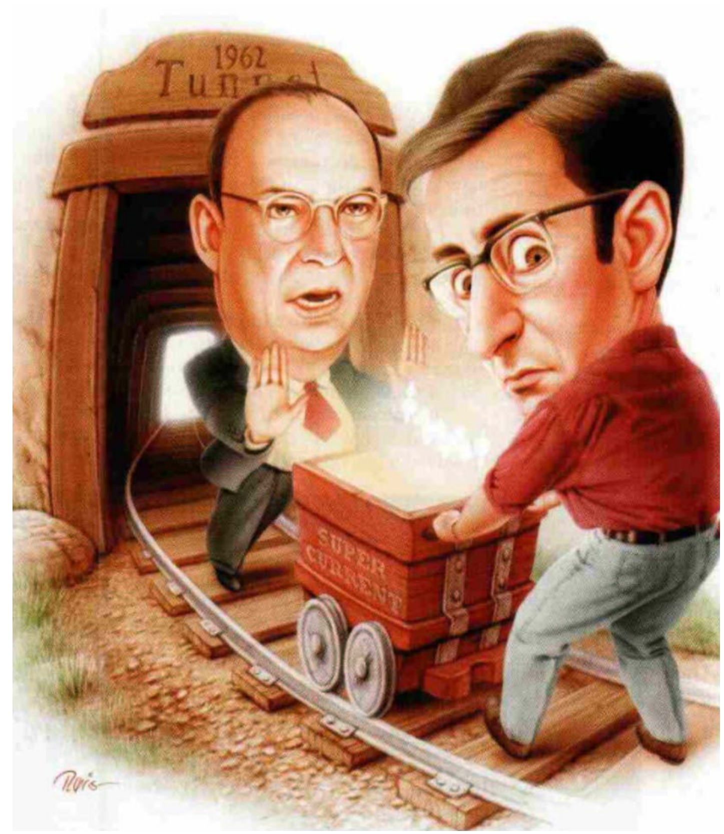
Direct current Josephson effect (DC)

$$\underline{V = 0} \Rightarrow \phi = const = \alpha \Rightarrow \underline{I_J = I_c \sin\alpha = const}$$

Amplitude current Josephson effect (AC)

$$\underline{V = V_0} \Rightarrow \phi = \frac{2e}{\hbar}V_0 t + \alpha \Rightarrow \underline{I_J = I_c \sin(\frac{2\pi}{\Phi_0}t + \alpha)}$$

In 1962 (when his work was published in Physics Letters) Brian Josephson was 22 years old and was a student at Cambridge University.

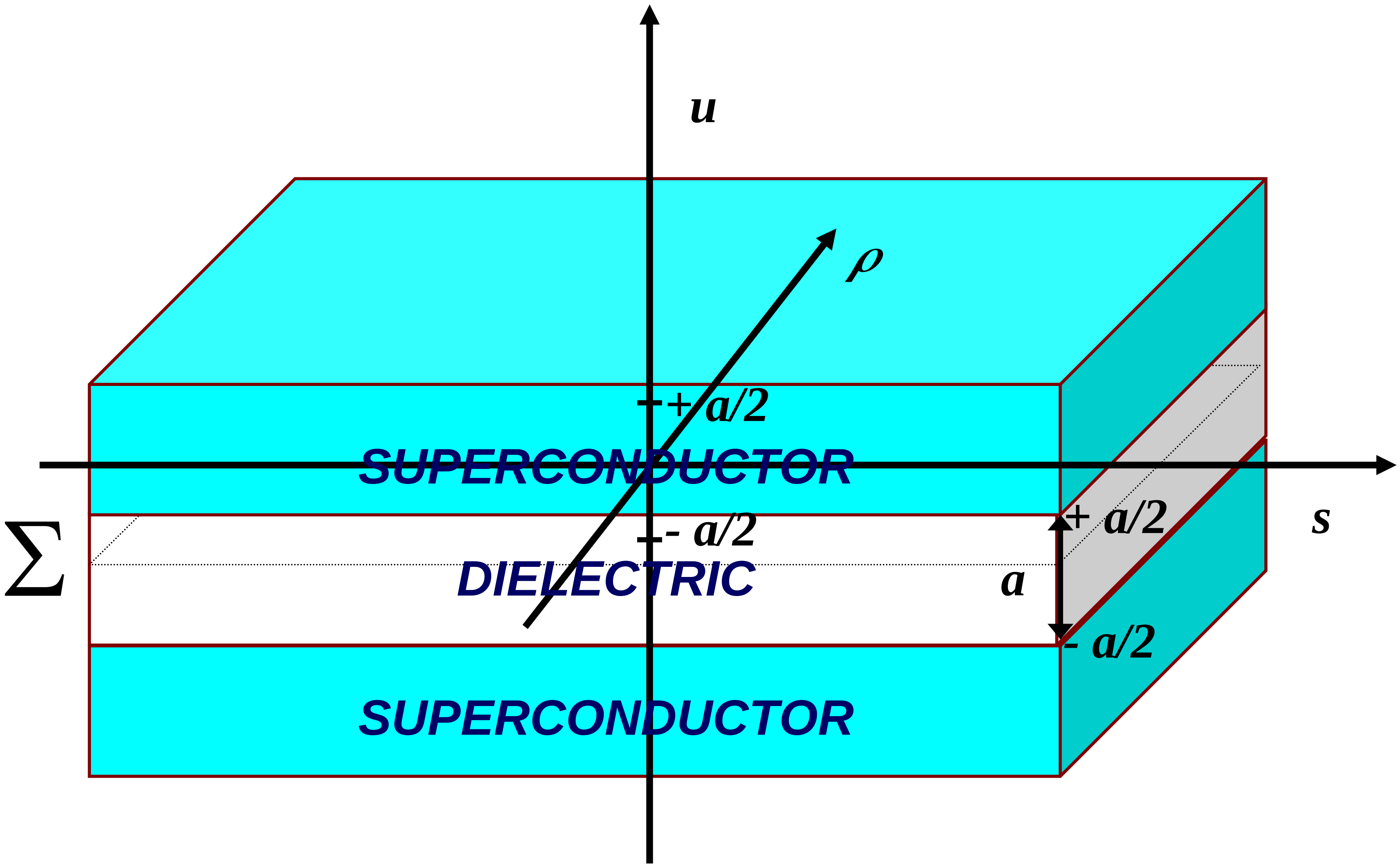


At this time, John Bardeen was 52 years. He was awarded the Nobel Prize in 1956 for his discovery of the transistor. Moreover, in 1957, along with Cooper and Schrieffer he developed a microscopic theory of superconductivity. Due to this discovery he was almost certain candidate for a second Nobel Prize.

He published his work in Physical Review Letters.

Indeed, ten years later, i.e. in 1972, Bardeen receives the Nobel Prize for BCS.

In 1973, Josephson receives the Nobel Prize for prediction of the tunneling of Cooper pairs.



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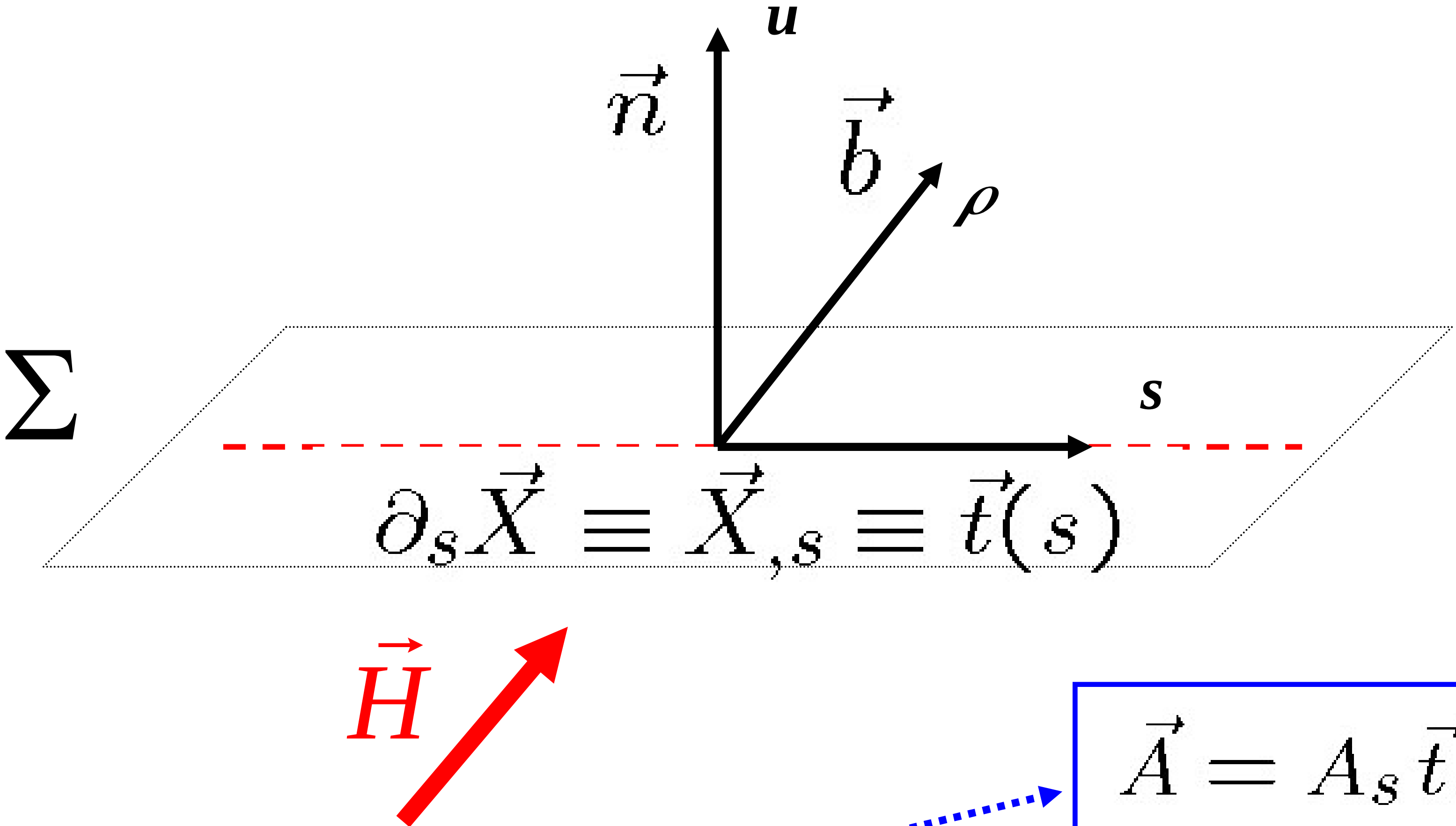
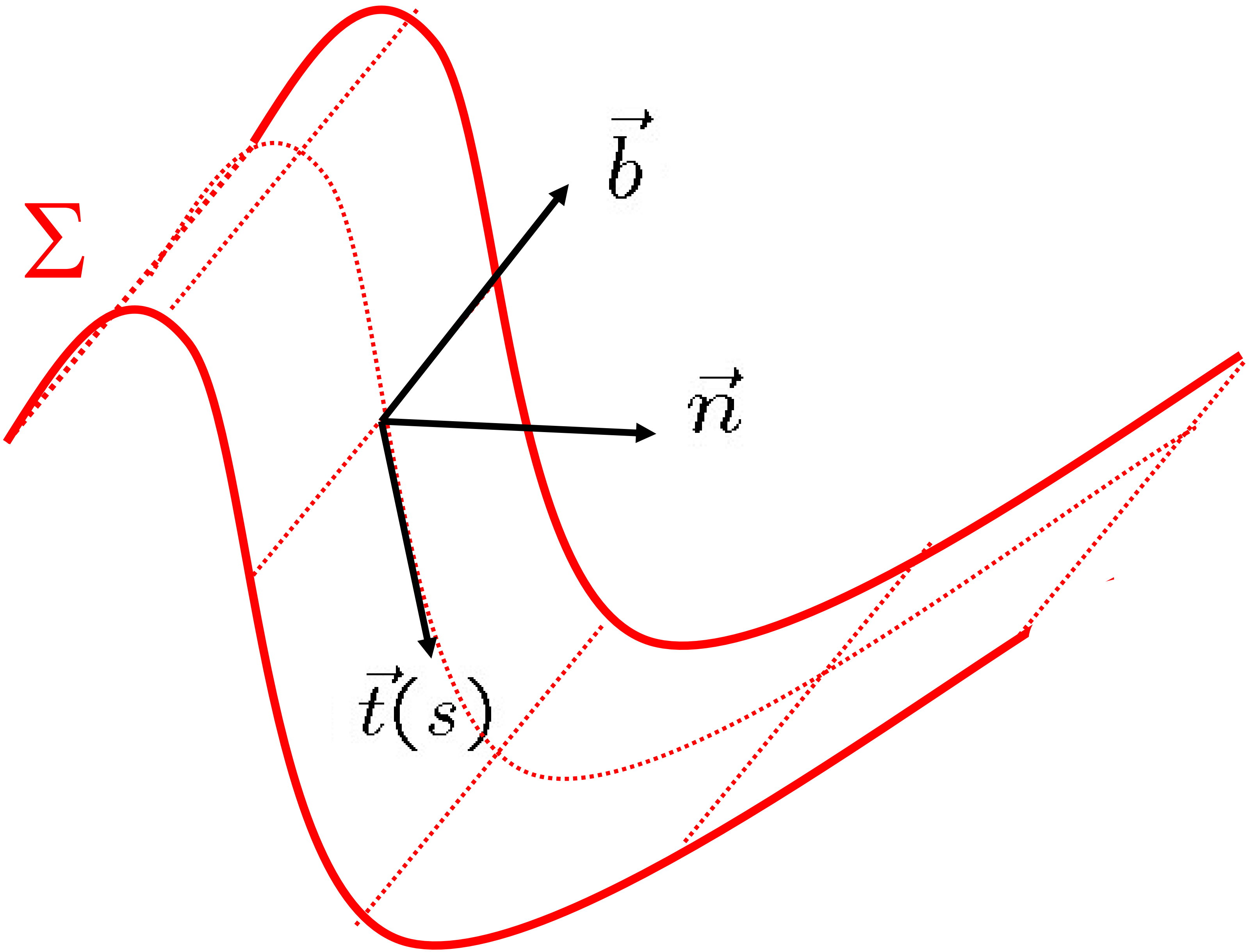


$$\vec{A} = A_s \, \vec{t}.$$

$$\boxed{\vec{H} = H_\rho \, \vec{b} \equiv H \, \vec{b}.}$$

$$\vec{E} = E_s \, \vec{t} + E_u \, \vec{n}.$$


Σ
$\vec{b}$
$\vec{n}$
$\vec{t}(s)$

The fields in the dielectric

Ampere's law

$$\boxed{\mathbf{curl}\,\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{\varepsilon}{c}\partial_t\vec{E}}$$

$$\begin{cases} -\partial_u H = \dfrac{4\pi}{c} J_s + \dfrac{\varepsilon_I}{c}\partial_t E_s \\[2ex] \dfrac{1}{\mathcal{G}}\partial_s H = \dfrac{4\pi}{c} J_u + \dfrac{\varepsilon_I}{c}\partial_t E_u \\[2ex] \dfrac{1}{\mathcal{G}}\left(E_s\partial_u\mathcal{G} + \mathcal{G}\partial_u E_s - \partial_s E_u\right) = -\dfrac{\mu}{c}\partial_t H \end{cases}$$

$$\mathcal{G} = 1 - uK$$

$$\boxed{\mathbf{curl}\,\vec{E} = -\frac{\mu}{c}\partial_t\vec{H}}$$

Maxwell-Faraday law

$$H(s,u,t) = \int dk d\omega \, e^{i(ks-\omega t)} \, h(k,u,\omega)$$

$$E_\alpha(s,u,t) = \int dk d\omega \, e^{i(ks-\omega t)} \, e_\alpha(k,u,\omega)$$

$$\left\{ \begin{array}{l} -\partial_u h = \cancel{\frac{4\pi}{c} j_s} + \frac{\varepsilon_I}{c}(-i\omega)e_s \\ \frac{1}{\mathcal{G}} ikh = \frac{4\pi}{c} j_u + \frac{\varepsilon_I}{c}(-i\omega)e_u \\ \frac{1}{\mathcal{G}} e_s \partial_u \mathcal{G} + \partial_u e_s - \frac{1}{\mathcal{G}} ike_u = -\frac{\mu}{c}(-i\omega)h \end{array} \right.$$

$$\boxed{e_s = -\frac{ic}{\varepsilon_I \omega}\partial_u h}$$

$$\boxed{e_u = -\frac{\omega}{ck}\mathcal{G}\left[\mu h + \frac{c^2}{\varepsilon_I \omega^2}\left(\partial_u^2 h + \frac{1}{\mathcal{G}}(\partial_u \mathcal{G})(\partial_u h)\right)\right]}$$

$$\boxed{\varepsilon \,\mathbf{div}\vec{E} = 4\pi\varrho} = \underline{\underline{0}}$$

Gauss law

$$\partial_s E_s + \partial_u(\mathcal{G}E_u) = 0$$

$$ike_s = -\partial_u(\mathcal{G}e_u)$$

$$\mathcal{G}^2\partial_u^2 h - K\mathcal{G}\partial_u h - \left(\gamma^2 + \alpha^2(1-\mathcal{G}^2)\right)h = -\gamma^2 h_0$$

$$\boxed{\begin{array}{l} \gamma^2 \equiv k^2 - \alpha^2 \\ \alpha^2 = \frac{\mu\varepsilon_I \omega^2}{c^2} \end{array}}$$

$$\mathcal{G} = 1 - uK$$

$$u = au'$$

$$\mathcal{G}^2\partial_{u'}^2 h - aK\mathcal{G}\partial_{u'} h - \left(a^2\gamma^2 + a^2\alpha^2(1-\mathcal{G}^2)\right)h =$$

$$= -a^2\gamma^2 h_0$$

$$\boxed{a^2\alpha^2 \sim 10^{-12}}$$

$$\boxed{\mathcal{G}^2\partial_u^2 h - K\mathcal{G}\partial_u h - \gamma^2 h = -\gamma^2 h_0}$$

$$h(k,u,\omega) = h_+(k,\omega)\left[\gamma(1-uK)\right]^{-\gamma/K} + $$

$$+ h_-(k,\omega)\left[\gamma(1-uK)\right]^{\gamma/K} + h_0(k,\omega)$$

$$\boxed{e_s = -\frac{ic}{\varepsilon_I \omega}\partial_u h}$$

$$\boxed{e_u = -\frac{\omega}{ck}\mathcal{G}\left[\mu h + \frac{c^2}{\varepsilon_I \omega^2}\left(\partial_u^2 h + \frac{1}{\mathcal{G}}(\partial_u \mathcal{G})(\partial_u h)\right)\right]}$$

The current in isolator

$$\frac{1}{\mathcal{G}}ikh = \frac{4\pi}{c}j_u + \frac{\varepsilon_I}{c}(-i\omega)e_u$$

$$\int dkd\omega e^{i(ks-\omega t)}\left(\frac{1}{\mathcal{G}}ikh + i\omega\frac{\varepsilon_I}{c}e_u\right) = \frac{4\pi}{c}J_u$$

$$\mathcal{F} = \frac{1}{aK}\ln\left(\frac{2+aK}{2-aK}\right)$$

$$\boxed{-\int dkd\omega e^{i(ks-\omega t)}\frac{1}{ik}\mathcal{F}\gamma^2 h_0 = \frac{4\pi}{c}J_m \sin\phi}$$

The fields in superconductors

$$\mathbf{curl}\,\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{\varepsilon_S}{c}\partial_t\vec{E}$$

$$\vec{E} = \partial_t\left(\Lambda\vec{J}\right) = \frac{4\pi\lambda_L^2}{c^2}\partial_t\vec{J}$$

$$\Lambda = \frac{4\pi\lambda_L^2}{c^2}$$

$$\vec{J} = \vec{f}_1(H)$$

$$m\frac{d\vec{V}}{dt} = q\vec{E}$$

$$n_C q\frac{d\vec{V}}{dt} = \frac{n_C q^2}{m}\vec{E}$$

$$\Bigg| \quad \vec{J} = n_C q\vec{V}$$

$$\frac{d\vec{J}}{dt} = \frac{1}{\Lambda}\vec{E}$$

$$\vec{J} = \frac{q^*}{m^*}\left[\frac{1}{2}\imath\,\hbar\,(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{q^*}{c}\vec{A}\,\psi\psi^*\right]$$

$$\vec{J} = |\psi|^2\frac{q^*}{m^*}\left[\hbar\nabla\varphi - \frac{2q^*}{c}\vec{A}\right]$$

$$\nabla\varphi = \vec{f}_2(\vec{J}) = \vec{f}(H)$$

$$\begin{cases} \dfrac{\Phi_0}{2\pi}\partial_s\varphi = -\int dk d\omega e^{i(ks-\omega t)}\left[\bar{\lambda}_L^2\mathcal{G}\partial_u h - \mu\int_0^u du'\mathcal{G}h\right] \\ \dfrac{\Phi_0}{2\pi}\partial_u\varphi = \int dk d\omega e^{i(ks-\omega t)}\left[ik\bar{\lambda}_L^2\dfrac{1}{\mathcal{G}}h\right] \end{cases}$$

$$\lambda_L = \frac{m^* c^2}{4\pi(q^*)^2|\psi|^2},$$

$$\begin{cases} \dfrac{\Phi_0}{2\pi}\partial_u\partial_s\varphi = -\int dk d\omega e^{i(ks-\omega t)}\left[\bar{\lambda}_L^2\partial_u\left(\mathcal{G}\partial_u h\right) - \mu\mathcal{G}h\right] \\ \dfrac{\Phi_0}{2\pi}\partial_s\partial_u\varphi = -\int dk d\omega e^{i(ks-\omega t)}\left[k^2\bar{\lambda}_L^2\dfrac{1}{\mathcal{G}}h\right] \end{cases}$$

$$\mathcal{G}^2\partial_u^2 h - K\mathcal{G}\partial_u h - \Gamma^2 h = 0$$

$$\Gamma^2 = k^2 + \left(\mu/\bar{\lambda}_L^2\right)$$

$$\boxed{h(k,u,\omega) = \begin{cases} h_<(k,\omega)\left[\Gamma(1-uK)\right]^{-\frac{\Gamma}{K}} & \text{dla } u \le -\frac{a}{2} \\ h_>(k,\omega)\left[\Gamma(1-uK)\right]^{+\frac{\Gamma}{K}} & \text{dla } u \ge +\frac{a}{2} \end{cases}}$$

Matching of the fields

The magnetic fields

In superconductors

$$h = \begin{cases} h_< \left[\Gamma(1-uK)\right]^{-\frac{\Gamma}{K}} \\ h_> \left[\Gamma(1-uK)\right]^{+\frac{\Gamma}{K}} \end{cases}$$

In dielectric

$$h = h_+ \left[\gamma(1-uK)\right]^{-\gamma/K} + h_- \left[\gamma(1-uK)\right]^{\gamma/K} + h_0$$

$$-\frac{a}{2} \leq u \leq \frac{a}{2}$$

The electric fields

$$e_s = \frac{i\omega\bar{\lambda}_L^2}{c} \begin{cases} h_< \left(\frac{\Gamma}{1-uK}\right) \left[\Gamma(1-uK)\right]^{-\frac{\Gamma}{K}} & \text{for } u \leq -\frac{a}{2} \\ -h_> \left(\frac{\Gamma}{1-uK}\right) \left[\Gamma(1-uK)\right]^{+\frac{\Gamma}{K}} & \text{for } u \geq +\frac{a}{2} \end{cases}$$

$$e_s = -\frac{ic}{\varepsilon_I \omega} \partial_u h =$$

$$= -\frac{ic}{\varepsilon_I \omega} \left\{ \gamma h_+ \frac{\left[\gamma(1-uK)\right]^{-\frac{\gamma}{K}}}{1-uK} - \gamma h_- \frac{\left[\gamma(1-uK)\right]^{\frac{\gamma}{K}}}{1-uK} \right\}$$

Matching on two surfaces gives four equations with unknowns $\{h_>, h_<, h_+, h_-\}$

The dynamics of the gauge invariant phase difference

$$\phi(s,t) \equiv \varphi(s,\frac{a}{2},t) - \varphi(s,-\frac{a}{2},t) - \frac{q^*}{\hbar c}\int_{-a/2}^{a/2} du\, A_u =$$

$$= \varphi(s,\frac{a}{2},t) - \varphi(s,-\frac{a}{2},t)$$

$$\frac{\Phi_0}{2\pi}\varphi(s,u,t) = -\int dk d\omega e^{i(ks-\omega t)} \frac{1}{ik}\left[\bar{\lambda}_L^2 \mathcal{G}\partial_u h - \mu \int_0^u du' \mathcal{G} h\right]$$

$$\boxed{\phi(s,t) = \frac{2\pi}{\Phi_0}\int dk d\omega e^{i(ks-\omega t)}\frac{\gamma^2}{ik}\frac{(2\lambda_L+\mu a)}{\mathcal{F}k^2-\frac{\omega^2}{\bar{c}^2}}\mathcal{F}h_0}$$

$$\downarrow$$

$$\frac{1}{\bar{c}^2} = \varepsilon_I\left(\frac{2\lambda_L}{a}+1\right)\frac{1}{c^2}$$

$$\boxed{\begin{aligned}&-\frac{1}{\bar{c}^2}\partial_t^2\phi(s,t)+\mathcal{F}\partial_s^2\phi(s,t) = \\ &= -\int dk d\omega e^{i(ks-\omega t)}\frac{\gamma^2}{ik}\frac{2\pi(2\lambda_L+\mu a)}{\Phi_0}\mathcal{F}h_0\end{aligned}}$$

$$-\frac{1}{\bar{c}^2}\partial_t^2\phi(s,t) + \mathcal{F}\partial_s^2\phi(s,t) =$$

$$= -\int dk d\omega e^{i(ks-\omega t)} \frac{\gamma^2 2\pi(2\lambda_L + \mu a)}{ik\,\Phi_0}\mathcal{F}h_0$$

$$-\int dk d\omega e^{i(ks-\omega t)} \frac{1}{ik}\mathcal{F}\gamma^2 h_0 = \frac{4\pi}{c} J_m \sin\phi$$

$$-\frac{1}{\bar{c}^2}\partial_t^2\phi(s,t) + \mathcal{F}\partial_s^2\phi(s,t) = \frac{1}{\lambda_J^2}\sin\phi$$

$$\frac{1}{\lambda_J^2} = \frac{8\pi^2(2\lambda_L + \mu a)}{c\Phi_0} J_m.$$

$$(s \to \frac{1}{\lambda_J}s,\ t \to \frac{\bar{c}}{\lambda_J}t)$$

$$\partial_t^2\phi(s,t) - \mathcal{F}\partial_s^2\phi(s,t) + \sin\phi = 0$$

sine-Gordon model on the flat curve

$$\boxed{\mathcal{L} = \frac{1}{2}\eta_M^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)}$$

$$V(\phi) = 1 - \cos\phi$$

$$x^i \longrightarrow \frac{x^i}{\lambda_J} \qquad t \rightarrow \omega_P t$$

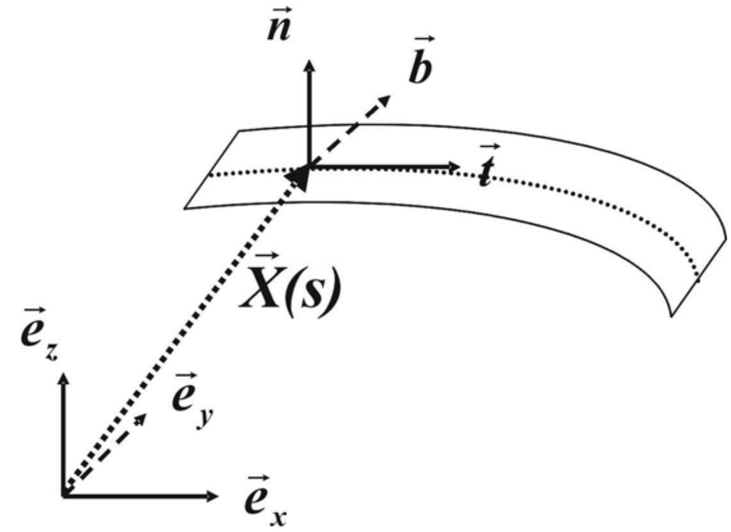
$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\eta_E^{ij}(\partial_i\phi)(\partial_j\phi) - V(\phi)$$

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}G^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - V(\phi)}$$

$$\xi^\alpha = (\xi^1, \xi^2, \xi^3) = (s, \rho^1, \rho^2) = (s, \rho, u)$$

Connection between curved and Cartesian coordinates

$$\xi^\alpha = (\xi^1, \xi^2, \xi^3) = (s, \rho^1, \rho^2) = (s, \rho, u)$$



$$\boxed{\vec{x} = \vec{X}(s) + \rho^j \vec{n}_j(s)}$$

$$G_{\alpha\beta} = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^j}{\partial \xi^\beta}\eta^E_{ij}$$

$$G_{ij} = \delta_{ij}, \quad G_{is} = 0, \quad G_{ss} = \mathcal{G}^2 = (1 - uK(s))^2$$

$$G^{ij} = \delta^{ij}, \quad G^{is} = 0 \quad G^{ss} = \frac{1}{G}$$

$$\boxed{G = \mathcal{G}^2 = (1 - uK)^2}$$

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2G}(\partial_s\phi)^2 - V(\phi)$$

$$L_{eff} = \int ds\,d\rho\,du\,\sqrt{G}\,\mathcal{L}$$

$$L_{eff} = \int ds\mathcal{L}_{eff}$$

$$\int_{-a/2}^{a/2} du(1 - uK) = a$$

$$\int_{-a/2}^{a/2} du\frac{1}{1-uK} = a\mathcal{F}$$

$$\mathcal{L}_{eff} = ab\left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\mathcal{F}(\partial_s\phi)^2 - V(\phi)\right)$$

$$\partial_t\left(\frac{\delta\mathcal{L}_{eff}}{\delta(\partial_t\phi)}\right) + \partial_s\left(\frac{\delta\mathcal{L}_{eff}}{\delta(\partial_s\phi)}\right) - \frac{\delta\mathcal{L}_{eff}}{\delta\phi} = 0$$

$$\boxed{\partial_t^2\phi - \mathcal{F}\,\partial_s^2\phi + \sin\phi = 0}$$

Instead of painstaking considerations on the basis of Maxwell's equations it is sufficient to reduce the sine-Gordon model to the lower dimensional manifold *(at least for slowly varying curvatures).*

The effective dynamics of the fluxion

$$L = \int_0^l ds \int_{-a/2}^{+a/2} du \int_{-b/2}^{+b/2} d\rho \sqrt{G}\, \mathcal{L}(\phi_K).$$

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi_K)^2 - \frac{1}{2G}(\partial_s \phi_K)^2 - V(\phi_K).$$

$$G = \mathcal{G}^2 = (1 - uK)^2$$

$$\phi_K = 4 \arctan\left[e^{s-S(t)}\right].$$

$$L = T - U.$$

The effective lagrangian

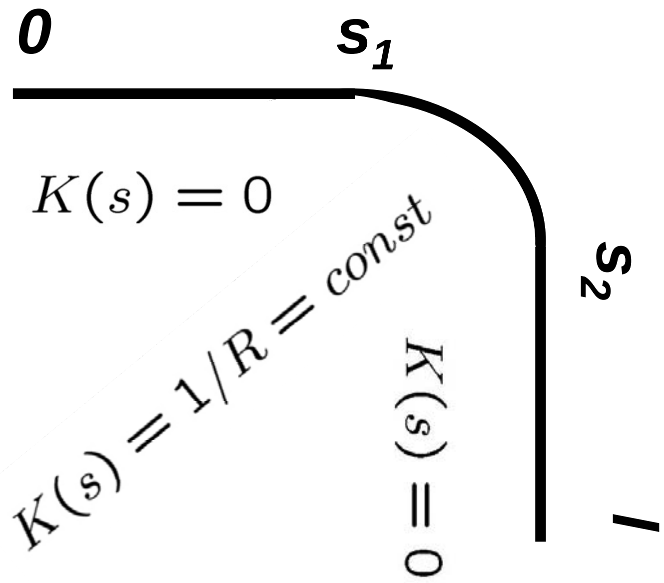
$$T = 2abJ(S)\dot{S}^2 \approx 4ab\dot{S}^2.$$

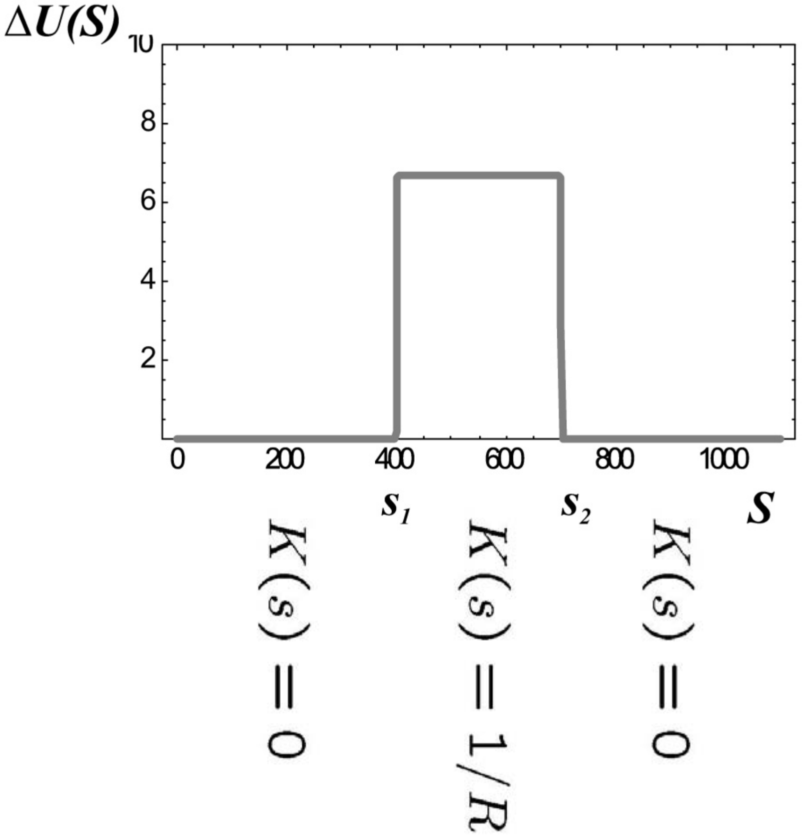
$$J(S) = \tanh(l - S) + \tanh(S) \approx 2$$

$$K(s) << 1/a$$

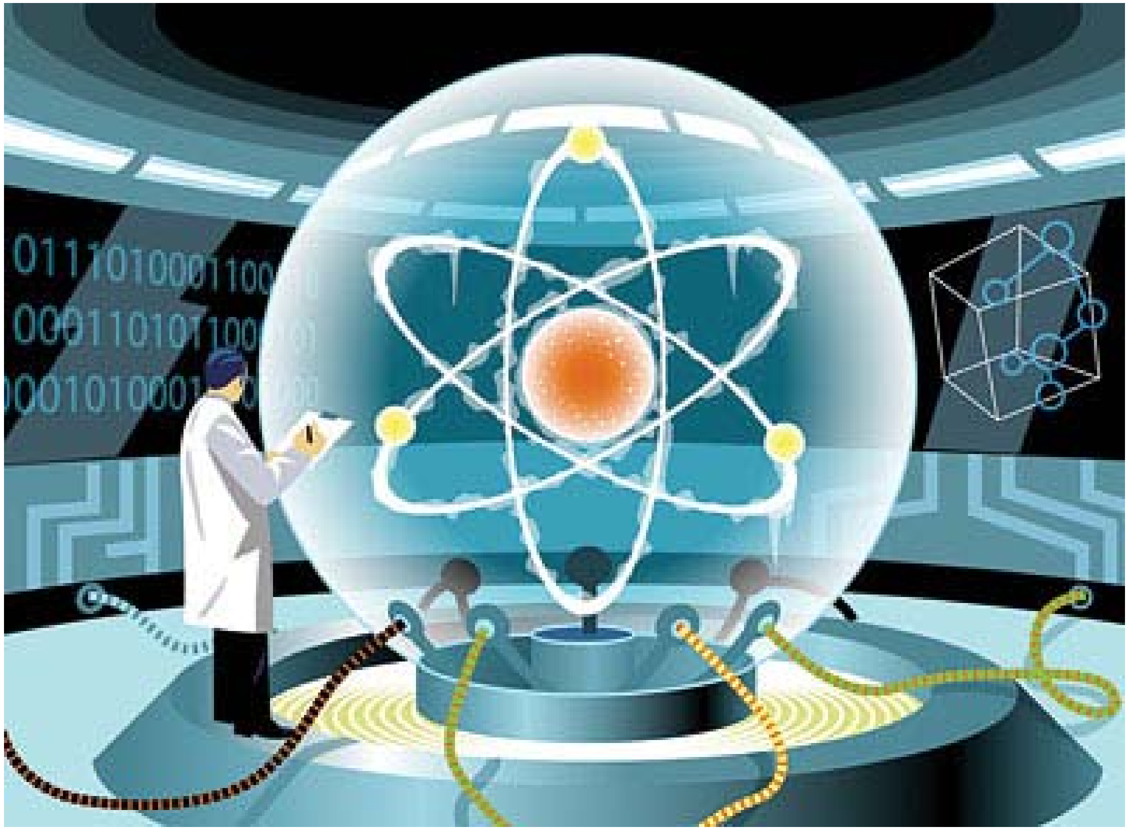
$$U \approx 8ab + \frac{ba^3}{6}\int_0^l ds \frac{K(s)^2}{\cosh^2(s - S)} \equiv 8ab + \Delta U.$$

Example





$$\Delta U \approx \frac{ba^3}{6R^2} \frac{\sinh(s_2 - s_1)}{\cosh(S - s_1)\cosh(s_2 - S)}.$$



$$N = 1000 \qquad 2^N \approx 1.07 \cdot 10^{301}$$