The quad graph equation with a nonstandard generalized symmetry structure.

R.N. Garifullin[1], A.V. Mikhailov[2] and R.I. Yamilov[1]

[1] Ufa Institute of Mathematics, Ufa, Russia
[2] Department of Applied Mathematics, University of Leeds, Leeds, UK

Varna-2013

Quad graph equations

A quad graph equation can be defined as

$$Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0. \qquad (1)$$

Q is defined function,
$n, m \in \mathbb{Z}$ are "‘independent"’ variables,
$u : \mathbb{Z}^2 \to \mathbb{C}$ is "‘dependent"’ variable or unknown function. **Shift operators**

$$T_n g(n,m) = g(n+1,m), \quad T_m g(n,m) = g(n,m+1)$$

Example: Equation

$$u_{n+1,m+1} - u_{n,m+1} - u_{n+1,m} + u_{n,m} = 0$$

has a common solution

$$u_{n,m} = \alpha(n) + \beta(m),$$

where $\alpha, \beta : \mathbb{Z} \to \mathbb{C}$ are arbitrary functions.

Comparing quad graph equations with hyperbolic one

$u_{xy} = f(u, u_x, u_y)$ $D_y = \frac{d}{dy}$, $D_x = \frac{d}{dx}$	$u_{n+1,m+1} = f(u_{n,m}, u_{n+1,m}, u_{n,m+1})$ $T_n g(n, m) = g(n+1, m)$, $T_m g(n, m) = g(n, m+1)$
Darboux integrable equations $u_{xy} = e^u$ $W_1 = u_{xx} - u_x^2/2,\ D_y W_1 = 0$ $W_2 = u_{yy} - u_y^2/2,\ D_x W_2 = 0$	 $u_{n+1,m+1}u_{n,m} - u_{n+1,m}u_{n,m+1} = 1$ $W_1 = \frac{u_{n+1,m}+u_{n-1,m}}{u_{n,m}},\ (T_2 - 1)W_1 = 0$ $W_2 = \frac{u_{n,m+1}+u_{n,m-1}}{u_{n,m}},\ (T_1 - 1)W_2 = 0$
Sin-Gordon type equations $u_{xy} = \sin u$ $u_t = u_{xxx} + u_x^3/2$ $u_\tau = u_{yyy} + u_y^3/2$	$u_{n+1,m+1}u_{n,m+1} + u_{n+1,m}u_{n,m}$ $+u_{n,m}u_{n,m+1} = 0$ $u_{n,m,t} = \frac{u_{n+1,m}u_{n,m}}{u_{n-1,m}}$ $u_{n,m,\tau} = \frac{u_{n,m}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}}$
Tzitzeica equation $u_{xy} = e^u - e^{-2u}$ $u_t = u_{xxxxx} + \dots$ $u_\tau = u_{yyyyy} + \dots$	$(u_{0,0} + u_{1,1})u_{0,1}u_{1,0} + 1 = 0$ Mikhailov&Xenitidis 13 $u_t = \Psi(u_{n-2,m}, \dots, u_{n+2,m})$ $u_\tau = \Phi(u_{n,m-2}, \dots, u_{n,m+2})$

In [Garifullin, Yamilov 12] the equation has been found

$$u_{n+1,m+1}(u_{n,m} - u_{n,m+1}) - u_{n+1,m}(u_{n,m} + u_{n,m+1}) + 2 = 0. \quad (2)$$

This equation has one of generalized symmetries of the form:

$$\frac{d}{dt_2} u_{n,m} = (-1)^n \frac{u_{n,m+1}u_{n,m-1} + u_{n,m}^2}{u_{n,m+1} + u_{n,m-1}}. \quad (3)$$

$$\frac{d}{dt_1} u_{n,m} = h_{n,m}h_{n-1,m}(a_n u_{n+2,m} - a_{n-1}u_{n-2,m}), \quad (4)$$

$$h_{n,m} = u_{n+1,m}u_{n,m} - 1, \quad a_{n+2} = a_n.$$

The symmetry depends on an arbitrary two-periodic function a_n which can be expressed in the form:

$$a_n = \tilde{a} + \hat{a}(-1)^n. \quad (5)$$

There are here the autonomous particular case $a_n = 1$ and the non-autonomous one $a_n = (-1)^n$, and they generate all the other possible subcases as linear combinations. For instance, we can obtain

$$a_n = \frac{1 + (-1)^n}{2} = \begin{cases} 0, & n = 2k+1; \\ 1, & n = 2k. \end{cases} \quad (6)$$

For constructing $L - A$ for eq. (2) we use the fact that the symmetry (4) is equivalent to a known system of two equations. That is eq.(3.13) of [Tsuchida 02] and up to addition of a point symmetry and up to point transformation

$$v_k \to (-1)^k v_k, \quad w_k \to (-1)^{k+1} w_k,$$

it can be written in the form

$$\begin{aligned} \frac{d}{dt_1} v_k &= (\alpha v_{k+1} - \beta v_{k-1})(v_k w_k - 1)(v_k w_{k+1} - 1) \\ \frac{d}{dt_1} w_k &= (\beta w_{k+1} - \alpha w_{k-1})(v_k w_k - 1)(v_{k-1} w_k - 1). \end{aligned} \tag{7}$$

The system (7) is integrable discretization of one of derivative nonlinear Schrödinger equations, introduced in [Ablowitz, Ramani, Segur 80, Gerdjikov and Ivanov 83].
Eq. (4) and the system (7) are related by

$$v_k = u_{2k,m}, \quad w_k = u_{2k-1,m}, \quad \alpha = -a_{2k}, \quad \beta = -a_{2k-1}. \tag{8}$$

Using transform (8), we are going to obtain $L - A$ pair for (4), rewriting a known $L - A$ pair for (7) given in [Tsuchida 02].

$L-A$ pairs

This $L-A$ pair is standard and is given by the following:

$$T_k\Phi_k = U_k\Phi_k, \quad D_{t_1}\Phi_k = V_k\Phi_k, \tag{9}$$

where Φ_k is two-component vector function, U_k, V_k is 2×2 matrices depending on spectral parameter, T_k: $T_k h_k = h_{k+1}$. Using (8), we obtain for eq. (4) an $L-A$ pair of a little bit different structure:

$$T_n^2\Psi_{n,m} = N_{n,m}\Psi_{n,m}, \quad D_{t_1}\Psi_{n,m} = A_{n,m}\Psi_{n,m}. \tag{10}$$

Here T_n is the $n-$shift, and 2×2 matrices $N_{n,m}, A_{n,m}$ read:

$$N_{n,m} = \begin{pmatrix} h_{n,m}(1-\lambda)-2\lambda & u_{n+1,m}(\lambda-1) \\ -2\lambda u_{n,m}h_{n+1,m} & h_{n+1,m}(\lambda-1) \end{pmatrix} \tag{11}$$

$$A_{n,m} = \begin{pmatrix} \begin{matrix} h_{n-1,m}\left(a_{n-1}u_{n+1,m}u_{n-2,m} + a_n\frac{\lambda-1}{\lambda+1}\right) \\ +a_{n-1}\frac{2\lambda}{\lambda-1} \end{matrix} & \begin{matrix} -a_n u_{n-1,m}\frac{\lambda-1}{\lambda+1} \\ -a_{n-1}u_{n+1,m} \end{matrix} \\ 2\lambda h_{n-1,m}\left(\frac{u_{n,m}a_n}{1+\lambda} + \frac{u_{n-2,m}a_{n-1}}{\lambda-1}\right) & \begin{matrix} h_{n,m}(a_n u_{n-1,m}u_{n+2,m} - a_{n-1}) \\ -a_n u_{n,m}u_{n-1,m}\frac{2\lambda}{1+\lambda} \end{matrix} \end{pmatrix} \tag{12}$$

The compatibility condition for eqs. (10) has in this case the form:

$$D_{t_1} N_{n,m} = (T_n^2 A_{n,m}) N_{n,m} - N_{n,m} A_{n,m}, \tag{13}$$

and this relation is equivalent to eq. (4). Passing to 4×4 matrices we can rewrite this $L - A$ pair in the standard form.

$$T_n \begin{pmatrix} \Psi_{n,m} \\ \Psi_{n+1,m} \end{pmatrix} = \begin{pmatrix} 0 & E \\ N_{n,m} & 0 \end{pmatrix} \begin{pmatrix} \Psi_{n,m} \\ \Psi_{n+1,m} \end{pmatrix},$$

$$D_{t_1} \begin{pmatrix} \Psi_{n,m} \\ \Psi_{n+1,m} \end{pmatrix} = \begin{pmatrix} A_{n,m} & 0 \\ 0 & A_{n+1,m} \end{pmatrix} \begin{pmatrix} \Psi_{n,m} \\ \Psi_{n+1,m} \end{pmatrix}.$$

As for the second symmetry (3), it is equivalent for any fixed $n \in \mathbb{Z}$ up to transformation $\widetilde{z}_m = i^m u_{n,m}$ to well-known equation, see [Y 06]:

$$\frac{dz_m}{dt_2} = \frac{z_{m+1} z_{m-1} + z_m^2}{z_{m+1} - z_{m-1}}. \qquad (14)$$

For such type of equations there exists linear problem :

$$T_m \Psi_{n,m} = M_{n,m} \Psi_{n,m}, \quad D_{t_2} \Psi_{n,m} = B_{n,m} \Psi_{n,m}, \qquad (15)$$

with 2×2 matrices $B_{n,m}(u_{n,m-1}, u_{n,m}, u_{n,m+1})$, $M_{n,m}(u_{n,m}, u_{n,m+1})$ and with the compatibility condition

$$D_{t_2} M_{n,m} = (T_m B_{n,m}) M_{n,m} - M_{n,m} B_{n,m}. \qquad (16)$$

Lax pair for eq. (14) and therefore for (3) can be obtained by direct calculation, using this ansatz.
First equations of (10,15) constitute Lax pair for discrete equation (2) if vector-functions $\Psi_{n,m}$ are the same and matrices $N_{n,m}, \ M_{n,m}$ are consistent up to a gauge transformation $\widetilde{M}_{n,m} = \Omega_{n,m+1}^{-1} M_{n,m} \Omega_{n,m}$ and change of the spectral parameter.

However, instead of searching for $\Omega_{n,m}$, we use the known matrix $N_{n,m}$ and relation

$$(T_n^2 M_{n,m})N_{n,m} = (T_m N_{n,m})M_{n,m} \tag{17}$$

in order to find the correct form of 2×2 matrix $M_{n,m}$:

$$M_{n,m} = \begin{pmatrix} \lambda & \frac{1-\lambda}{u_{n,m}+u_{n,m+1}} \\ \lambda(u_{n,m+1} - u_{n,m}) & \frac{\lambda(u_{n,m}-u_{n,m+1})}{u_{n,m}+u_{n,m+1}} \end{pmatrix}. \tag{18}$$

Now corresponding matrix $B_{n,m}$ defining Lax pair for eq. (3) is constructed by relation (16):

$$B_{n,m} = \frac{(-1)^n}{u_{n,m+1} + u_{n,m-1}} \begin{pmatrix} (1-\lambda)(u_{n,m-1} - u_{n,m}) & 1-\lambda \\ \lambda(u_{n,m}^2 - u_{n,m+1}^2) & \begin{array}{c} \lambda(u_{n,m} + u_{n,m+1}) \\ -(u_{n,m} + u_{n,m-1}) \end{array} \end{pmatrix}. \tag{19}$$

The discrete compatibility condition (17) can be rewritten in standard form:

$$(T_n \widetilde{M}_{n,m})\widetilde{N}_{n,m} = (T_m \widetilde{N}_{n,m})\widetilde{M}_{n,m} \tag{20}$$

in terms of 4×4 matrices $\widetilde{M}_{n,m}$, $\widetilde{N}_{n,m}$.

Conservation laws

If we transform vector-function $\Psi_{n,m}$ by some matrix $\Omega_{n,m}$:

$$\widetilde{\Psi}_{n,m} = \Omega_{n,m} \Psi_{n,m}$$

then matrices defining $L - A$ pairs (13,16,17) are transformed as follows:

$$\begin{aligned}
\widetilde{B}_{n,m} &= \Omega_{n,m}^{-1} B_{n,m} \Omega_{n,m} - \Omega_{n,m}^{-1} \partial_{t_2} \Omega_{n,m}, \\
\widetilde{M}_{n,m} &= \Omega_{n,m+1}^{-1} M_{n,m} \Omega_{n,m}, \\
\widetilde{N}_{n,m} &= \Omega_{n+2,m}^{-1} N_{n,m} \Omega_{n,m}, \\
\widetilde{A}_{n,m} &= \Omega_{n,m}^{-1} A_{n,m} \Omega_{n,m} - \Omega_{n,m}^{-1} \partial_{t_1} \Omega_{n,m}.
\end{aligned} \tag{21}$$

We want to find matrix $\Omega_{n,m}$ which diagonalize the matrix of differential operator $B_{n,m}$ or $A_{n,m}$ We follow the formal diagonalization scheme developed in [Drinfel'd and Sokolov 84] for partial differential equations.

We are going to use the following Lemma taken from [Drinfel'd and Sokolov 84]:

Lemma

If the matrix $\frac{\partial B_{n,m}}{\partial \lambda}$ has different eigenvalues, then there exists formal series

$$\Omega_{n,m} = \Omega^{*}_{n,m} \left(E + \sum_{j=1}^{\infty} \lambda^{-j} \Omega_{n,m}^{(-j)} \right),$$

where E is unit matrix and $\Omega_{n,m}^{(-j)}, j \geq 1$, are anti-diagonal matrices, such that matrix $\widetilde{B}_{n,m}$ of the form:

$$\widetilde{B}_{n,m} = \lambda B_{n,m}^{(1)} + \sum_{j=0}^{\infty} \lambda^{-j} B_{n,m}^{(-j)},$$

related to $\Omega_{n,m}$ by the first of relations (21), has diagonal coefficients $B_{n,m}^{(l)}$.

As the matrix $\frac{\partial B_{n,m}}{\partial \lambda}$ has different eigenvalues, it can be transformed into the Jordan normal form which will be diagonal. That can be done by matrix $\Omega^*_{n,m}$ of the form

$$\Omega^*_{n,m} = \begin{pmatrix} 1 & 1 \\ u_{n,m} - u_{n,m-1} & u_{n,m} + u_{n,m-1} \end{pmatrix}. \qquad (22)$$

We obtain the matrix $B^{(1)}_{n,m}$:

$$B^{(1)}_{n,m} = \frac{\partial \widetilde{B}_{n,m}}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ 0 & (-1)^{n+1} \end{pmatrix}. \qquad (23)$$

For calculating matrices $\Omega^{(l)}_{n,m}$, $B^{(l)}_{n,m}$ we rewrite the first of eqs. (21) in the form

$$\Omega_{n,m}\widetilde{B}_{n,m} = B_{n,m}\Omega_{n,m} - \partial_{t_2}\Omega_{n,m}$$

and collect coefficients at the same powers of λ. A relation at λ^1 is satisfied identically due to the choice of $\Omega^*_{n,m}$. At the other powers of λ, we obtain some recurrence formulae for the other coefficients of $\widetilde{B}_{n,m}$ and $\Omega_{n,m}$, from which those coefficients are found explicitly.

Using eq. (16) and the formula (23) for $\frac{\partial \widetilde{B}_{n,m}}{\partial \lambda}$, we can prove by induction that matrix $\widetilde{M}_{n,m}$ is diagonal. From (21) we can find:

$$\widetilde{M}_{n,m} = -\lambda \frac{u_{n,m+1} + u_{n,m-1}}{u_{n,m} + u_{n,m+1}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{(u_{n,m+2}+u_{n,m+1})u_{n,m-1}-u_{n,m}(u_{n,m+1}+u_{n,m+2})}{(u_{n,m+2}+u_{n,m})(u_{n,m}+u_{n,m+1}} & 0 \\ 0 & \frac{u_{n,m+1}-u_{n,m}}{u_{n,m}+u_{n,m+2}} \end{pmatrix} + \ldots .$$

Using eq. (17) and the coefficient of $\widetilde{M}_{n,m}$ at λ, we can prove by induction that matrix $\widetilde{N}_{n,m}$ is diagonal. From (21) we can find:

$$\widetilde{N}_{n,m} = -\lambda \begin{pmatrix} 1 + u_{n+1,m}u_{n,m-1} & 0 \\ 0 & 1 - u_{n+1,m}u_{n,m+1} \end{pmatrix} - \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + u_{n,m-1}} \begin{pmatrix} 1 + u_{n+1,m}u_{n,m-1} & 0 \\ 0 & -1 + u_{n+1,m}u_{n,m+1} \end{pmatrix} + \ldots$$

Matrixes $\widetilde{M}_{n,m}$, $\widetilde{N}_{n,m}$ are diagonal and their coefficients are formal series in powers of λ^{-1}.

Eq. (17) for these matrixes can be rewritten as:

$$(T_n^2 - 1)\log \widetilde{M}_{n,m} = (T_m - 1)\log \widetilde{N}_{n,m}. \tag{24}$$

Here we suppose that

$$\log \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \log \alpha & 0 \\ 0 & \log \beta \end{pmatrix}.$$

The folowing diagonal elements can be represented as:

$$(\log \widetilde{M}_{n,m})_{1,1} = \log \lambda + \sum_{j=0}^{\infty} \lambda^{-j} p_{n,m}^{(j)}, \quad (\log \widetilde{N}_{n,m})_{1,1} = \log \lambda + \sum_{j=0}^{\infty} \lambda^{-j} q_{n,m}^{(j)},$$

and we obtain an infinite hierarchy of conservation laws:

$$(T_n^2 - 1)p_{n,m}^{(j)} = (T_m - 1)q_{n,m}^{(j)}, \quad j \geq 0. \tag{25}$$

These conservation laws can be expressed in the standard form:

$$(T_n - 1)p_{n,m} = (T_m - 1)q_{n,m} \tag{26}$$

as $T_n^2 - 1 = (T_n - 1)(T_n + 1)$.

First two conservation laws are given by:

$$
\begin{aligned}
p^{(0)}_{n,m} &= \log \frac{u_{n,m+1} + u_{n,m-1}}{u_{n,m} + u_{n,m+1}},\\
q^{(0)}_{n,m} &= \log(1 + u_{n+1,m}u_{n,m+1});\\
p^{(1)}_{n,m} &= \frac{(u_{n,m+1} + u_{n,m+2})(u_{n,m} - u_{n,m-1})}{(u_{n,m+2} + u_{n,m})(u_{n,m+1} + u_{n,m-1})},\\
q^{(1)}_{n,m} &= \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + u_{n,m-1}}.
\end{aligned}
$$

From second diagonal element we obtain the same in a sense (see explanation below) conservation laws except for the first step, where we get:

$$
\begin{aligned}
\hat{p}^{(0)}_{n,m} &= \log \frac{u_{n,m} - u_{n,m+1}}{u_{n,m} + u_{n,m+2}},\\
\hat{q}^{(0)}_{n,m} &= \log(1 - u_{n+1,m}u_{n,m+1}).
\end{aligned}
$$

All conserved densities $p^{(j)}_{n,m}$ of this hierarchy of conservation laws (25) depend on a finite number of functions:

$$u_{n,m+k}, \;\; k \in \mathbb{Z}. \tag{27}$$

The matrix $A_{n,m}$ (12) has poles at the points $\lambda = \pm 1$, and we can carry out the formal diagonalization in terms of the formal series in powers of $\lambda + 1$ or $\lambda - 1$. The results will be the same and we restrict ourselves by the case $\lambda = -1$. In this case

$$\Omega_{n,m} = \begin{pmatrix} u_{n-1,m} & 1 \\ -h_{n-1,m} & u_{n,m} \end{pmatrix} \begin{pmatrix} 1 & \frac{\lambda+1}{2}\frac{1-h_{n,m}h_{n+1,m}}{u_{n+1,m}} + \dots \\ \frac{\lambda+1}{2} u_{n-3,m}h_{n-1,m}h_{n-2,m} + \dots & 1 \end{pmatrix},$$

$$\widetilde{N}_{n,m} = \begin{pmatrix} 2 + (\lambda+1)(h_{n,m}u_{n-1,m}u_{n+2,m} - h_{n-1,m} - 2) & 0 \\ 0 & -(\lambda+1)h_{n,m}h_{n+1,m} \end{pmatrix} + \dots$$

$$\widetilde{M}_{n,m} = \begin{pmatrix} 1 - (\lambda+1)\frac{u_{n,m}(1+u_{n,m+1}u_{n-1,m})}{u_{n,m}+u_{n,m+1}} & 0 \\ 0 & \frac{u_{n,m+1}-u_{n,m}}{u_{n,m+1}+u_{n,m}}\left(1 + (\lambda+1)\frac{h_{n-1,m}u_{n,m+1}}{u_{n,m+1}+u_{n,m}}\right) \end{pmatrix} + \dots.$$

At the order $(\lambda+1)^0$ we obtain the conservation law from previous series and at the orders $(\lambda+1)^1$, $(\lambda+1)^2$ the following conservation laws:

$$\begin{aligned}
\check{p}^{(1)}_{n,m} =& \frac{2h_{n-1,m}u_{n,m+1}}{u_{n,m+1}+u_{n,m}},\\
\check{q}^{(1)}_{n,m} =& u_{n-1,m}(u_{n+2,m}h_{n,m}-u_{n,m}).
\end{aligned}$$

$$\begin{aligned}
\check{p}^{(2)}_{n,m} =& 4h_{n-1,m}\left(\frac{h_{n-1,m}(u_{n+2,m}h_{n,m}+u_{n,m})}{u_{n,m}+u_{n,m+1}} - \frac{u^2_{n,m}h_{n-1,m}}{(u_{n,m}+u_{n,m+1})^2}\right)\\
&+4h_{n-1,m}-4u_{n-1,m}u_{n+2,m}h_{n-1,m}h_{n,m},\\
\check{q}^{(2)}_{n,m} =& 2u_{n-1,m}u_{n+4,m}h_{n,m}h_{n+1,m}h_{n+2,m}+(h_{n-1,m}h_{n+1,m}-u_{n-1,m}u_{n+2,m}\\
&-2h_{n-1,m}h_{n+1,m}-(h_{n+1,m}+2)^2.
\end{aligned}$$

In this way, we can construct a hierarchy of conservation laws of the form (25), where the functions $q^{(j)}_{n,m}$ depend on a finite number of functions

$$u_{n+k,m},\ k\in\mathbb{Z}. \tag{28}$$

Master symmetry and recursion operator

In this section we discuss the problem of construction of generalized symmetries for the discrete equation (2). There are two hierarchies of symmetries in $n-$ and $m-$directions.
For $m-$direction case we construct master symmetry. It generate the hierarchy of generalized symmetries in this direction.
For $n-$direction case we construct recursion operator.

As it has been said above, the symmetry (3) is equivalent to the known equation (14) of the Volterra type. Equations similar to (14) have master symmetries which generate for them generalized symmetries and conservation laws [Adler Shabat Yamilov 2000]. For such type of master symmetry, we first need to introduce a generalization of (3) depending on a parameter τ which will be the time of the master symmetry:

$$\frac{d}{dt_2^{(1)}} u_{n,m} = \frac{(-1)^n}{\cosh \tau} \frac{u_{n,m}^2 + u_{n,m-1} u_{n,m+1}}{u_{n,m-1} + u_{n,m+1}} + \tanh \tau \frac{u_{n,m}(u_{n,m+1} - u_{n,m-1})}{u_{n,m-1} + u_{n,m+1}} \tag{29}$$

Eq. (29) at $\tau = 0$ is coincides with eq. (3). Corresponding master symmetry reads:

$$\frac{d}{d\tau} u_{n,m} = m\Psi_{n,m}^{(1)}(\tau) = \Psi_{n,m}^{*}. \tag{30}$$

It generates the hierarchy of symmetries

$$\frac{d}{dt_2^{(j)}} u_{n,m} = \Psi^{(j)}_{n,m}(\tau) \tag{31}$$

by the rule:

$$\begin{aligned}\Psi^{(j+1)}_{n,m}(\tau) =& [\Psi^*_{n,m}, \Psi^{(j)}_{n,m}] = D_\tau \Psi^{(j)}_{n,m}(\tau) - D_{t_2^{(j)}} \Psi^*_{n,m} \\ =& \frac{\partial \Psi^{(j)}_{n,m}}{\partial \tau} + \sum_{k\in\mathbb{Z}} \left(\Psi^*_{n,m+k} \frac{\partial \Psi^{(j)}_{n,m}(\tau)}{\partial u_{n,m+k}} - \Psi^{(j)}_{n,m+k} \frac{\partial \Psi^*_{n,m}}{\partial u_{n,m+k}} \right).\end{aligned}$$

We can put $\tau = 0$ in eqs. (31) and obtain an hierarchy of generalized symmetries for eq. (3). These equations are generalized symmetries of discrete equation (2) too.
For example, for $j = 2, \tau = 0$:

$$\frac{d}{dt_2^{(2)}} u_{n,m} = \frac{(u_{n,m+2} - u_{n,m-2})(u^2_{n,m+1} - u^2_{n,m})(u^2_{n,m} - u^2_{n,m-1})}{(u_{n,m} + u_{n,m-2})(u_{n,m+1} + u_{n,m-1})^2(u_{n,m+2} + u_{n,m}).}$$

It can be checked by direct calculation that this equation is compatible not only with (3) but also with discrete equation (2).

In case of n-direction we provide the construction of generalized symmetries by using a recursion operator. Eq. (4) is related to the system (7), for which a recursion operator has been constructed by Mikhailov and his PhD student. Using the relation (8), we just rewrite that operator in the scalar form suitable for eq. (4).
It is convenient in this case to construct the recursion operator R in the form

$$R = H \circ S, \tag{32}$$

where H is a Hamiltonian operator and S is a symplectic one. These operators read:

$$S = (-1)^n \left(\frac{1}{h_{n,m}} T_n + \frac{1}{h_{n-1,m}} T_n^{-1} \right), \tag{33}$$

$$\begin{aligned} H = & h_{n,m} h_{n-1,m} (c_n u_{n+2,m} - c_{n-1} u_{n-2,m}) (T_n - 1)^{-1} (-1)^n u_{n,m} \\ & + (-1)^n u_{n,m} T_n (T_n - 1)^{-1} h_{n,m} h_{n-1,m} (c_n u_{n+2,m} - c_{n-1} u_{n-2,m}) \\ & - (-1)^n h_{n-1,m} h_{n,m} (c_n h_{n+1,m} T_n + c_{n-1} h_{n-2,m} T_n^{-1}), \end{aligned} \tag{34}$$

where c_n is arbitrary two-periodic n-dependent function.

These operators satisfy the following equations

$$\begin{aligned} \frac{dS}{dt_1} + S \circ f^*_{n,m} + f^{*\perp}_{n,m} \circ S = 0, \\ \frac{dH}{dt_1} = f^*_{n,m} \circ H + H \circ f^{*\perp}_{n,m}. \end{aligned} \tag{35}$$

Here operators $f^*_{n,m}$, $f^{*\perp}_{n,m}$ are expressed in terms of the right hand side $f_{n,m}$ of eq. (4):

$$\frac{d}{dt_1} u_{n,m} = f_{n,m}. \tag{36}$$

The discrete Frechet derivative $f^*_{n,m}$ of $f_{n,m}$ is given by

$$f^*_{n,m} = \sum_{k=-2}^{2} \frac{\partial f_{n,m}}{\partial u_{n+k,m}} T_n^k,$$

coressponding to $f^*_{n,m}$ adjoint operator $f^{*\perp}_{n,m}$ is defined by

$$f^{*\perp}_{n,m} = \sum_{k=-2}^{2} \frac{\partial f_{n+k,m}}{\partial u_{n,m}} T_n^k.$$

From (35) it follows that $R = H \circ S$ satisfies the following Lax equation

$$\frac{dR}{dt_1} = [f^*_{n,m}, R], \tag{37}$$

where $[A, B] = A \circ B - B \circ A$. All these formulae are standard and can be found e.g. in [Y 06].

The operator R satisfying (37) allows one to construct conservation laws and generalized symmetries for eq. (4). In particular, eq. (37) implies that equations

$$\frac{\partial}{\partial t_1^{(k)}} u_{n,m} = R^{k-1}(f_{n,m}) = f^{(k)}_{n,m}, \quad k \geq 2, \tag{38}$$

are generalized symmetries of (4,36). The case $k = 1$ corresponds to eq. (36) itself. For example

$$\begin{aligned} f^{(2)}_{n,m} =& \hat{f}^{(2)}_{n,m} - (c_n + c_{n-1}) f_{n,m} + (c_n a_{n-1} - c_{n-1} a_n)(-1)^n u_{n,m}, \\ \hat{f}^{(2)}_{n,m} =& h_{n,m} h_{n-1,m} (b_n h_{n+1,m} h_{n+2,m} u_{n+4,m} - b_{n-1} h_{n-2,m} h_{n-3,m} u_{n-4,m} \\ & + u_{n,m}(b_n u_{n+2,m} h_{n-2,m} u_{n-3,m} - b_{n-1} u_{n-2,m} h_{n+1,m} u_{n+3,m}) \\ & + (u_{n-1,m} h_{n,m} - u_{n+1,m})(b_n u^2_{n+2,m} - b_{n-1} u^2_{n-2,m}) \end{aligned}$$

The formula (38) allows one to construct symmetries of the form:

$$\frac{\partial}{\partial t_1^{(k)}} u_{n,m} = f_{n,m}^{(k)}(u_{n+2k,m}, u_{n+2k-1,m}, \ldots, u_{n-2k+1,m}, u_{n-2k,m}),$$

which have even orders $2k$. In [GY 12] it has been shown that the symmetry of the first order does not exist. Probably there are no generalized symmetries of odd orders in this case.
We can see that generalized symmetries constructed for eqs. (4) and (2) depend on arbitrary two-periodic n-dependent functions. The same is true the Hamiltonian and recursion operators. Such picture is unusual for scalar differential-difference equations like (4) and probably appears for the first time. From the viewpoint of systems similar to the Tsuchida system (7), this is the case of relativistic Toda type equations, where symmetries and operators depend on two parameters. Such properties of symmetries are discussed in [Adler Shabat Yamilov 2000] and the case of operators is discussed in [Yamilov 2007].

Hyperbolic systems

Here we derive some hyperbolic systems of equations together with their $L-A$ pairs from symmetries of the discrete equation (2). At first we consider two compatible symmetries of the form (4) with $a_n=\chi_n$ and $a_n=\chi_{n-1}$, where $\chi_n=(1+(-1)^n)/2$:

$$\begin{aligned} \partial_x u_{n,m} &= h_{n,m}h_{n-1,m}(\chi_n u_{n+2,m}-\chi_{n-1}u_{n-2,m}),\\ \partial_y u_{n,m} &= h_{n,m}h_{n-1,m}(\chi_{n-1} u_{n+2,m}-\chi_{n}u_{n-2,m}). \end{aligned} \tag{40}$$

We can obtain from (10,12) the following system of linear equations

$$D_x\Psi_{n,m}=A^{(1)}_{n,m}\Psi_{n,m},\quad D_y\Psi_{n,m}=A^{(2)}_{n,m}\Psi_{n,m},$$

where $A^{(1)}_{n,m}$ is $A_{n,m}$ with $a_n=\chi_n$ and $A^{(2)}_{n,m}$ is $A_{n,m}$ with $a_n=\chi_{n-1}$, which is compatible in virtue of the system (40).

Then we modify the matrices $A^{(1)}_{n,m}$ and $A^{(2)}_{n,m}$, expressing the functions $u_{n\pm 2,m}$ via $u_{n,m}, u_{n\pm 1,m}$ and $\partial_x u_{n,m}$ or $\partial_y u_{n,m}$. We can do that by using the following consequences of (40):

$$\begin{aligned} \chi_n u_{n+2,m} &= \frac{\chi_n \partial_x u_{n,m}}{h_{n,m} h_{n-1,m}}, \qquad \chi_{n-1} u_{n-2,m} = -\frac{\chi_{n-1}\partial_x u_{n,m}}{h_{n,m} h_{n-1,m}}, \\ \chi_{n-1} u_{n+2,m} &= \frac{\chi_{n-1} \partial_y u_{n,m}}{h_{n,m} h_{n-1,m}}, \qquad \chi_n u_{n-2,m} = -\frac{\chi_n \partial_y u_{n,m}}{h_{n,m} h_{n-1,m}}. \end{aligned} \tag{41}$$

To avoid an explicit dependence on n, we pass to odd or even values of n. Let $n = 2k-1$, and let us introduce the notations:

$$p = u_{2k-1,m}, \quad q = u_{2k,m}, \quad r = u_{2k-2,m}. \tag{42}$$

Then the matrices take the form:

$$A^{(1)} = \begin{pmatrix} \frac{qp_x}{1-pq} + \frac{2\lambda}{\lambda-1} & -q \\ \frac{2\lambda p_x}{(\lambda-1)(1-pq)} & 1-pq \end{pmatrix}, \tag{43}$$

$$A^{(2)} = \begin{pmatrix} \frac{(\lambda-1)(pr-1)}{1+\lambda} & \frac{(1-\lambda)r}{1+\lambda} \\ \frac{2\lambda p(pr-1)}{1+\lambda} & \frac{rp_y}{pr-1} - \frac{2\lambda pr}{1+\lambda} \end{pmatrix}.$$

Corresponding linear equations can be rewritten in the form

$$D_x\Psi = A^{(1)}\Psi, \qquad D_y\Psi = A^{(2)}\Psi,$$

and their compatibility condition reads:

$$D_xA^{(2)} - D_yA^{(1)} = [A^{(1)}, A^{(2)}].$$

This matrix equation is equivalent to the following hyperbolic system:

$$\begin{aligned} &\frac{\partial^2 \log p}{\partial x \partial y} + \frac{p_x p_y}{p^2(pq-1)(pr-1)} + (pq-1)(pr-1) = 0,\\ &(pr-1)q_y + qrp_y - r(pq-1)(pr-1) = 0,\\ &(pq-1)r_x + qrp_x + q(pq-1)(pr-1) = 0. \end{aligned} \tag{44}$$

So we have derived an integrable system of three hyperbolic equations together with its $L-A$ pair. If $u_{n,m}(x,y)$ is a common solution of eqs. (40), then (42) is a solution of this system for any k, m. The system (44) can be derived directly from eqs. (40) without using their $L-A$ pairs.
In the case $n = 2k$ we obtain the same hyperbolic system up to the transformation $x \leftrightarrow y$.

In both case the first equation of (44) can be written in the form:

$$\frac{\partial^2 \log u_{n,m}}{\partial x \partial y} + \frac{\partial_x u_{n,m} \partial_y u_{n,m}}{u_{n,m}^2 (u_{n,m} u_{n-1,m} - 1)(u_{n,m} u_{n+1,m} - 1)} + (u_{n,m} u_{n-1,m} - 1)(u_{n,m} u_{n+1,m} - 1) = 0, \tag{45}$$

and this is a $2 + 1$-dimensional lattice equation similar to two-dimensional Toda lattice. Any common solution $u_{n,m}(x, y)$ of eqs. (40) provide a particular solution of the lattice (45). The problem whether this lattice equation (45) is integrable remains open.

Let us now consider the following two symmetries:

$$\begin{aligned} \partial_x u_{n,m} &= h_{n,m} h_{n-1,m} (\chi_n u_{n+2,m} - \chi_{n-1} u_{n-2,m}), \\ \partial_z u_{n,m} &= (-1)^n \frac{u_{n,m}^2 + u_{n,m+1} u_{n,m-1}}{u_{n,m+1} + u_{n,m-1}}, \end{aligned} \tag{46}$$

which are compatible on solutions of the discrete eq. (2).
Corresponding auxiliary linear problem reads:

$$D_x \Psi_{n,m} = A^{(1)}_{n,m} \Psi_{n,m}, \quad D_z \Psi_{n,m} = B_{n,m} \Psi_{n,m},$$

where $A^{(1)}_{n,m}$ is defined above, while $B_{n,m}$ is given by (19).
The matrix $A^{(1)}_{n,m}$ is modified by using the same formulae (41), and in the matrix $B_{n,m}$ we exclude $u_{n,m-1}$ by using the second of eqs. (46). To avoid an explicit dependence on n, we pass to $n = 2k - 1$ and introduce the notations:

$$p = u_{2k-1,m}, \quad q = u_{2k,m}, \quad r = u_{2k-1,m+1}. \tag{47}$$

As a result we obtain the matrix $A^{(1)}$ given by (43) and the following matrix B :

$$B = \frac{1}{p-r}\begin{pmatrix} (\lambda-1)(p_z+p) & \frac{(1-\lambda)(p_z+r)}{p+r} \\ \lambda(p+r)(p_z+p) & \frac{(p-r)(p_z-p)}{p+r} - \lambda(p_z+r) \end{pmatrix}.$$

In this case we have the matrix relation

$$D_x B - D_z A^{(1)} = [A^{(1)}, B]. \tag{48}$$

It can be checked that (48) is equivalent to the following hyperbolic system:

$$\begin{gathered} \frac{\partial^2 \log p}{\partial x \partial z} + \frac{(p_z-p)(p-r)p_x}{p^2(p+r)(pq-1)} + \frac{(p_z+p)(p+r)(pq-1)}{(p-r)p} = 0, \\ (p^2-r^2)q_z - 2(qr-1)p_z - q(p^2+r^2) + 2r = 0, \\ (pq-1)r_x - (qr-1)p_x - (p+r)(pq-1)(qr-1) = 0. \end{gathered} \tag{49}$$

As we can choose the odd or even n and as we can exclude $u_{n,m+1}$ instead of $u_{n,m-1}$ in $B_{n,m}$, we are led here to four different hyperbolic systems. However, all these systems are equivalent up to some simple point transformations.

Thank You for Attention