Berezin(-Toeplitz) “quantization” - the harmonic case

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An attempt at motivation

Quantization = a recipe for passing (or trying to pass) from a classical system to (the) quantum system be means of the mapping

$$f \stackrel{Q}{\mapsto} Q_f$$

where $f \in C^\infty(\Omega)$ (or generally smooth functions on a symplectic manifold - a phase space), Q_f is an operator on a fixed (separable, infinite-dimensional) Hilbert space whose spectrum is usually interpreted as the possible outcomes of measuring a quantity (position, velocity, momentum, energy) represented by (the classical observable) f in an experiment (the G_f's are then called quantum observables). We usually insist that the mapping Q is subject to the following conditions:

1. The map Q is linear.

2. For any polynomial $\phi : \mathbb{R} \to \mathbb{R}$ we should have

$$Q_{\phi \circ f} = \phi(Q_f).$$

3. $[Q_f, Q_g] = -\frac{\mathrm{i}h}{2\pi} Q_{\{f,g\}}$, where

$$\{f, g\} := \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right).$$

An Example

The space $\mathcal{F}_h$ of functions that are *entire* on $\mathbb{C}^n$ and square-integrable with respect to

$$\mathrm{d}\mu_h(z) := \frac{1}{(\pi h)^n} \mathrm{e}^{-|z|^2/h} \mathrm{d}z = \left(\frac{\alpha}{\pi}\right)^n \mathrm{e}^{-\alpha|z|^2} \mathrm{d}z = \rho_h(z)\mathrm{d}z$$

with the "weight" $\rho_h(z) = \mathrm{e}^{-\alpha|z|^2}(\alpha/\pi)^n$, where $\alpha := \pi/h > 0$ (in which case the "classical limit" $h \to 0$ corresponds to $\alpha \to \infty$), $\mathrm{d}z$ being the Lebesgue volume measure on $\mathbb{C}^n$

Reproducing kernel:

$$K_h(x,y) = \mathrm{e}^{\langle x,y\rangle/h} = \mathrm{e}^{\alpha\langle x,y\rangle}$$

and the reproducing property reads:

$$f(x) = \int_{\mathbb{C}^n} f(y)K_h(x,y)\rho_h(y)\mathrm{d}y = \langle f, K_{h,x}\rangle,$$

where $K_{h,x} := K_h(\cdot, x)$. This space is usually called the **Segal-Bargmann** or **Fock** space.

Closely related to the Berezin and/or Berezin-Toeplitz deformation quantization is the Berezin transform defined for $f \in L^\infty(\mathbb{C}^n)$ by the formula

$$\begin{aligned} B_h f(x) &= \int_{\mathbb{C}^n} f(y) \frac{|K_h(x,y)|^2}{K_h(x,x)} \rho_h(y)\, \mathrm{d}y \\ &= \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} f(y)\, \mathrm{e}^{-\alpha|x-y|^2}\, \mathrm{d}y \\ &= \mathrm{e}^{\frac{\Delta}{4\alpha}} f, \end{aligned}$$

which is precisely the heat solution operator at $t = \frac{1}{4\alpha}$.

For this operator there is the well-established asymptotic expansion

$$\begin{aligned} B_h f(x) &= f(x) + \frac{\Delta f(x)}{4\alpha} + \frac{\Delta^2 f(x)}{2!(4\alpha)^2} + \dots \\ &= f(x) + \frac{h\Delta f(x)}{4} + \frac{h^2\Delta^2 f(x)}{2!4^2} + \dots \end{aligned}$$

as $\alpha \to \infty$ or, respectively, $h \to 0$.

This means that $B_h f$ is an *approximate identity* as $\alpha \to \infty$ $(h \to 0)$:

$$
\begin{aligned}
B_h &= I + \frac{Q_1}{\alpha} + \frac{Q_2}{\alpha^2} + \ldots \\
&= I + hQ_1 + h^2 Q_2 + \ldots, \qquad (1)
\end{aligned}
$$

where I is the identity operator and Q_j are generally certain differential operators. This neatly points to the fact that, in a sense, for $h \to 0$ the classical physics is recovered.

The possibility of expanding B_h in an asymptotic expansion of the form (1) applies even in much more general situations of domains Ω in $\mathbb{C}^n$ on which a strictly plurisubharmonic real-valued smooth function Φ defines a Kahler metric $g_{i\overline{j}}$ by

$$g_{i\overline{j}} = \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j}.$$

Then we have the associated volume element $\mathrm{d}\mu(z) = \det(g_{i\bar{j}})\mathrm{d}z$ and for any $h > 0$ there are the **weighted Bergman spaces** $L^2_{\mathrm{hol}}(\Omega, \mathrm{e}^{-\frac{\Phi}{h}}\mathrm{d}\mu) =: L_{\mathrm{hol},h}$ of all holomorphic functions in $L^2(\Omega, \mathrm{e}^{-\frac{\Phi}{h}}\mathrm{d}\mu)$ and also the corresponding reproducing kernels $K_h(x, y)$ as well as Berezin transforms B_h, and, in case $\Omega \subset \mathbb{C}^n$ is

1. smoothly bounded and strictly pseudoconvex and $e^{-\Phi}$ is a defining function for Ω [Bordemann, Meinrenken, Schlichenmaier (1994)], [Engliš (2002)], or

2. Ω is a bounded symmetric domain in $\mathbb{C}^n$ and e^{Φ} is the unweighted Bergman kernel of Ω [Berezin (1974)], [Borthwick, Lesniewski, Upmeier (1993)], [Coburn (1992)], or, of course

3. $\Omega = \mathbb{C}^n$ with $\Phi(z) = |z|^2$,

then, as $\alpha \to \infty$ ($h \to 0$), there are the asymptotic expansions of the form

$$B_h f \approx \sum_{j=0}^{\infty} \frac{Q_j f}{\alpha^j} = \sum_{j=0}^{\infty} h^j Q_j f.$$

The harmonic case

the weighted Bergman spaces $L^2_{\text{hol},h}$, however, have a "drawback" from the point of view of the preceding applications: their definition requires a holomorphic structure so that, as a matter of fact, only complex manifolds are involved. On the other hand, the Berezin transform as well as some other ingredients that are present in the theory of deformation quantization (e. g. Toeplitz operators) still make perfect sense also in any subspace of L^2 that possesses the reproducing kernel.

In this respect, it is of interest to find out whether or not, and, eventually, to what extent the analogues of the preceding facts can be pushed forward. Even the fragmentary results known up to now and their proofs in particular show that this line of investigation is (if not related directly to quantization) quite a challenging enterprise. Namely, for the **harmonic** (rather than holomorphic) Segal-Bargmann (Fock) space on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, i.e. the space

$$\mathcal{H}_h := \{f \in L^2(\mathbb{R}^{2n}, \mathrm{d}\mu_h) : \Delta f = 0\},$$

we have the following result:

Theorem 1 (M. Engliš, J. Math. Anal. Appl. 367 (2010)**).** *Let $\mathcal{R}$ be an operator given by the formula*

$$\mathcal{R} = \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right) = \sum_{j=1}^{m} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right),$$

where $z_j = x_j + \mathrm{i} y_j$, acting on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Then there are linear differential operators $R_0, R_1, \dots$ on $\mathbb{R}^{2n} \setminus \{0\}$ of the form

$$R_j = \sum_{\substack{k,l \geq 0 \\ k+2l \leq 2j}} \rho_{jkl} |y|^{2l-2j} \mathcal{R}^k \Delta^l$$

with some constants ρ_{jkl} (depending only on n), such

(Theorem 1, continued) that for any $y \neq 0$ and for any $f \in L^\infty(\mathbb{R}^{2n})$ smooth in a neighbourhood of y, the harmonic Berezin transform B_h^{harm} associated to the spaces $\mathcal{H}_h$ has the asymptotic expansion

$$B_h^{\text{harm}} f(y) = \sum_{j=0}^{\infty} \frac{R_j f(y)}{\alpha^j} \quad as\ \alpha \to \infty. \tag{2}$$

Furthermore, R_0 is the identity operator,

$$R_1 = \frac{\Delta}{4(2n-1)} + \frac{(4n-3)(n-1)}{2(2n-1)|y|^2}\mathcal{R} + \frac{n-1}{2(2n-1)|y|^2}\mathcal{R}^2,$$

and

$$B_h^{\mathrm{harm}} f(0) \approx \sum_{j=0}^{\infty} \frac{\Delta^j f(0)}{j!(4\alpha)^j} \quad as \ \alpha \to \infty. \tag{3}$$

(End of Th.1) Note that (2) doesn't actually reduce to (3) in case $y = 0$, (where even the operator R_1 is singular) which suggests that B_h^{harm} enjoys quite an abruptive behaviour at $y = 0$ (this is in fact a manifestation of the so called Stokes phenomenon).

SoP

As a first step, using some standard facts about spherical harmonics, an explicit formula for the reproducing kernel $K_h(x, y)$ of the harmonic Fock space

$$\mathcal{H}_h = L^2(\mathbb{R}^n, \mathrm{d}\mu_h), \quad \mathrm{d}\mu_h(x) = (\pi h)^{-n/2} \mathrm{e}^{-|x|^2/h} \mathrm{d}x$$

can be derived (note that this result is true for general $n \in \mathbb{N}$, not necessarily even, the specification to $\mathbb{R}^{2n}$ comes later).

Indeed, denote by $\mathbf{S}^{n-1} = \partial\mathbf{B}^n$ the unit sphere in $\mathbb{R}^n$ and let $\mathcal{Y}_k$ be the space of all harmonic polynomials on $\mathbb{R}^n$ homogeneous of degree k equipped $\mathcal{Y}_k$ with the inner product

$$\langle f, g\rangle_{\mathcal{Y}_k} := \int_{\mathbf{S}^{n-1}} f(\zeta)\,\overline{g(\zeta)}\,\mathrm{d}\sigma(\zeta),$$

where $\mathrm{d}\sigma$ stands for the normalized surface measure on $\mathbf{S}^{n-1}$. Now,

$$Y_k(x,y) = |x|^k|y|^k\left(\frac{n}{2}+k+1\right)\sum_{j=0}^{[k/2]}\frac{(-)^j 2^{k-2j}(\frac{n}{2})_{k-j-1}}{j!(k-2j)!}\left(\frac{\langle x,y\rangle}{|x||y|}\right)^{k-2j}, \;\; x,y\in\mathbb{R}^n,$$

where $(a)_k := a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the classical Pochhammer symbol. Since harmonic polynomials are dense in $\mathcal{H}_h$, each $f \in \mathcal{H}_h$ has the homogeneous decomposition

$$f = \sum_{k=0}^{\infty} f_k \qquad \text{(convergence in } \mathcal{H}_h\text{).} \tag{4}$$

By a quite routine but tedious calculations we can observe that in fact

$$\|f_k\|^2_{\mathcal{H}_k} = c_k \|f_k\|^2_{\mathbf{S}^{n-1}} \quad c_k := \left(\frac{n}{2}\right)_k h^k.$$

But of course, the proportionality of norms implies the proportionality of the corresponding products and using the decomposition (4) we thus have

$$\begin{aligned}\langle f, H_{h,x}\rangle_{\mathcal{H}_h} &= f(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty} \langle f_k, Y_{k,x}\rangle_{\mathbf{S}^{n-1}} \\ &= \sum_{k=0}^{\infty} \frac{\langle f_k, Y_{k,x}\rangle_{\mathcal{H}_h}}{c_k} = \langle f, \sum_{k=0}^{\infty} \frac{Y_{k,x}}{c_k}\rangle_{\mathcal{H}_h}.\end{aligned}$$

This in turn implies that

$$H_h(x,y) = \sum_{k=0}^{\infty} \frac{Y_k(x,y)}{c_k} = \sum_{k=0}^{\infty} \frac{Y_k(x,y)}{h^k \left(\frac{n}{2}\right)_k}. \tag{5}$$

Recalling that the $k-$th **Gegenbauer polynomial** C_k^ν is given by the formula

$$C_k^\nu(z) = \sum_{j=0}^{[k/2]} \frac{(-)^j (\nu)_{k-j}}{j!(k-2j)!}(2z)^{k-2j},$$

we have for all $k \geq 0$

$$Y_k(x,y) = |x|^k |y|^k \frac{\frac{n}{2}+k-1}{\frac{n}{2}-1} C_k^{\frac{n}{2}-1}\left(\frac{\langle x, y\rangle}{|x||y|}\right).$$

Let's introduce the complex number

$$z = \frac{\langle x, y\rangle + \mathrm{i}\sqrt{|x|^2|y|^2 - \langle x, y\rangle^2}}{h},$$

so that

$$z = r\mathrm{e}^{\mathrm{i}\phi} \quad \text{with } r = \frac{|x||y|}{h},\ \cos\phi = \frac{\langle x, y\rangle}{|x||y|}$$

and using the following formula from [Bateman-Erdélyi, 10.9 (17)]:

$$C_k^{\nu}(\cos\phi) = \sum_{j=0}^{k} \frac{(\nu)_j(\nu)_{k-j}}{j!(k-j)!}\mathrm{e}^{-\mathrm{i}(k-2j)\phi},$$

we obtain

$$\begin{aligned}\frac{Y_k(x,y)}{h^k(\frac{n}{2})_k} &= \frac{|x|^k|y|^k}{h^k(\frac{n}{2}-1)_k} C_k^{\frac{n}{2}-1}\left(\frac{\langle x,y\rangle}{|x||y|}\right)\\ &= \frac{r^k}{(\frac{n}{2}-1)_k}\sum_{j=0}^{k}\frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_{k-j}}{j!(k-j)!}\mathrm{e}^{-(k-2j)\mathrm{i}\phi}.\end{aligned}$$

Inserting this into (5) and making the change of variables from k to $l = k - j$, we end up with:

$$
\begin{aligned}
H_h(x,y) &= \sum_{k=0}^{\infty} \frac{r^k}{(\frac{n}{2}-1)_k} \sum_{j=0}^{k} \frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_{k-j}}{j!(k-j)!} \mathrm{e}^{-(k-2j)\mathrm{i}\phi} \\
&= \sum_{j,l=0}^{\infty} \frac{r^{j+l}}{(\frac{n}{2}-1)_{j+l}} \frac{(\frac{n}{2}-1)_j(\frac{n}{2}-1)_l}{j!(l)!} \mathrm{e}^{(j-l)\mathrm{i}\phi} \\
&= \Phi_2 \left(\begin{matrix} \frac{n}{2}-1 & \frac{n}{2}-1 \\ \multicolumn{2}{c}{\frac{n}{2}-1} \end{matrix} \middle| r\mathrm{e}^{\mathrm{i}\phi}, r\mathrm{e}^{-\mathrm{i}\phi} \right),
\end{aligned}
$$

where, of course

$$\Phi_2\left(\begin{matrix} a & & b \\ & c & \end{matrix}\,\middle|\, z,w\right) = \sum_{j,k=0}^{\infty} \frac{(a)_j(b)_k}{(c)_{j+k}j!k!} z^j w^j$$

is the hypergeometric function Φ_2 of two variables from Horn's list [Bateman-Erdélyi, 5.7.1]. So the point here is that in fact we have the following expression for the reproducing kernel $H_h(x,y)$:

$$H_h(x,y) = \Phi_2\left(\begin{matrix} \frac{n}{2}-1 & & \frac{n}{2}-1 \\ & \frac{n}{2}-1 & \end{matrix}\,\middle|\, \frac{t_1+\mathrm{i}t_2}{h}, \frac{t_1-\mathrm{i}t_2}{h}\right)$$

where

$$t_1 = \langle x, y \rangle, \quad t_2 = \sqrt{|x|^2|y|^2 - \langle x, y \rangle^2}.$$

The harmonic Berezin transform associated to the harmonic Fock space $\mathcal{H}_h$ takes the form

$$B_h f(x) = \frac{(\pi h)^{-n/2}}{H_h(x, x)} \int_{\mathbb{R}^n} f(y) |H_h(x, y)|^2 \mathrm{e}^{-|y|^2/h} \mathrm{d}y.$$

Now, to establish the asymptotic behaviour of B_h, it would be nice to know the behaviour of $H_h(x, y)$ as $h \to 0$.

This could be quite easily done for $x = y$, in which case we are essentially faced with the task to asymptotically expand the confluent hypergeometric function ${}_1F_1$. This yields

$$H_h(x,x) \approx \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(n-2)} \mathrm{e}^{|x|^2/h} \frac{|x|^{n-2}}{h^{\frac{n}{2}-1}} \sum_{j=0}^{\infty} \frac{(1-\frac{n}{2})_j (3-n)_j}{j!} \frac{h^j}{|x|^{2j}}$$

for $x \neq 0$. NOTE: $H_h(x,x) = 1$ for every h for $x = 0$. Unfortuntely, for $x \neq y$, no analogous asymptotic expansion seems to exist in the literature and the standard integral representation for Φ_2 :

$$\Phi_2\left(\begin{matrix} a & & b \\ & c & \end{matrix}\,\middle|\, z,w\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b-a)} \int\limits_{\substack{u,v\geq 0\\ u+v\leq 1}} u^{a-1}v^{b-1}(1-u-v)^{c-a-b-1}\mathrm{e}^{uz+vw}\,\mathrm{d}u\,\mathrm{d}v,$$

is of little use, since it is valid only for Re $a > 0$, Re $b > 0$, Re $(c-a-b) > 0$, while in our case we have $a = b = c = \frac{n}{2}-1$. This looks like troubles and the remedy is the following *ad hoc* contour representation:

for any $z, w \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ with Re $\alpha > 0$, it can be shown that

$$\oint_\gamma a^{2a-1}(a-1)^{\beta-2\alpha-1} \int_{-1}^{-1} (1-t^2)^{\alpha-1} \mathrm{e}^{\frac{1+t}{2}az+\frac{1-t}{2}aw} \mathrm{d}t \ \mathrm{d}a$$
$$= \frac{-\pi \mathrm{i} 4^\alpha \Gamma(\alpha)^2}{\Gamma(\beta)\Gamma(1+2\alpha-\beta)} \Phi_2 \left(\begin{matrix} \alpha & & \alpha \\ & \beta & \end{matrix} \middle| z, w \right).$$

Taking $\alpha = \beta$ we finally obtain

$$H_h(x,y) = \frac{\mathrm{i}}{\pi}\frac{n-2}{2^{n-1}}\oint_\gamma a^{n-3}(a-1)^{-\frac{n}{2}}\int_{-1}^{-1}(1-t^2)^{\frac{n}{2}-2}$$
$$\mathrm{e}^{a\frac{\langle x,y\rangle+\mathrm{i}tV(x,y)}{h}}\mathrm{d}t\ \mathrm{d}a,$$

where for the sake of brevity we write

$$V(x,y) = \sqrt{|x|^2|y|^2 - \langle x,y\rangle^2}.$$

From this representation, the asymptotic expansion for B_h can be established by some technical manipulations, but unfortunately at the cost that we restrict ourselves to the case of $n \geq 3$ even, i.e.

$$n = 2N + 2, \quad N = 1, 2, 3, \ldots .$$

(...)

The half-space

A few explicit formulas for reproducing kernels have been known for function spaces of harmonic functions square-integrable on other domains than the harmonic Fock space (with respect to some appropriately weighted measure), namely, let H be the half-space in $\mathbb{R}^{n+1}$:

$$H := \{(x, y); x \in \mathbb{R}^n \wedge y \in \mathbb{R}_+\}.$$

Denote by $\mathcal{H}_\alpha$ the Hilbert space of all harmonic functions that are square integrable with respect to the measure

$$y^\alpha \, \mathrm{d}x \, \mathrm{d}y,$$

where dx and dy are the standard Lebesgue measures on $\mathbb{R}^n$ and $\mathbb{R}_+$, respectively, and $\alpha \geq 0$ is an arbitrary nonnegative real number.

Again, by the volume version of the mean-value property for harmonic functions, the (linear) functional $e_{(x,y)} : \mathcal{H}_\alpha \ni f \mapsto f(x,y) \in \mathbb{C}$ is bounded for each fixed $(x,y) \in \mathbb{R}^n \times \mathbb{R}_+$ and therefore continuous. Thus there is the reproducing kernel $K_\alpha : H \times H \to \mathbb{C}$ with $K_\alpha(\cdot, h) \in \mathcal{H}_\alpha \ \forall h = (x,y) \in H$ and the reproducing property: for all $f \in \mathcal{H}_\alpha$ and $(x,y) \in H$,

$$
\begin{aligned}
f(x,y) &= \langle f, K_\alpha(\,\cdot\,;(x,y))\rangle_{L_2} \\
&\equiv \int_{\mathbb{R}^n} \int_0^\infty f(z,w)\overline{K_\alpha(z,w;x,y)}\; w^\alpha \,\mathrm{d}w \,\mathrm{d}z.
\end{aligned}
$$

Last but not least, *the harmonic Berezin transform* B_α is this time defined by the formula

$$B_\alpha f(x,y) := \frac{1}{K_\alpha(x,y;x,y)} \int_{\mathbb{R}^n} \int_0^\infty f(z,w) |K_\alpha(x,y;z,w)|^2 \, w^\alpha \, \mathrm{d}w \, \mathrm{d}z.$$

Now, in this setting we have the following result

Theorem 2 (JJ, J. Math. Anal. Appl. (2013))**.** *For any* $f \in L^\infty(H) \cap C^\infty(H)$,

$$(B_\alpha f)(x,y) \approx \sum_{j=0}^{\infty} \frac{R_j f(x,y)}{\alpha^j} \tag{6}$$

as $\alpha \longrightarrow +\infty$ *for every* $(x,y) \in H$, *where* R_j *are certain differential operators, with* $R_0 = I$ *and*

$$R_1 f(x,y) = y^2 \frac{\Delta f}{n}(x,y) + (1-n) y \frac{\partial f}{\partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y)$$

for arbitrary $(x,y) \in H$ *with* $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

The **proof** again uses some tricks and observations that help to simplify the situation. First we note that if for any function $f : H \longrightarrow \mathbb{C}$ and for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}_+$ we denote $f^{a,b}(x,y) := f(bx+a, by)$. Then we have the following lemma:

Lemma 1. *Let* $f : H \longrightarrow \mathbb{C}$, $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}_+$ *be given. Then*

$$(B_\alpha f)(a,b) = (B_\alpha f^{a,b})(0,1)$$

whose corollary will prove useful:

Corollary 1. *In proving Theorem 2 we can confine ourselves to the case* $(x, y) = (0, 1)$.

Proof. Pick $(a, b) \in H$ and suppose that $\left(B_\alpha f^{a,b}\right)(0, 1) \approx \sum_{j=0}^{\infty} \frac{R_j f^{a,b}(0,1)}{\alpha^j}$. Then, because $\left(B_\alpha f^{a,b}\right)(0, 1) = B_\alpha f(a, b)$, we see that

$$(B_\alpha f)(a, b) \approx \sum_{j=0}^{\infty} \frac{R_j f(a, b)}{\alpha^k}, \quad \alpha \longrightarrow +\infty,$$

as desired, with $R_j f(a, b) = R_j f^{a,b}(0, 1)$. $\square$

In view of the previous corollary, it's quite reasonable to adopt the notation $K_\alpha(x, y; 0, 1) := H_\alpha(x, y)$. Then, by means of Fourier transform, we can give an explicit formula for $H_\alpha(x, y)$ in the form:

$$H_\alpha(x, y) = \frac{1}{(2\pi)^n \Gamma(\alpha + 1)} \int_{\mathbb{R}^n} (2|\xi|)^{\alpha+1} \mathrm{e}^{-(y+1)|\xi|} \mathrm{e}^{i\xi\cdot x} \mathrm{d}\xi. \tag{7}$$

Using polar coordinates $\xi = r\zeta$ and some changes of variables, (7) can be transformed into

$$H_\alpha(x, y) = c_{\alpha,y} \int_0^\infty \int_{\mathbf{S}^{n-1}} r^{\alpha+n} \mathrm{e}^{(1-r)\alpha} \mathrm{e}^{\frac{i\alpha r\zeta\cdot x}{y+1}} \, \mathrm{d}\sigma(\zeta) \, \mathrm{d}r, \tag{8}$$

where $c_{\alpha,y} = \frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)(y+1)^{\alpha+n+1}\mathrm{e}^{\alpha}}$. Inserting this into the very definition of the Berezin transform (with the kernel given by (8)), we obtain

$$B_\alpha f(0,1) = \frac{1}{H_\alpha(0,1)} d_\alpha^2 \int_0^\infty \int_{\mathbf{R}^n} f(z,w) g_\alpha(w) h_\alpha(z,w)\, w^\alpha \,\mathrm{d}w\,\mathrm{d}z, \tag{9}$$

where

$$d_\alpha = \frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)\mathrm{e}^{\alpha}},$$

$$g_\alpha(w) = \frac{w^\alpha}{(w+1)^{2\alpha+2n+2}},$$

and

$$h_\alpha(z,w) =$$
$$= \int_0^\infty \int_0^\infty \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} r^n s^n \left(\frac{r}{\mathrm{e}^{r-1}} \frac{s}{\mathrm{e}^{s-1}}\right)^\alpha \mathrm{e}^{\frac{i\alpha(r\zeta - s\eta)\cdot z}{w+1}}$$
$$\mathrm{d}\sigma(\zeta)\ \mathrm{d}\sigma(\eta)\ \mathrm{d}s\ \mathrm{d}r.$$

Now a little trick: the orthogonal group $O(n)$ is a compact Lie group with a normalized left and right invariant Haar measure dg on $O(n)$ such that for every function F that is continuous on $\mathbf{S}^{n-1}$ we have

$$\int_{\mathbf{S}^{n-1}} F(\zeta)\ \mathrm{d}\sigma(\zeta) = \omega_n \int_{O(n)} F(ge_1)\ \mathrm{d}g, \tag{10}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, $g \in O(n)$ and $e_1 = (1, 0, \ldots, 0) \in \mathbf{S}^{n-1}$. Now we can apply (10) to some of the integrals from (9), namely:

$$\begin{aligned}
\mathcal{I}(\alpha, r, s, w) &:= \int_{\mathbf{R}^n} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} f(z, w) \mathrm{e}^{\frac{i\alpha(r\zeta - s\eta)\cdot z}{w+1}} \, \mathrm{d}\sigma(\zeta) \, \mathrm{d}\sigma(\eta) \, \mathrm{d}z \\
&= \omega_n^2 \int_{O(n)} \int_{O(n)} \int_{\mathbf{R}^n} f(z, w) \mathrm{e}^{\frac{i\alpha(rge_1 - she_1)\cdot z}{w+1}} \, \mathrm{d}z \, \mathrm{d}g \, \mathrm{d}h \\
&= \omega_n^2 \int_{O(n)} \int_{O(n)} \int_{\mathbf{R}^n} f(hz, w) \mathrm{e}^{\frac{i\alpha(rhge_1 - she_1)\cdot hz}{w+1}} \, \mathrm{d}z \, \mathrm{d}g \, \mathrm{d}h \\
&= \omega_n^2 \int_{O(n)} \int_{O(n)} \int_{\mathbf{R}^n} f(hz, w) \mathrm{e}^{\frac{i\alpha(rge_1 - se_1)\cdot z}{w+1}} \, \mathrm{d}z \, \mathrm{d}g \, \mathrm{d}h,
\end{aligned}$$

where in the third and fourth equality we respectively used the changes of variables $g \mapsto hg$, $z \mapsto hz$ and the fact that h preserves the inner product on $\mathbf{R}^n$. Moreover, denoting $f^*(t,w) := \int_{O(n)} f(hte_1, w)\ \mathrm{d}h = \frac{1}{\omega_n}\int_{\mathbf{S}^{n-1}} f(t\zeta, w)\ \mathrm{d}\sigma(\zeta)$, we can turn the last integral into

$$\mathcal{I}(\alpha, r, s, w) = \omega_n^2 \int_{O(n)} \int_{\mathbf{R}^n} f^*(|z|, w) \mathrm{e}^{\frac{i\alpha(rge_1 - se_1)\cdot z}{w+1}}\ \mathrm{d}z\ \mathrm{d}g.$$

Finally, using spherical coordinates $z = t\tau$ and (10) again, we obtain

$$\mathcal{I}(\alpha, r, s, w) = \omega_n^2 \int_0^\infty \int_{\mathbf{S}^{n-1}} \int_{O(n)} f^*(t, w) t^{n-1} \mathrm{e}^{\frac{i\alpha(rge_1 - se_1)\cdot t\tau}{w+1}}$$

$$\mathrm{d}g \ \mathrm{d}\sigma(\tau) \ \mathrm{d}t$$

$$= \ \omega_n \int_0^\infty \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} f^*(t, w) t^{n-1} \mathrm{e}^{\frac{i\alpha(r\zeta - se_1)\cdot t\tau}{w+1}} \ \mathrm{d}\sigma(\zeta) \ \mathrm{d}\sigma(\tau) \ \mathrm{d}t$$

The next step is to insert this last expression for the integral $\mathcal{I}(\alpha, r, s, w)$ into the formula (9) for $B_\alpha f(0, 1)$:

$$B_\alpha f(0,1) =$$

$$= \frac{\omega_n d_\alpha^2}{H_\alpha(0,1)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \frac{f^*(t,w) t^{n-1} r^n s^n}{(w+1)^{2n+1}} \left(\frac{w}{(w+1)^2} \frac{r}{\mathrm{e}^{r-1}} \frac{s}{\mathrm{e}^{s-1}} \mathrm{e}^{\frac{i(r\zeta - se_1)\cdot t\tau}{w+1}} \right)^\alpha \mathrm{d}\sigma(\zeta)\ \mathrm{d}\sigma(\tau)\ \mathrm{d}t\ \mathrm{d}r\ \mathrm{d}s\ \mathrm{d}w.$$

Finally, let's undo the polar coordinates, say, $r\zeta = y$ and $t\tau = z$ to obtain

$$
\begin{aligned}
&B_\alpha f(0,1) = \\
&= \frac{\omega_n d_\alpha^2}{H_\alpha(0,1)} \int_0^\infty \int_0^\infty \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{f^*(|z|,w)|y|s^n}{(w+1)^{2n+1}} \\
&\quad \left(\frac{w}{(w+1)^2} \frac{|w|}{\mathrm{e}^{|w|-1}} \frac{s}{\mathrm{e}^{s-1}} \mathrm{e}^{\frac{i(y-se_1)\cdot z}{w+1}} \right)^\alpha \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}s \, \mathrm{d}w.
\end{aligned}
$$

This is essentially an integral of the form

$$
\mathcal{J}(\alpha) = \int_\Omega F(x) \mathrm{e}^{\alpha S(x)} \, \mathrm{d}x,
$$

where we may take $\Omega := \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$, $x = (w,s,z,y) \in \Omega$, $F(x) := \frac{f^*(|z|,w)|y|s^n}{(w+1)^{2n+1}}$ and

$S(x) = \ln \frac{w}{(w+1)^2} + (\ln s + 1 - s) + \frac{i(w - se_1)\cdot z}{w+1} + (\ln|y| + 1 - |y|)$, to which the so called multidimensional Laplace method [Fedoryuk (1987)] applies and we thus obtain the following asymptotic expansion

$$\mathcal{J}(\alpha) \approx \frac{e^{\alpha S(x_0)}}{\alpha^{n+1}} \sum_{j=0}^{\infty} \frac{Q_j F(x_0)}{\alpha^j},$$

where the Q_j's are the differential operators mentioned above.

The ball

Consider the harmonic Bergman space $L^2_{harm}(\mathbb{B}^n, \mathrm{d}\mu^n_\alpha)$ on the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, consisting of all functions that are harmonic and square integrable with respect to the measure

$$\mathrm{d}\mu^n_\alpha(y) := c_\alpha(1 - |y|^2)^\alpha \mathrm{d}^n y, \qquad \alpha > -1,$$

where $\mathrm{d}^n y$ is the usual n-dimensional Lebesgue measure and the coefficient c_α is chosen so that $\mathbb{B}^n$ has measure 1, namely $c_\alpha = \frac{\Gamma(\alpha+\frac{n}{2}+1)}{\pi^{n/2}\alpha!}$t. Again, it is known that such a space possesses the reproducing kernel R_α, $R_\alpha : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{R}$ harmonic and square integrable in both arguments satisfying $\forall f \in L^2_{harm}\left(\mathbb{B}^n, \mathrm{d}\mu^n_\alpha\right)$, $\forall x \in \mathbb{B}^n$:

$$\int\limits_{\mathbb{B}^n} f(y) R_\alpha(x,y) \mathrm{d}\mu^n_\alpha(y) = f(x). \tag{11}$$

It was shown by C. Liu in 2007 that if $n = 2$ then for $f \in C(\overline{\mathbb{B}^2})$,

$$B_\alpha f \to f \qquad \text{as} \quad \alpha \to \infty.$$

This result can be extended to full asymptotic expansion, as stated in the following theorem:

Theorem 3 (**P. Blaschke**, (submitted))**.** *For* $x \neq 0$, $n > 1$, *and* $f \in C^\infty(\mathbb{B}^n)$, *there exist differential operators* $Q_i := Q_i\big(\Delta, x \cdot \nabla, |x|^2\big)$, *involving only the Laplace operator* Δ, *the directional derivative* $x \cdot \nabla$ *and the quantity* $|x|^2$, *such*

(Theorem 3, continued) that

$$(B_\alpha f)(x) := \int_{\mathbb{B}^n} f(y)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}\mathrm{d}\mu_\alpha^n(y) \approx \sum_{i=0}^{\infty} \frac{Q_i f(x)}{\alpha^i} \qquad (\alpha \to \infty),$$

where $Q_0 = 1$ and

$$Q_1 = \frac{n-2}{2} \cdot \frac{1-|x|^2}{|x|^2}\; x\cdot\nabla + \frac{(n-2)(1-|x|^2)^2}{4(n-1)\,|x|^2}(x\cdot\nabla)^2 + \\ + \frac{1}{4(n-1)}(1-|x|^2)^2\Delta.$$

Finally, for $x = 0$

$$(B_\alpha f)(0) \approx \sum_{i=0}^{\infty} \frac{\Delta^i f(0)}{4^i \left(\alpha + \frac{n}{2} + 1\right)_i} \qquad (\alpha \to \infty).$$

(end of theorem).

Note that, again, the Stokes phenomenon occurs! The proof goes making heavy use of **hypergeometric functions**. Unfortunately, it's too much complicated to be given here. The highlights are: if for a real (or complex) function f of a real argument we define its **hypergeometrization** by the series

$$ {}_pf_q\left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; t\right) := \sum_{m=0}^{\infty} \frac{t^m f^{(m)}(0)}{m!} \frac{(a_1)_m \dots (a_p)_m}{(c_1)_m \dots (c_q)_m}, $$

and for a real function $f(x)$ of a vector argument, $x \in \mathbb{R}^n$, $n > 1$ we define

$$ {}_pf_q\left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; x\right) := {}_pf_q\left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; tx\right)\Bigg|_{t=1}, $$

then the following formula for the "generalized reproducing property" can be given

Lemma 2. *For* $p \in \mathbb{N}_0$, $\beta \geq \alpha$ *and* $f \in C^p(\mathbb{B}^n)$ *such that* $\Delta f = 0$ *we have:*

$$\int_{\mathbb{B}^n} R_\alpha(x,y) f(y)(x\cdot y)^p \mathrm{d}\mu^n_\beta(y)$$

$$= \frac{p!}{2^p} \sum_{j+2l+m=p} \frac{|x|^{2(j+l)} (\tilde{\alpha})_j (2b)_j}{j!m!l!(\tilde{\beta})_{j+m+l}(b)_j}$$

$$_3((x\cdot\nabla)^m f)_3 \left(\begin{matrix} \tilde{\alpha}+j & 2b+j & b \\ \tilde{\beta}+j+l+m & b+j & 2b \end{matrix} ; x \right).$$

The rest can be already deduced from here but the details are quite messy and we shall omit it.

Concluding remarks

Thank you for your attention!