POSITIVE DEFINITE KERNELS AND QUANTIZATION

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<th>Hamiltonian Dynamical Systems</th>
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<td>• Symplectic manifold $(M, \omega)$</td>
<td>• $\mathcal{H}$ — Hilbert space</td>
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<td>• Hamiltonian flow $\sigma_t : M \to M$</td>
<td>• Unitary flow $U_t = e^{it\hat{F}}$</td>
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<td>• defined by Hamilton equation $X \lhd \omega = dF$, where $F \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$ is tangent to ${\sigma}_{t \in \mathbb{R}}$.</td>
<td>where $\hat{F}$ — a selfadjoint operator $\hat{F} : \mathcal{D}(\hat{F}) \to \mathcal{H}$ unbounded in general</td>
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Quantization

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Example

\((\mathcal{H}, e^{it\hat{F}}, \hat{F} : \mathcal{D}(\hat{F}) \to \mathcal{H}) \iff (M, \omega, F)\)

\[ \hat{F} = \int \lambda dE(\lambda) \] — selfadjoint operator with semisimple spectrum

Thus

- \(\mathcal{H} \cong L^2(\mathbb{R}, d\sigma)\) where \(d\sigma(\lambda) = \langle 0 | dE(\lambda) | 0 \rangle\) and \(|0\rangle\) — cyclic for \(\hat{F}\)
- \(|n\rangle := P_n(\hat{F})|0\rangle, n = 0, 1, \ldots\) — orthonormal basis in \(\mathcal{H}\), where \(P_n\) — orthogonal polynomials with respect to \(d\sigma\)
We assume the condition

\[ \limsup_{n \to \infty} \frac{\sqrt[n]{|\mu|}}{n} < +\infty \]

on the absolute moments

\[ |\mu|_n := \int_{\mathbb{R}} |\omega|^n d\sigma(\omega) = \frac{1}{P_0^2} \langle 0 | |F| |0 \rangle \]

of the operator \( \hat{F} \).
Then, there exists the open strip $\Sigma \subset \mathbb{C}$ in complex plane $\mathbb{C}$, which is invariant under the translations

$$\tau_t z := z + t$$

$t \in \mathbb{R}$ and such that the characteristic functions

$$\chi(s) = \int_{\mathbb{R}} e^{-i\omega s} d\sigma(\omega),$$

$s \in \mathbb{R}$, of the measure $d\sigma$ posses holomorphic prolongation $\chi_{\Sigma}$ on $\Sigma$. 
Hence, one has the positive definite kernel on $\Sigma$

$$K_\Sigma(\bar{z}, \nu) := \chi_\Sigma(\bar{z} - \nu).$$

The map $K_\Sigma : \Sigma \to \mathcal{H} \cong B(\mathbb{C}, \mathcal{H})$ defined by

$$K_\Sigma(z) := \sum_{n=0}^{\infty} \chi_n(z) |n\rangle$$

where

$$\chi_n(z) := \int e^{-iz\omega} P_n(\omega) d\sigma(\omega),$$

for $z \in \Sigma$, gives factorization

$$K_\Sigma(\bar{z}, \nu) = K_\Sigma(z)^* K_\Sigma(\nu)$$

of the kernel $K_\Sigma$. 
One has

$$e^{-it\hat{F}}\mathcal{K}_\Sigma(z) = \mathcal{K}_\Sigma(z + t).$$

The states $\mathcal{K}_\Sigma(z), z \in \mathbb{Z}$, span an essential domain $\mathcal{D}(\hat{F})$ of $\hat{F}$ and

$$\hat{F}\mathcal{K}_\Sigma(z) = i \frac{d}{dz}\mathcal{K}_\Sigma(z).$$

The function

$$F = (\log \circ \chi_\Sigma)'(\bar{z} - z).$$

and the vector field tangent to the translation flow $\tau(t)$

$$X = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

satisfy

$$X \perp \Omega_\Sigma = dF$$

for symplectic form

$$\Omega_\Sigma = i\partial\bar{\partial}(\log \circ K_\Sigma)(\bar{z}, z) = i(\log \circ \chi_\Sigma)''(\bar{z} - z)d\bar{z} \wedge dz.$$
Applying the geometric quantization to Hamiltonian system 
\((M = \Sigma, \omega = \Omega_\Sigma, F)\) we back to the initial quantum system 
\((\mathcal{H}, e^{it\hat{F}}, \hat{F} : \mathcal{D}(\hat{F}) \to \mathcal{H})\).
Positive definite kernels on the principal bundles

- \( P \) — a set
- \( V \) and \( \mathcal{H} \) — Hilbert spaces
- \( \mathcal{B}(V, \mathcal{H}) \) — Banach space of bounded linear operators from \( V \) into \( \mathcal{H} \)

**i)** The \( \mathcal{B}(V) \)-valued **positive definite kernels**, i.e. maps \( K : P \times P \rightarrow \mathcal{B}(V) \) such that for any finite sequences \( p_1, \ldots, p_J \in P \) and \( v_1, \ldots, v_J \in V \) one has

\[
\sum_{i,j=1}^{J} \langle v_i, K(p_i, p_j)v_j \rangle \geq 0,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( V \).

One has

\[
K(q, p) = K(p, q)^*
\]

for each \( q, p \in P \).
(ii) The maps $\mathcal{K} : P \to \mathcal{B}(V, \mathcal{H})$ satisfying the condition

$$\{ \mathcal{K}(p)v : p \in P \text{ and } v \in V \}^\perp = \{0\}.$$ 

(iii) The Hilbert spaces $\mathcal{K} \subset V^P$ realized by the functions $f : P \to V$ such that evaluation functionals

$$E_pf := f(p)$$

are continuous maps of Hilbert spaces $E_p : \mathcal{K} \to V$ for every $p \in P$. 
There exist functorial equivalences between the categories of the object defined above.

• Equivalence between (ii) and (iii) is given as follows. For \( \mathcal{R} : P \to B(V, \mathcal{H}) \) we define monomorphism of vector spaces \( J : \mathcal{H} \to V^P \) by

\[
J(\psi)(p) := \mathcal{R}(p)^*\psi,
\]

and

\[
\mathcal{R}(p) := E_p^*,
\]

where \( \psi \in \mathcal{H}, \ p \in P \).

• The passage from (ii) to (i) is given by

\[
K(q, p) := \mathcal{R}(q)^*\mathcal{R}(p).
\]

• In order to show the implication (i) \( \Rightarrow \) (iii) let us take vector subspace \( \mathcal{K}_0 \subset V^P \) consisting of the following functions

\[
f(p) := \sum_{i=1}^l K(p, p_i)v_i,
\]

defined for the finite sequences \( p_1, \ldots, p_l \in P \) and \( v_1, \ldots, v_l \in V \).
Due to positive definiteness of the kernel $K : P \times P \to \mathcal{B}(V)$ we define a scalar product between $g(\cdot) = \sum_{j=1}^{J} K(\cdot, q_j) w_j \in \mathcal{K}_0$ and $f \in \mathcal{K}_0$ as follows

$$\langle g | f \rangle := \sum_{i=1}^{I} \sum_{j=1}^{J} \langle K(p_i, q_j) w_j, v_i \rangle.$$  

We obtain $\mathcal{K} \subset V^P$ as a closure of $\mathcal{K}_0$ with respect to the norm given by the above scalar product.
Proposition

Let $P$ be a smooth manifold and $V$ a finite dimensional complex Hilbert space. Then the following properties are equivalent:

(a) The positive definite kernel $K : P \times P \to \mathcal{B}(V)$ is a smooth map.

(b) The map $\hat{K} : P \to \mathcal{B}(V, \mathcal{H})$ is smooth.

(c) The Hilbert space $\mathcal{K} \subset V^P$ defined in (iii) consists of smooth functions, i.e. $\mathcal{K} \subset C^\infty(P, V)$. 

From now let us assume that $P$ is a principal bundle

$$G \rightarrow P \xrightarrow{\pi} M$$

over the smooth manifold $M$ with some Lie group $G$ as the structural group. Additionally we introduce a faithful representation of $G$

$$T : G \rightarrow \text{Aut}(V)$$

in Hilbert space $V$ and suppose that positive definite kernel $K : P \times P \rightarrow \mathcal{B}(V)$ has equivariance property

$$K(p, qg) = K(p, q) T(g)$$

where $p, q \in P$ and $g \in G$. This property is equivalent to each of the following two properties

$$\mathcal{K}(pg) = \mathcal{K}(p) T(g)$$

and

$$f(pg) = T(g^{-1}) f(p)$$

for $f \in \mathcal{K}$. 
Using the action of $G$ on $P \times V$ defined by

$$P \times V \ni (p, v) \mapsto (pg, T(g^{-1})v) \in P \times V$$

one obtains the $T$-associated vector bundle

$$
\begin{array}{ccc}
V & \longrightarrow & \mathbb{V} \\
\downarrow \pi & & \downarrow \tilde{\pi} \\
M & & \\
\end{array}
$$

over $M$ with the quotient manifold $\mathbb{V} := (P \times V)/G$ as its total space.
Given $\pi(p) = m$, $\pi(q) = n$, we define by

$$K_T(m, n)([(p, v)], [(q, w)]) := \langle v, K(p, q)w \rangle,$$

the section

$$K_T : M \times M \longrightarrow pr_1^*\overline{V}^* \otimes pr_2^*V^*$$

of the bundle $pr_1^*\overline{V}^* \otimes pr_2^*V^* \to M \times M$.

The diagonal $K_T|_{\Delta}$ of the kernel $K_T$ determines positive semi-definite hermitian structure $H_K := K_T|_{\Delta}$ on the bundle $\tilde{\pi} : \tilde{\mathbb{V}} \to M$. 
One has $I : \mathcal{H} \to C^\infty(M, \mathbb{V})$ a linear monomorphism of vector spaces defined by

$$I(\psi)(\pi(p)) := [(p, \mathbb{K}(p)^*\psi)] = [(p, J(\psi)(p))].$$

Apart of hermitian structure $H_K$ the positive hermitian kernel $K$ defines on $P$ a $\mathcal{B}(\mathbb{V})$-valued differential one-form

$$\vartheta(p) := (\mathbb{K}(p)^*\mathbb{K}(p))^{-1}\mathbb{K}(p)^*d\mathbb{K}(p) = K(p, p)^{-1}d_qK(p, q)|_{q=p},$$

which satisfy

$$\vartheta(pg) = T(g^{-1})\vartheta(p)T(g)$$

and

$$\langle v, K(p, p)\vartheta(p)w \rangle + \langle \vartheta(p)v, K(p, p)w \rangle = d\langle v, K(p, p)w \rangle.$$

Thus we conclude that $\vartheta \in C^\infty(P, T^*P \otimes \mathcal{B}(\mathbb{V}))$ is the one-form of the metric connection $\nabla_K$ consistent with the hermitian structure $H_K$. 
One-parameter groups of automorphisms and prequantization

Let $\xi \in C^\infty(P, TP)$ be the vector field tangent to the flow of automorphisms $\tau : (\mathbb{R}, +) \to \text{Aut}(P, \vartheta)$ of the principal bundle

$$
\tau_t(pg) = \tau_t(p)g,
$$

where $g \in G$ and $p \in P$, which preserve the connection form $\vartheta$

$$
\tau^*_t \vartheta = \vartheta.
$$

Then one has

$$
\xi(pg) = DR_g(p)\xi(p),
$$

and

$$
\mathcal{L}_\xi \vartheta = 0,
$$

where $R_g(p) := pg$, $DR_g(p)$ is the derivative of $R_g$ at $p$ and $\mathcal{L}_\xi$ is Lie derivative with respect to $\xi$. 
The space of vector fields preserving connection we denoted by $\mathcal{E}_G^0 \subset C_G^\infty(P, TP)$.

For connection 1-form $\vartheta$ and the $DT(e)(g)$-valued pseudotensorial 0-form, i.e. $DT(e)(g)$-valued function such that

$$F(pg) = T(g^{-1})F(p)T(g),$$

one has

$$\Omega := D\vartheta = d\vartheta + \frac{1}{2}[\vartheta, \vartheta],$$

$$DF = dF + [\vartheta, F].$$
$C^\infty_G(P, DT(e)(g))$ — the space of $DT(e)(g)$-valued functions satisfying equivariance condition

Now let us investigate the Lie algebra $P_G$ which consists of pairs $(F, \xi) \in C^\infty_G(P, DT(e)(g)) \times C^\infty_G(P, TP)$ such that

$$\xi_L \Omega = DF \quad \iff \quad L_{\xi} \theta = D(F + \theta(\xi))$$

with the bracket $[\cdot, \cdot] : P_G \times P_G \rightarrow P_G$ defined for $(F, \xi), (G, \eta) \in P_G$ by

$$[(F, \xi), (G, \eta)] := ([F, G], [\xi, \eta]),$$

where

$$\{F, G\} := 2\Omega(\xi, \eta) + DG(\xi) - DF(\eta) + [F, G] =$$

$$= -2\Omega(\xi, \eta) + [F, G] = DG(\xi) + [F, G]$$

and $[\xi, \eta]$ is the commutator of vector fields.
• Let $\mathcal{E}_G$ be the Lie algebra of vector fields $\xi \in C^\infty_G(P, TP)$ for which exists $F \in C^\infty_G(P, DT(e)(g))$ such that $(F, \xi) \in \mathcal{P}_G$.

• Denote by $\mathcal{N}_G$ the set of $F \in C^\infty_G(P, DT(e)(g))$ such that $DF = 0$.

• The subspace $\mathcal{P}_G^0 \subset \mathcal{P}_G$ of such elements $(F, \xi) \in \mathcal{P}_G$ that $\xi \in \mathcal{E}_G^0$ and $F = F_0 - \vartheta(\xi)$, where $DF_0 = 0$.

Summing up we have

\[
\begin{align*}
0 & \rightarrow \mathcal{N}_G \xrightarrow{\iota_1} \mathcal{P}_G \xrightarrow{pr_2} \mathcal{E}_G \rightarrow 0, \\
0 & \rightarrow \mathcal{N}_G \xrightarrow{\iota_1} \mathcal{P}_G^0 \xrightarrow{pr_2} \mathcal{E}_G^0 \rightarrow 0,
\end{align*}
\]

where horizontal arrows form the exact sequences of Lie algebras and vertical arrows are Lie algebra monomorphisms.

\[
\iota_1(F) := (F, 0), \quad pr_2(F, \xi) := \xi.
\]
From now on we will assume that $M$ is a connected manifold and denote by $P(p)$ the set of elements of $P$ which one can join with $p$ by curves horizontal with respect to the connection $\vartheta$. By $G(p)$ we denote the subgroup $G(p) \subset G$ consisting of those $g \in G$ for which $pg \in P(p)$, i.e. $G(p)$ is the holonomy group based at $p$. Let us recall that for connected base manifold $M$ all holonomy groups $G(p)$ and their Lie algebras $\mathfrak{g}(p)$ are conjugated in $G$ and $\mathfrak{g}$, respectively. Recall also that Lie algebra $\mathfrak{g}(p)$ is generated by $\Omega_{p'}(X(p'), Y(p'))$, where $p' \in P(p)$ and $X(p'), Y(p') \in T_{p'}P$.

After these preliminary remarks we conclude that for $(F, \xi) \in \mathcal{P}_G$ the function $F$ takes values $F(p')$ in $\mathfrak{g}(p)$ if $p' \in P(p)$. In the special case if $F \in \mathcal{N}_G$, i.e. when $DF = 0$, function $F$ is constant on $P(p)$ and $F(p) \in DT(e)(\mathfrak{g}(p)) \cap DT(e)(\mathfrak{g}'(p))$, where $\mathfrak{g}'(p)$ is the centralizer of the Lie subalgebra $\mathfrak{g}(p)$ in $\mathfrak{g}$.
In order to describe the Lie algebra $\mathcal{P}_G^0$ we define the linear monomorphism $\Phi : \mathcal{E}_G^0 \to \mathcal{P}_G^0$ of Lie algebras by

$$\Phi(\xi) := (-\vartheta(\xi), \xi).$$

One has the decomposition

$$\mathcal{P}_G^0 = \iota_1(\mathcal{N}_G) \oplus \Phi(\mathcal{E}_G^0)$$

of $\mathcal{P}_G^0$ into the direct sum of Lie subalgebra $\Phi(\mathcal{E}_G^0)$ and ideal $\iota_1(\mathcal{N}_G)$ of central elements of $\mathcal{P}_G^0$. 
Now let us define the following Lie subalgebra

\[ \mathcal{H}_G^0 := D\pi(\mathcal{E}_G^0), \]

of \( C^\infty(M, TM) \), where \( D\pi : TP \to TM \) is the tangent map of the bundle map \( \pi : P \to M \).

We define the vector subspace \( \mathcal{F}_G^0 \subset C^\infty(G, DT(e)(g)) \times \mathcal{H}_G^0 \) consisting of such elements \( (F, X) \in C^\infty(G, DT(e)(g)) \times \mathcal{H}_G^0 \) which satisfy the condition (Hamilton equation)

\[ X^* \cdot \Omega = DF, \]

where \( X^* \) is the horizontal lift of \( X \) with respect to \( \vartheta \).

One has

\[ \xi = X^* - F^* \in \mathcal{E}_G^0, \]

where \( F^* \) is a vertical field defined by the function

\( F \in C^\infty(G, DT(e)(g)) \)
Proposition

One has the Lie algebras isomorphism between \( (\mathcal{E}_G^0, [\cdot, \cdot]) \) and \( (\mathcal{F}_G^0, \{\cdot, \cdot\}) \), where the Lie bracket of \((F, X), (G, Y) \in \mathcal{F}_G^0\) is defined by

\[
\{ (F, X), (G, Y) \} := (-2\Omega(X^*, Y^*) + [F, G], [X, Y]).
\]

The following exact sequence of Lie algebras has place

\[
0 \to \mathcal{N}_G \xrightarrow{\iota_1} \mathcal{F}_G^0 \xrightarrow{pr_2} \mathcal{H}_G^0 \to 0,
\]

where \(\iota_1(F) := (F, 0)\) and \(pr_2(F, X) := X\).
The integration of the horizontal part $\xi^h = X^*$ of $\xi \in \mathcal{E}_G^0$ gives the flow $\{\tau^h_t\}_{t \in \mathbb{R}}$ being the horizontal lift of the flow

$$\sigma : (\mathbb{R}, +) \longrightarrow \text{Diff}(M)$$

defined by the projection of $\{\tau_t\}_{t \in \mathbb{R}}$ on the base $M$ of the principal bundle $P$. The vector field $X \in \mathcal{H}_G^0$ is the velocity vector field of $\{\sigma_t\}_{t \in \mathbb{R}}$. 
The flow
\[ \tilde{\tau}_t[(p, v)] := [(\tau_t(p), v)] \]
defines
\[ (\tilde{\Sigma}_t \psi)(\pi(p)) := \tilde{\tau}_t \psi(\sigma_{-t} \circ \pi(p)) = \tilde{\tau}_t \psi(\pi(\tau_{-t}(p))) = \tilde{\tau}_t \psi(\pi(\tau^h_{-t}(p))), \]
where \( \psi \in C^\infty(M, V) \).

The generator \( Q_{(F,X)} \) of the one parameter group \( \tilde{\Sigma}_t \) is \( G \)-version of Kostant–Souriau prequantization operator
\[ Q_{(F,X)} := -(\nabla_X + \tilde{F}), \]
where \((F, X) \in F_G^0\) and
\[ \tilde{F}([(p, v)]) := [(p, F(p)v)]. \]
For 
\[ Q : \mathcal{F}_G^0 \longrightarrow \text{End}(C^\infty(M, \nabla)) \]

one has prequantization property

\[ [Q_{(F,X)}, Q_{(G,Y)}] = Q_{\{ (F,X),(G,Y) \}}. \]

In the non-degenerate case, i.e. when \((F,X)\) is defined by \(F\) we have

\[ [Q_F, Q_G] = Q_{\{F,G\}}, \]

where \(Q_F := Q_{(F,X_F)}\) and the bracket \(\{F, G\}\) is defined by

\[ \{F, G\} := -2\Omega(X_F^*, Y_G^*) + [F, G]. \]
Quantization

We will quantize those flows which preserve $\mathcal{B}(V)$-valued positive definite kernel $K$

$$K(\tau_t(p), \tau_t(q)) = K(p, q), \quad \text{for } p, q \in P \text{ and } t \in \mathbb{R}$$

i.e. $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, K) \subset \text{Aut}(P, \vartheta)$

Theorem
The flow $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, K)$ if and only if there exists an unitary flow $U_t : \mathcal{H} \to \mathcal{H}$ on the Hilbert space $\mathcal{H}$ such that

$$\mathcal{K}(\tau_t(p)) = U_t \mathcal{K}(p),$$

where the map $\mathcal{K} : P \to \mathcal{B}(V, \mathcal{H})$ satisfies conditions of the definition (ii) and factorizes the kernel $K(p, q) = \mathcal{K}(p)^* \mathcal{K}(q)$.

The unitary flow $\{U_t\}_{t \in \mathbb{R}}$ is defined by $\{\tau_t\}_{t \in \mathbb{R}}$ in a unique way.
Theorem

The vector space $\mathcal{H}_0 := \text{span}\{\mathcal{K}(p)(v), p \in P, v \in V\}$ is the essential domain of the generator $\hat{F}$, where $\hat{F}$ is generator of $U_t = e^{it\hat{F}}$.

One has the filtration

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \subset \mathcal{U}_\infty \subset D(\hat{F})$$

of the domain $D(\hat{F})$ of the operator $\hat{F}$ onto its essential domains, where

$$\mathcal{U}_l := \mathcal{U}_{l-1} + \hat{F}(\mathcal{U}_{l-1}), \quad \mathcal{U}_0 := \mathcal{H}_0.$$

This filtration is preserved by the flow $\{U_t\}_{t \in \mathbb{R}}$. Moreover

$$\hat{F}\mathcal{U}_l \subset \mathcal{U}_{l+1}$$

and

$$\mathcal{U}_\infty \subset D(\hat{F}^l),$$

for $l \in \mathbb{N} \cup \{0\}$. 
The following relations are valid

\[ U_t = l^{-1} \circ \tilde{\Sigma}_t \circ l \]

and

\[ \hat{F} = il^{-1} \circ Q_{(F,X)} \circ l. \]

One also has

\[ F(p) = i(\mathcal{R}(p)^* \mathcal{R}(p)^{-1} \mathcal{R}^*(p) \hat{F} \mathcal{R}(p)). \]
For the further investigation of \( \hat{F} \) we will describe its representation in a trivialization

\[
s_\alpha : \Omega_\alpha \to P, \quad \pi \circ s_\alpha = \text{id}_{\Omega_\alpha}
\]

of \( \pi : P \to M \), where \( \bigcup_{\alpha \in A} \Omega_\alpha = M \) is a covering of \( M \) by the open subsets.

We note that on \( \pi^{-1}(\Omega_\alpha) \) one has

\[
\Omega(p) = T(h^{-1}) \left( dv_\alpha(m) + \frac{1}{2} [v_\alpha(m), v_\alpha(m)] \right) T(h),
\]

\[
DF(p) = T(h^{-1}) (dF_\alpha(m) + [v_\alpha(m), F_\alpha(m)]) T(h),
\]

for \( p = s_\alpha(m)h \), where

\[
v_\alpha := s_\alpha^* v \quad \text{and} \quad F_\alpha := F \circ s_\alpha.
\]
We find that for $\xi = X^* - F^* \in \mathcal{E}_G^0$ and for $\varphi_\alpha := F_\alpha + \vartheta_\alpha(X)$ we have

$$\mathcal{L}_X \vartheta_\alpha = d\varphi_\alpha + [\vartheta_\alpha, \varphi_\alpha].$$

The positive definite kernel $K : P \times P \rightarrow \mathcal{B}(V)$ in the trivialization is described by

$$K_\alpha(m) := \mathcal{K} \circ s_\alpha(m),$$

$$K_{\alpha\beta}(m, n) := K_\alpha^*(m)K_\beta(n),$$

for $m \in \Omega_\alpha$ and $n \in \Omega_\beta$ and connection form by

$$\vartheta_\alpha(m) = (K_\alpha(m)^*K_\alpha(m))^{-1} K_\alpha(m)^* d\mathcal{K}_\alpha(m).$$
We find that

\[ i\hat{F}\mathcal{K}_\alpha(m)\nu = (X\mathcal{K}_\alpha)(m)\nu + \mathcal{K}_\alpha(m)\varphi_\alpha(m)\nu, \]  

(1)

where \( \nu \in V, \ m \in \Omega_\alpha \).

The selfadjointness of \( \hat{F} \) implies the following relation

\[ \mathcal{K}_\beta(n)^*(X\mathcal{K}_\alpha)(m) + (X\mathcal{K}_\beta)(n)^*\mathcal{K}_\alpha(m) + \mathcal{K}_\beta(n)^*\mathcal{K}_\alpha(m)\varphi_\alpha(m) + \varphi_\beta(n)^*\mathcal{K}_\beta(n) \equiv 0 \]

between the kernel map \( \mathcal{K}_\alpha : \Omega_\alpha \to \mathcal{B}(V, \mathcal{H}) \) and \( (F, X) \in \mathcal{F}_G^0 \).
In the $s_\alpha$-gauge section $I(\psi) \in C^\infty(M, \nabla)$ and $Q_{(F,X)}I(\psi)$ are given by

$$I(\psi)(m) = [(s_\alpha(m), K_\alpha^*(m)K_\beta(n)\nu)]$$

and by

$$(Q_{(F,X)}I(\psi))(m) = il(\hat{F}\psi)(m) = [(s_\alpha(m), K_\alpha^*(m)\hat{F}K_\beta(n)\nu)]$$

respectively, $m \in \Omega_\alpha$. Hence we obtain the expression on $Q_{(F,X)}$ in terms of the kernel $K_{\bar{\alpha}\beta}(m, n)$:

$$Q_{(F,X)}(K_{\bar{\alpha}\beta}(\cdot, n))(m)\nu = -(XK_{\bar{\alpha}\beta})(\cdot, n)(m)\nu - \phi_\alpha(m)^* K_{\bar{\alpha}\beta}(m, n)\nu.$$


