Symmetries and properties of gauged and ungauged D=5 supergravities

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Plan of the talk

- General introduction in the mathematics of hidden symmetries
- Symmetries and properties of D=5 minimal ungauged supergravity
- Symmetries and properties of D=5 minimal gauged supergravity
- The Chong-Cvetič-Lü-Pope black hole as an example of D=5 minimal gauged supergravity
- Almost-BPS solutions in multi-center Taub-NUT

About spacetime symmetries and internal ones

- Hidden symmetries are characterized by Killing-Yano and Stäckel-Killing tensors
- The spacetime symmetries (isometries) are characterized by Killing vectors
- Both types of symmetries are important for solving the dynamics of the system (see example)

Motivation for the study of Killing-Yano and Stäckel-Killing tensors

- They are correlated with the internal symmetries of spacetimes, facilitating their study and the integration of the equations of motion on the respective geodesics
- They help factorize and solve the Dirac and Klein-Gordon equations on curved spacetimes
- They are naturally associated with supersymmetries and they bridge the classical and quantum symmetries
- They lead to a better understanding of the Physics of black holes of various types and their associated symmetries in an arbitrary number of dimensions
- They lead to the construction of some operators of Dirac type, which are conserved and construct new superalgebras

Motivation for the study of Killing-Yano and Stäckel-Killing tensors

- They signal the presence of supersymmetries in a semi-classical system (particle with spin, charged particle with spin) in curved spacetime
- They help construct Killing spinors, well-known for classifying vacuums of supergravity theories
- They lead to the understanding and discovery of new internal, "hidden" symmetries of physical systems

Definition of Killing-Yano tensors

Definition

A diferential form of order p, $Y \in \Omega^p(\mathcal{M})$ is a Killing-Yano tensor if $\nabla Y \in \Omega^{p+1}(\mathcal{M})$, i.e. $\nabla_\mu Y_{\alpha_1\alpha_2\cdots\alpha_p}$ is totally antisymmetric.

Remarks and conventions:

- $\mathcal{M}$ is a Riemannian manifold and ∇ is the associated Levi-Civita connection
- the Killing-Yano tensors depend on the metric of the manifold and we shall assume that the torsion of the connection is null and that $\nabla g = 0$

Definition of Stäckel-Killing tensors

Definition

A contravariant totally symmetric tensor K of rang p on a manifold $\mathcal{M}$ is a Stäckel-Killing tensor if $\nabla^{(\mu}K^{\alpha_1\alpha_2\cdots\alpha_p)} = 0$

Properties of Killing vectors - which are tensors of rank 1:

- Killing vectors are in one-to-one correspondence with continuous symmetries of the respective manifold
- The existence of each Killing vector assumes the existence of some conserved quantities, associated with a geodesic motion, such that the metric doesn't change in the direction of the Killing vector

Conformal Stäckel-Killing tensors and conformal Killing-Yano tensors

Definition

A totaly symmetric tensor is a conformal Stäckel-Killing tensor if it obeys the following equation:

$$K_{(\alpha_1\alpha_2\cdots\alpha_p;\beta)} = g_{\beta(\alpha_1}\tilde{K}_{\alpha_2\cdots\alpha_p)} \tag{1}$$

Definition

A totally antisymmetric tensor is a conformal Killing-Yano tensor if it obeys the following equation:

$$\nabla_{(\alpha_1}h_{\alpha_2)\alpha_3\cdots\alpha_{p+1}} = g_{\alpha_1\alpha_2}\tilde{h}_{\alpha_3\cdots\alpha_{p+1}} - (p-1)g_{[\alpha_3(\alpha_1}\tilde{h}_{\alpha_2)\cdots\alpha_{p+1}]} \tag{2}$$

Properties of the Dirac operator with torsion

Hence the Dirac operator with torsion is:

$$D^A_\mu \gamma^\mu = D_\mu \gamma^\mu + \frac{1}{12}\gamma^\mu \gamma^\nu \gamma^\rho A_{\mu\nu\rho}. \tag{3}$$

The square of the Dirac operator as a function of the torsion A is:

$$D^{2A} = -\Delta^A - \frac{dA}{4} - \frac{s}{4} - \frac{||A||^2}{24}, \tag{4}$$

where

$$\Delta^A = \nabla^A_{X_a} \nabla^A_{X^a} + \nabla^A_{\nabla^A_{X_a} X^a}, \tag{5}$$

and s is the scalar curvature of the connection with torsion-

$$s = -X^a \lrcorner R(X_a, X_b) e^b. \tag{6}$$

Dirac-type operators

We introduce a Dirac-type operator that anticommutes with the standard Dirac operator:

$$D_f = i\gamma^\mu (f_\mu{}^\nu \nabla_\nu - \frac{1}{6} f_{\mu\nu;\rho} \gamma^\nu \gamma^\rho), \tag{7}$$

where $f_{\mu\nu}$ are second rank Killing-Yano tensors.

Remarkable superalgebras of Dirac-type operators can be produced by second order Killing-Yano tensors that represent square roots of the metric tensor.

Killing spinors

The equation of symplectic Majorana 5-dimensional Killing spinors is:

$$D_\mu \epsilon_i = i M_{ij} \frac{a}{2} \gamma_\mu \epsilon_j \tag{8}$$

where

where $M = \vec{x}\vec{\sigma}$ with $\vec{\sigma}$ the Pauli matrices and

$$\vec{x} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta) \tag{9}$$

If a Kiliing spinor ψ exists for a (pseudo) Riemannian manifold then a Killing-Yano tensor of rank-p ω_p exists and it can be constructed as:

$$\omega_p(X_1 \cdots X_p) =< [X_1 \wedge \cdots \wedge X_p] \vee \psi, \psi > \tag{10}$$

N=1 D=5 ungauged supergravity - main findings

- supersymmetric solutions fall into two classes depending whether the Killing vector obtained from the Killing spinor is time-like or null
- the null case: the solution is a plane-fronted wave
- the time-like case: the solution is a hyper-Kähler manifold
- 1/2 supersymmetry survives in both cases
- an example is the 5 -dimensional charged rotating Beckenridge, Myers, Peet, Vafa black hole spacetime

Some properties of the timelike case

- there are CTCs (closed time-like curves) in the solution
- in fact the solution is similar to the 4-dimensional Gödel solution (which is a homogeneous solution with trivial $\mathbf{R}^4$ topology which contains CTCs through every point)
- if we use a Gibbons-Hawking base the most general solution is specified by 4 arbitrary harmonic functions on $\mathbf{R}^3$

The D=5 minimal gauged supergravity

- the solutions fall into 2 classes depending whether the Killing-vector is time-like or null
- the time-like solution: 4-dimensional Kähler base manifold with a U(2) structure and preserves 1/4 supersymmetry
- null case: preserves 1/4 of the supersymmetry and is fixed up to three harmonic functions

The Chong-Cvetič-Lü-Pope black hole

- it is a D=5 minimal supergravity solution, a rotating, charged non-extremal solution
- it is endowed with torsion: the Chern-Simons 2-form, *F, is assimilated with Killing-Yano torsion
- it is a solution characterized by four parameters: mass, charge, and 2 independent rotation parameters
- in Boyer-Lindquist coordinates this solution of D=5 minimal gauged supergravity is static (non-rotating) asymptotically

The Chong-Cvetič-Lü-Pope black hole metric

The metric is given by:

$$g = \sum_{\mu=x,y} (\omega^\mu \omega^\mu + \tilde{\omega}^\mu \tilde{\omega}^\mu) + \omega^\epsilon \omega^\epsilon, \tag{11}$$

$$A = \sqrt{3}(A_q + A_p). \tag{12}$$

And-

$$\omega^x = \sqrt{\frac{x-y}{4X}}dx, \qquad \tilde{\omega^x} = \frac{\sqrt{X}(dt + yd\phi)}{\sqrt{x(y-x)}}, \tag{13}$$

$$\omega^y = \sqrt{\frac{y-x}{4Y}}dy, \qquad \tilde{\omega^y} = \frac{\sqrt{Y}(dt + xd\phi)}{\sqrt{y(x-y)}}, \tag{14}$$

The metric

$$\omega^{\epsilon} = \frac{1}{\sqrt{-xy}}[\mu dt + \mu(x+y)d\phi + xyd\psi - yA_q - xA_p], \tag{15}$$

$$A_q = \frac{q}{x-y}(dt + yd\phi), \qquad A_p = \frac{-p}{x-y}(dt + xd\phi), \tag{16}$$

and

$$X = (\mu + q)^2 + Ax + CX^2 + \frac{1}{12}\Lambda x^3, \tag{17}$$

$$Y = (\mu + p)^2 + By + Cy^2 + \frac{1}{12}\Lambda y^3. \tag{18}$$

The generalized conformal Killing-Yano and Stäckel-Killing tensors of the spacetime

$$F = \sqrt{-x}\tilde{\omega}^x \wedge \omega^x + \sqrt{-y}\tilde{\omega}^y \wedge \omega^y \tag{19}$$

$$K^{(F)} = y(\omega^x\omega^x + \tilde{\omega}^x\tilde{\omega}^x) + x(\omega^y\omega^y + \tilde{\omega}^y\tilde{\omega}^y) + (x+y)\omega^\epsilon\omega^\epsilon. \tag{20}$$

$$h^{(\psi)}{}_{ab} = 4\omega_{[a}(\partial_\psi)_{b]} \tag{21}$$

$$K^\psi_{ab} = 16\omega_{[a}(\partial_\psi)_{c]}\omega_{[b}(\partial_\psi)^{c]} - 4g_{ab}(\omega_d\omega^d(\partial_\psi)_c(\partial_\psi)^c - \omega_d\omega^c(\partial_\psi)_c(\partial_\psi)^d), \tag{22}$$

$$h^{(\phi)}{}_{ab} = 4\omega_{[a}(\partial_\phi)_{b]}. \tag{23}$$

$$K^\phi_{ab} = 16\omega_{[a}(\partial_\phi)_{c]}\omega_{[b}(\partial_\phi)^{c]} - 4g_{ab}(\omega_d\omega^d(\partial_\phi)_c(\partial_\phi)^c - \omega_d\omega^c(\partial_\phi)_c(\partial_\phi)^d).$$

Dirac-type operators for this spacetime with Killing-Yano torsion

$$\hat{Q}_Y^A = \gamma^\mu Y_\mu{}^\nu D_\nu^A - \frac{1}{6}\gamma^\mu\gamma^\nu\gamma^\rho\nabla_\mu Y_{\nu\rho}. \tag{25}$$

with

$$\{\hat{Q}_Y^A, D^A\} = 0 \tag{26}$$

Hence there are no gravitational anomalies in this case. Same for the higher rank operators:

$$\hat{Q}_Y^{A,p} = \gamma^{\mu_1}\dots\gamma^{\mu_{p-1}} Y_{\mu_1\cdots\mu_{p-1}}{}^\nu D_\nu^A - \frac{(-1)^p}{2(p+1)}\gamma^\nu\gamma^{\mu_1}\dots\gamma^{\mu_p}\nabla_\nu Y_{\mu_1\cdots\mu_p}. \tag{27}$$

Killing spinor equation and its solution in D=5 minimal gauged supergravity

$$[D_\alpha + \frac{1}{4\sqrt{3}}(\gamma_\alpha^{\beta\gamma} - 4\delta_\alpha^\beta\gamma^\gamma)F_{\beta\gamma}]\epsilon^a - \chi\epsilon^{ab}(\frac{1}{4\sqrt{3}}\gamma_\alpha - \frac{1}{2}A_\alpha)\epsilon^b = 0, \qquad (28)$$

And the solution to this equation is:

$$\epsilon_i = (e^{\frac{i}{2}\gamma^i x_i M})_j{}^k (\delta_i^j x^\alpha(\gamma_\alpha{}^{\beta\delta} - \delta_\alpha^\beta\gamma^\delta)F_{\beta\delta} + \frac{i\epsilon^{jl}}{2}\chi x^\alpha\gamma_\alpha(M_{il} - i\delta_{il}A_\alpha\gamma^\alpha))\xi_k \qquad (29)$$

where $M = \vec{x}\vec{\sigma}$ with $\vec{\sigma}$ the Pauli matrices and

$$\vec{x} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta) \qquad (30)$$

and ϵ^{jki} and ϵ^{ij} are the Levi-Civita tensors and ξ_k is a symplectic Majorana spinor.

Almost-BPS solutions in multi-center Taub-NUT

- in string theory the challenge is to construct black hole microstates compatible with macroscopic classical black hole
- it is possible to have this compatibility between horizonless microstates and classical black holes of finite, pozitive entropy
- this thing can be understood via the bubbling and mergers of horizonless microstates in the fuzzball approach
- a lot of work has been done with two and three-charge black holes microstates in 4 and 5 dimensions

Almost-BPS solutions in multi-center Taub-NUT

- a classification of BPS, almost-BPS or non-BPS solutions via nilpotent orbits of simple Lie algebras was given in 2012 by Bossard and Ruef
- we construct almost-BPS solutions in 4 dimensions describing a system equivalent to an arbitrary number of extremal colinear BPS three charge black holes embedded in a multi-center Taub-NUT, colinear with one of the rotating non-BPS center of the Taub-NUT spacetime in N=2 supergravity.
- the two-center Taub-NUT in 4 dimensions is equivalent to a 5-dimensional black-ring wrapped on the fiber of Taub-NUT
- we build regular and horizonless microstates that can be used to test the 'fuzzball' hypothesis, which means we calculate the entropy of the BPS supertubes placed in Taub-NUT, in accordance with the entropy enhancement mechanism

Our physical problem

- two non-BPS centers $\bar{D}6$ -D2 rotating black holes at the centers of the Taub-NUT spacetime
- we place a series of colinear BPS D4-D2-D0 black holes, colinear with one of the Taub-NUT centers
- this almost-BPS system preserves $1/4$ supersymmetry
- there is $\vec{E}X\vec{B}$ interaction between the BPS centers included in the solution that we present here

The framework of our problem and its solutions

- We start with a supergravity theory in 11 dimensions in the context of M-theory with three M2 (electric) and three M5 (magnetic) charges.
- We compactify this theory to a 4 dimensional Taub-NUT and N=2 supersymmetry.

The Ansatz for the metric and the gauge field for the M2 charges is:

$$ds_{11}^2 = -(Z_1Z_2Z_3)^{-2/3}(dt+k)+(Z_1Z_2Z_3)^{1/3}ds_4^2+(\frac{Z_2Z_3}{Z_1^2})^{1/3}(dx_1^2+dx_2^2) + (\frac{Z_1Z_3}{Z_2^2})^{1/3}(dx_3^2+dx_4^2)+(\frac{Z_1Z_2}{Z_3^2})^{1/3}(dx_5^2+dx_6^2), \quad (31)$$

$$C^{(3)} = (a^1 - \frac{dt+k}{Z_1}) \wedge dx_1 \wedge dx_2 + (a^2 - \frac{dt+k}{Z_2}) \wedge dx_3 \wedge dx_4 + + (a^3 - \frac{dt+k}{Z_3}) \wedge dx_5 \wedge dx_6. \quad (32)$$

The framework of our problem and its solutions

So we find solutions in the Taub-NUT spacetime with the well-known metric:

$$d^2 s_4 = (V^m)^{-1}(d\psi + A) + V^m ds_3^2 \tag{33}$$

and a Gibbons-Hawking potential

$$V^m = h + \frac{q}{r} + \frac{q'}{r'}, \qquad A = q cos\theta d\phi, \qquad d^2 s_3 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{34}$$

The Almost-BPS equations

- The almost-BPS solutions are found by solving the following equations derived from heuristics and constraints on the 11 dimensional metric:

$$\Theta^{(I)} = - *_4 \Theta^{(I)}, \tag{35}$$

$$d *_4 dZ_I = \frac{C_{IJK}}{2} \Theta^{(I)} \wedge \Theta^{(I)}, \tag{36}$$

$$dk - *_4 dk = Z_I \Theta^{(I)} \tag{37}$$

where

- Z_I are warp-factors of the metric,
- $\Theta^{(I)} = da^I$ are the dipolar magnetic fields of the theory (M5),
- k is the orbital angular momentum of the system.

Notations

- $V^u = h + \frac{q}{r}$, the uni-center Taub-NUT potential.
- $V^m = h + \frac{q}{r} + \frac{q'}{r'}$, the multi-center Taub-NUT potential.
- $\Sigma_i = \sqrt{r^2 + a_i^2 - 2ra_i cos\theta}$.
- d_i are magnetic dipoles.
- $K^{(I)} = \sum_i \frac{d_i^{(I)}}{\Sigma_i}$, the harmonic functions characterizing dipolar magnetic charges.
- $L_I = l_I + \sum_i \frac{Q_i^{(I)}}{\Sigma_i}$, the harmonic functions characterizing electric fields.

The generic solutions

$$\Theta^{(i)} = d[K^{(i)}(d\psi + A) + b^{(I)}]. \tag{38}$$

$$Z_I = L_I + \frac{|\epsilon_{IJK}|}{2} \sum_{j,k} (h + \frac{qr}{a_j a_k} + \frac{q' r'}{a_j a_k}) \frac{d_j^{(J)} d_k^{(K)}}{\Sigma_j \Sigma_k}. \tag{39}$$

$$k = \mu(d\psi + A) + \omega. \tag{40}$$

The equations in angular momentum

We start from the equation:

$$dk - *_4 dk = Z_I \Theta^{(I)} \qquad (41)$$

and we get:

$$d(V^m \mu) + *_3 d\omega = V^m Z_I dK^{(I)} = V^m \sum_i l_I d_i^{(I)} d\frac{1}{\Sigma_i} +$$
$$+(h+\frac{q}{r}+\frac{q'}{r'})\sum_{i,j} Q_i^{(I)} d_j^{(I)} \frac{1}{\Sigma_i} d\frac{1}{\Sigma_j} + \frac{|\epsilon_{IJK}|}{2} \sum_{i,j,k} d_i^{(I)} d_j^{(J)} d_k^{(K)} [h^2 + \frac{hq}{r} + \frac{hqr}{a_j a_k} +$$
$$+ \frac{q^2}{a_j a_k} + \frac{hq'}{r'} + \frac{hq' r'}{a_j a_k} + \frac{qq' r}{a_j a_k r'} + \frac{qq' r'}{a_j a_k r} + \frac{q'^2}{a_j a_k}] \frac{1}{\Sigma_j \Sigma_k} d\frac{1}{\Sigma_i}. \qquad (42)$$

The complete solution in μ

$$\mu(r,r',\theta,\phi)=\sum_i(l_I d_i^{(I)}\mu_i^{(1)}+Q_i^{(I)}d_i^{(I)}(h\mu_i^{(2)}+q\mu_i^{(4)}))+$$
$$+\sum_{i\neq j}Q_i^{(I)}d_j^{(I)}(h\mu_{ij}^{(3)}+q\mu_{ij}^{(5)})+\sum_i Q_i^{(I)}d_i^{(I)}q'\mu_i^{(4')}+\sum_{i\neq j}Q_i^{(I)}d_j^{(I)}q'\mu_{ij}^{(5')}+$$
$$+\sum_{i,j,k}d_i^{(1)}d_j^{(2)}d_k^{(3)}(h^2\mu_{ijk}^{(6)}+(q^2+q'^2)\mu_{ijk}^{(7)}+hq\mu_{ijk}^{(8)})+$$
$$+\sum_{i,j,k}d_i^{(1)}d_j^{(2)}d_k^{(3)}(q'h\mu_{ijk}^{(9)}+qq'(\mu_{ijk}^{(10)}+\mu_{ijk}^{(11)}))+\mu_{shift}+\mu^{(12)}, \quad (43)$$

The complete solution in ω

$$\omega(r,r',\theta,\phi)=\sum_i(l_I d_i^{(I)}\omega_i^{(1)}+Q_i^{(I)}d_i^{(I)}q\omega_i^{(4)})+\sum_{i\neq j}Q_i^{(I)}d_j^{(I)}(h\omega_{ij}^{(3)}+q\omega_{ij}^{(5)})+$$
$$+\sum_i Q_i^{(I)}d_i^{(I)}q'\omega_i^{(4')}+\sum_{i\neq j}Q_i^{(I)}d_j^{(I)}q'\omega_{ij}^{(5')}+$$
$$+\sum_{i,j,k}d_i^{(1)}d_j^{(2)}d_k^{(3)}(h^2\omega_{ijk}^{(6)}+(q^2+q'^2)\omega_{ijk}^{(7)}+hq\omega_{ijk}^{(8)})+$$
$$+\sum_{i,j,k}d_i^{(1)}d_j^{(2)}d_k^{(3)}(q'h\omega_{ijk}^{(9)}+qq'(\omega_{ijk}^{(10)}+\omega_{ijk}^{(11)}))+\omega_{shift}+\omega^{(12)}. \quad (44)$$

The Killing-Yano and Stäckel-Killing tensors of this spacetime

- there are three Killing-Yano tensors, the two-form magnetic field strengths:

$$\Theta^I = d[\sum_i \frac{d_i^{(I)}}{\Sigma_i}(d\psi + q cos\theta d\phi) + \sum_i \frac{d_i^{(I)}}{\Sigma_i}(h(rcos\theta) - a_i) + \\ + q\frac{(r - a_i cos\theta)}{a_i})d\phi] \quad (45)$$

- the associated Stäckel-Killing tensor is actually the metric tensor of the spacetime

$$g_{\mu\nu} = \Theta^{(I)}_{\mu\rho}\Theta^{(I)\rho}{}_{\nu} \quad (46)$$

Conclusions

- We reviewed the definitions and some of the properties of : Stäckel-Killing, Killing-Yano tensors, Dirac-type operators and Killing spinors.
- We reviewed some results for D=5 supergravities in the ungauged and gauged minimal case.
- We presented some new results regarding the hidden symmetries of the Chong-Cvetič-Lü-Pope black hole in 5 dimensions.
- We reviewed the almost-BPS solution for a physical problem in multi-center Taub-NUT in 5 dimensions, a new result as well.