Darboux transformations, discrete integrable systems and related Yang-Baxter maps

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Outline

- Darboux transformations for NLS type equations and related discrete integrable systems (with A.Mikhailov and P.Xenitidis)
- Introduction to Yang-Baxter maps: Definitions, examples etc.
- Derivation of Yang-Baxter maps from Darboux matrices (with A. Mikhailov)
- Conclusions - Future work
- References

Darboux transformations and related discrete integrable systems

Lax operators and Darboux transformations

- We consider Lax operators of the form

$$\mathfrak{L} := D_x + U(p, q; \lambda), \quad p = p(x),\ q = q(x),$$

where U is a 2×2 matrix which belongs in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

- By *Darboux transformation* we understand a map

$$\mathfrak{L} \rightarrow M\mathfrak{L}M^{-1} = \mathfrak{L}_1 := D_x + \underbrace{U(p_{10}, q_{10}; \lambda)}_{U_{10}},$$

Matrix M is called *Darboux matrix* and satisfies the following equation

$$D_x M + U_{10} M - MU = 0.$$

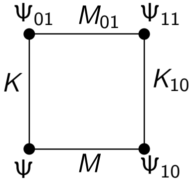
- Now, let $\Psi(x, \lambda)$ be a fundamental solution of the equation

$$\mathfrak{L}\Psi(x, \lambda) = 0.$$

- We can employ Darboux Matrices to derive two new fundamental solutions Ψ_{10} and Ψ_{01}:

$$\Psi_{10} = M(p, q, p_{10}, q_{10}; \lambda)\Psi \equiv M\Psi, \quad \Psi_{01} = M(p, q, p_{01}, q_{01}; \lambda)\Psi \equiv K\Psi$$

- A third one can be derived in the following



- The compatibility condition:

$$M_{01}K - K_{01}M = 0.$$

NLS equation

- We start with the Lax operator

$$\mathfrak{L}(p,q;\lambda) = D_x + \lambda\sigma_3 + \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix},$$

for the NLS equation

$$p_t = p_{xx} + 4p^2q, \quad q_t = -q_{xx} - 4pq^2.$$

- A Darboux transformation for $\mathfrak{L}$:

$$M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & p \\ q_{10} & 1 \end{pmatrix}.$$

Its entries obey the system of equations

$$\partial_x f = 2(pq - p_{10}q_{10}), \quad \partial_x p = 2(pf - p_{10}), \quad \partial_x q_{10} = 2(q - q_{10}f).$$

A first integral of the above system of differential equations is

$$\partial_x f = \partial_x (pq_{10}),$$

which implies that $\partial_x \det M = 0$.

- From the Darboux matrix

$$M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & p \\ q_{10} & 1 \end{pmatrix}.$$

we obtain a 2nd one by replacing $(f, q_{10}) \rightarrow (g, q_{01})$

$$K = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} g & p \\ q_{01} & 1 \end{pmatrix}.$$

- Their compatibility around the square, $M_{01}K = K_{01}M$, results to

$$\begin{aligned} & f_{01} - f - (g_{10} - g) = 0, \\ & f_{01}g - fg_{10} - p_{10}q_{10} + p_{01}q_{01} = 0, \\ & p\,(f_{01} - g_{10}) - p_{10} + p_{01} = 0, \\ & q_{11}\,(f - g) - q_{01} + q_{10} = 0. \end{aligned}$$

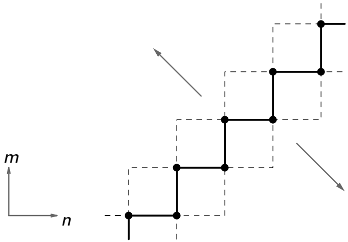
- This system can be soved either for $(p_{01}, q_{01}, f_{01}, g)$ or for $(p_{10}, q_{10}, f, g_{10})$

- In both cases it admits the trivial solution

$$p_{10} = p_{01}, \quad q_{10} = q_{01}, \quad f = g, \quad g_{10} = f_{01}.$$

- A non-trivial solution for $(p_{01}, q_{01}, f_{01}, g)$:

$$p_{01} = \frac{q_{10}p^2 + (g_{10} - f)p + p_{10}}{1 + pq_{11}}, \quad q_{01} = \frac{p_{10}{q_{11}}^2 + (f - g_{10})q_{11} + q_{10}}{1 + pq_{11}},$$
$$f_{01} = \frac{q_{11}(p_{10} + pg_{10}) + f - pq_{10}}{1 + pq_{11}}, \quad g = \frac{q_{11}(pf - p_{10}) + g_{10} + pq_{10}}{1 + pq_{11}}.$$



Derivative NLS equation (DNLS): $\mathbb{Z}_2$ reduction

- Consider the Lax operator

$$\mathfrak{L} = D_x + \lambda^2 \sigma_3 + \lambda \begin{pmatrix} 0 & 2\,p \\ 2\,q & 0 \end{pmatrix},$$

which is a Lax operator for DNLS equation

$$p_t = p_{xx} + 4\,(p^2\,q)_x, \quad q_t = -q_{xx} - 4\,(p\,q^2)_x.$$

- $\mathfrak{L}$ is invariant under the transformation

$$s_1(\lambda) : \mathfrak{L}(\lambda) \to \mathfrak{L}(-\lambda) = \sigma_3 \mathfrak{L}(\lambda) \sigma_3.$$

- A Darboux matrix for $\mathfrak{L}$

$$M := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ fq_{10} & 0 \end{pmatrix} + \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix},$$

where f, p and q_{10} satisfy

$$\begin{aligned} \partial_x p &= 2p(p_{10}q_{10} - pq) - \frac{2}{f}(c_2 p_{10} - c_1 p), \\ \partial_x q_{10} &= 2q_{10}(p_{10}q_{10} - pq) - \frac{2}{f}(c_1 q_{10} - c_2 q), \\ \partial_x f &= 2f(pq - p_{10}q_{10}). \end{aligned}$$

- The above system admits the 1st integral

$$\partial_x(f^2pq_{10} - c_2f) = 0.$$

- The compatibility of the Darboux matrices

$$M := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ fq_{10} & 0 \end{pmatrix} + \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix},$$

$$K := \lambda^2 \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & gp \\ gq_{01} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

implies the following system

$$\begin{aligned}
&fg_{10} - gf_{01} = 0, \\
&f_{01}q_{11} - fq_{10} - g_{10}q_{11} + gq_{01} = 0, \\
&f_{01}p_{01} - fp - g_{10}p_{10} + gp = 0, \\
&f_{01} - f - g_{10} + g - fg_{10}p_{10}q_{10} + gf_{01}p_{01}q_{01} = 0.
\end{aligned}$$

The above system admits a trivial solution and a non-trivial for $(p_{01}, q_{01}, f_{01}, g)$

$$
\begin{aligned}
p_{01} &= \frac{1}{f}\frac{A}{B^2}[g_{10}(p-p_{10})+fp(Aq_{10}-1)], \qquad f_{01}=f\frac{B}{A},\\
q_{01} &= \frac{1}{g_{10}}\frac{B}{A^2}[f(q_{11}-q_{10})+g_{10}q_{11}(Bp_{10}-1)], \quad g=g_{10}\frac{A}{B},
\end{aligned}
$$

where

$$A := (fp+g_{10}p_{10})q_{11}-1, \qquad B := (fq_{10}+g_{10}q_{11})p-1$$

Deformation of DNLS equation: $\mathbb{D}_2$ reduction group

- Consider the Lax operator

$$\mathfrak{L} = D_x + \lambda^2 \sigma_3 + \lambda \begin{pmatrix} 0 & 2\,p \\ 2\,q & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & 2\,q \\ 2\,p & 0 \end{pmatrix} - \frac{1}{\lambda^2} \sigma_3,$$

which is a Lax operator for the deformation of the DNLS equation

$$p_t = p_{xx} + 8\,(p^2\,q)_x - 4\,q_x, \quad q_t = -q_{xx} + 8\,(p\,q^2)_x - 4\,p_x.$$

- $\mathfrak{L}$ is invariant with respect to the following transformations

$$s_1(\lambda) : \mathfrak{L}(\lambda) \to \mathfrak{L}(-\lambda) = \sigma_3 \mathfrak{L}(\lambda) \sigma_3, \quad s_2(\lambda) : \mathfrak{L}(\lambda) \to \mathfrak{L}\left(\frac{1}{\lambda}\right) = \sigma_1 \mathfrak{L}(\lambda) \sigma_1.$$

- A Darboux transformation for $\mathfrak{L}$

$$M = f\left(\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} + \lambda \begin{pmatrix} 0 & p \\ q_{10} & 0 \end{pmatrix} + g \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & q_{10} \\ p & 0 \end{pmatrix}\right),$$

$$\begin{aligned} \partial_x p &= 2((p_{10} q_{10} - pq)p + (p - p_{10})g + q - q_{10}), \\ \partial_x q_{10} &= 2((p_{10} q_{10} - pq) q_{10} + p - p_{10} + (q - q_{10}) g), \\ \partial_x g &= 2((p_{10} q_{10} - pq) g + (p - p_{10}) p + (q - q_{10}) q_{10}), \\ \partial_x f &= -2(p_{10} q_{10} - pq)) f. \end{aligned}$$

- The above system admits two first integrals

$$\Phi_1 := f^2(g - pq_{10}) \qquad \Phi_2 := f^2(g^2 + 1 - p^2 - q_{10}^2).$$

- The Darboux matrices

$$M = f\left(\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^{-2}\end{pmatrix} + \lambda\begin{pmatrix}0 & p\\ q_{10} & 0\end{pmatrix} + g\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix}0 & q_{10}\\ p & 0\end{pmatrix}\right),$$

$$K = u\left(\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^{-2}\end{pmatrix} + \lambda\begin{pmatrix}0 & p\\ q_{01} & 0\end{pmatrix} + v\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix}0 & q_{01}\\ p & 0\end{pmatrix}\right).$$

- Their consistency around the square implies

$$
\begin{aligned}
g_{01} - g - v_{10} + v + p_{01}q_{01} - p_{10}q_{10} &= 0,\\
(g_{01} - v_{10})p + vp_{01} - gp_{10} + q_{01} - q_{10} &= 0,\\
(g - v)q_{11} + v_{10}q_{10} - g_{01}q_{01} - p_{01} + p_{10} &= 0,\\
g_{01}v - gv_{10} + p(p_{01} - p_{10}) + q_{11}(q_{01} - q_{10}) &= 0,
\end{aligned}
$$

and an equation for f and u

$$f_{01}u - u_{10}f = 0,$$

which using the first integrals can be written as

$$(S-1)ln(g - pq_{10}) = (T-1)ln(v - pq_{01}),$$

which can be easily verified that is a conservation law for the discrete system.

Introduction to Yang-Baxter maps

Quantum YB equation: Set-theoretical solutions

- The original Yang-Baxter (YB) equation (Quantum YB equation) contains a linear operator $Y : U \otimes U \to U \otimes U$ and has the form

$$Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12},$$

where Y^{ij} are maps

$$Y^{ij} : U \otimes U \otimes U \to U \otimes U \otimes U,$$

where U is a vector space.

- Drinfel'd, 1990: Set-theoretical solutions of YB equation
Replace

$$U \to A, \quad U \otimes U \to A \times A$$

Here A is an algebraic variety.

- Vesselov, 2003: Term "Yang-Baxter maps" for set-theoretical solutions of the YB equation.

Consider a set A and the map $Y : A \times A \to A \times A$,

$$Y : (x, y) \mapsto (u(x, y), v(x, y)),$$

and the functions $Y^{i,j} : A \times A \times A \to A \times A \times A$ for $i, j = 1, 2, 3, \; i \neq j$, given by

$$\begin{aligned} Y^{12}(x, y, z) &= (u(x, y), v(x, y), z), \\ Y^{13}(x, y, z) &= (u(x, z), y, v(x, z)), \\ Y^{23}(x, y, z) &= (x, u(y, z), v(y, z)), \end{aligned}$$

for $x, y, z \in A$.

Definition

The map Y is called Yang-Baxter map if it satisfies the YB equation

$$Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12}$$

Moreover, map Y is called reversible if

$$Y^{21} \circ Y^{12} = Id.$$

Definition

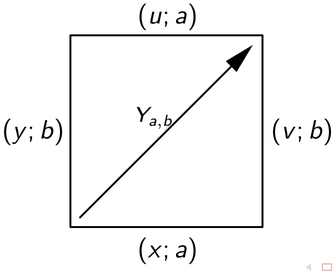
Parametric YB map is the following map

$$Y_{a,b} \equiv Y : (x, y; a, b) \mapsto (u(x, y; a, b), v(x, y; a, b)),$$

which satisfies the parametric YB equation

$$Y^{12}_{a,b} \circ Y^{13}_{a,c} \circ Y^{23}_{b,c} = Y^{23}_{b,c} \circ Y^{13}_{a,c} \circ Y^{12}_{a,b}.$$

We can represent a parametric YB map $Y_{a,b}(x, y) = (u, v)$ on the sides of a quadrilateral.



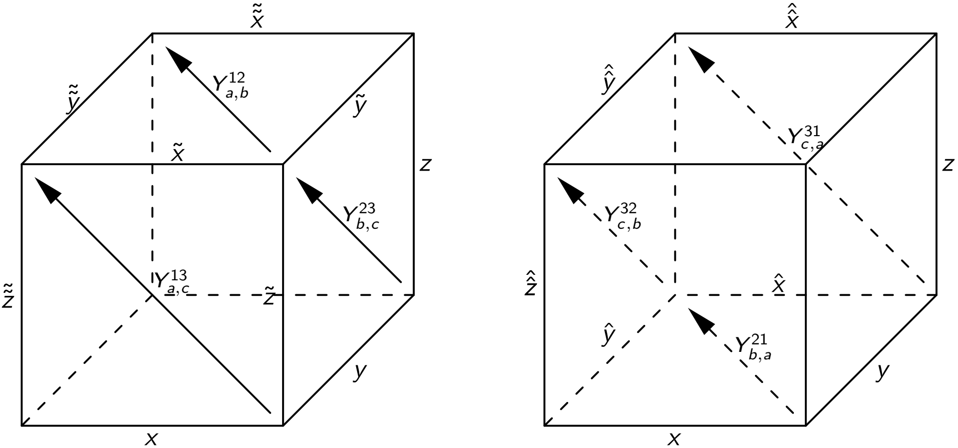
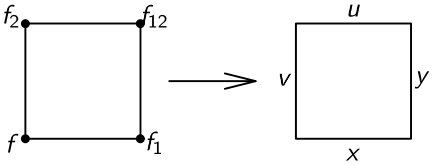


Figure: Representation of the YB equation $Y^{12}_{a,b} \circ Y^{13}_{a,c} \circ Y^{23}_{b,c} = Y^{23}_{b,c} \circ Y^{13}_{a,c} \circ Y^{12}_{a,b}$

Example: Adler's map

We can contruct YB maps from well-known discrete integrable equations. For instance, consider the discrete KdV equation

$$(f_{12} - f)(f_1 - f_2) + b - a = 0$$



We set the differences of the values on the corners to be the values on the sides of the quadrirateral.

$$x = f_1 - f, \;\; y = f_{12} - f_1, \;\; u = f_{12} - f_2, \;\; v = f_2 - f$$

Then,

$$x + y = u + v \;\; \text{and} \;\; (x + y)(x - v) = a - b.$$

Solving the above equations for (u, v) we obtain Adler's map

$$(x, y) \xrightarrow{Y_{a,b}} (y + \frac{a-b}{x+y}, x - \frac{a-b}{x+y})$$

which is a YB map.

Lax matrix for YB map

- Veselov and Suris (2003): Lax matrix for YB map.

Definition

A matrix $L = L(x; \alpha, \lambda)$ is a Lax matrix for the parametric YB map

$$Y_{a,b} : (x, y) \longmapsto (u(x, y; a, b), v(x, y; a, b))$$

if it satisfies the following Lax-equation

$$L(u; a, \lambda)L(v; b, \lambda) = L(y; b, \lambda)L(x; a, \lambda).$$

- If the Lax-equation is uniquely solvable for u, v, then the map $(x, y) \mapsto Y_{a,b}(x, y) = (u, v)$ is a reversible YB map (Vesselov, 2007).

- We use Darboux matrices as Lax matrices.

Example: Adler's map

For L given by the following matrix

$$L(x; a, \lambda) = \begin{pmatrix} x & 1 \\ x^2 + a - \lambda & x \end{pmatrix},$$

the Lax-equation

$$L(u; a, \lambda)L(v; b, \lambda) = L(y; b, \lambda)L(x; a, \lambda),$$

has a unique solution given by

$$u = y + \frac{a-b}{x+y}, \quad v = x - \frac{a-b}{x+y},$$

which is Adlers' map.

Derivation of Yang-Baxter maps using Darboux transformations

NLS equation

- Recall: Darboux matrix for NLS operator

$$M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & p \\ q_{10} & 1 \end{pmatrix}. \tag{1}$$

Its entries obey the system

$$\partial_x f = 2(pq - p_{10}q_{10}), \qquad \partial_x p = 2(pf - p_{10}), \qquad \partial_x q_{10} = 2(q - q_{10}f),$$

which admits the following first integral

$$\partial_x(f - pq_{10}) = 0.$$

- In correspondence with (1), define the matrix

$$M(\mathbf{x}, X; \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} X & x_1 \\ x_2 & 1 \end{pmatrix}$$

and substitute into the Lax-equation

$$M(\mathbf{u}, U; \lambda)M(\mathbf{v}, Y; \lambda) = M(\mathbf{y}, Y; \lambda)M(\mathbf{x}, X; \lambda)$$

to derive the following system of equations

$$v_1 = x_1, \ u_2 = y_2, \ U + V = X + Y, \ u_2 v_1 = x_1 y_2,$$
$$u_1 + Uv_1 = y_1 + x_1 Y, \ u_1 v_2 + UV = x_2 y_1 + XY, \ v_2 + u_2 V = x_2 + X y_2.$$

- Trivial solution

$$\mathbf{x} \mapsto \mathbf{u} = \mathbf{y}, \quad \mathbf{y} \mapsto \mathbf{v} = \mathbf{x}, \quad X \mapsto U = Y, \quad Y \mapsto V = X,$$

- Nontrivial solution

$$\begin{aligned} x_1 &\mapsto u_1 = \frac{y_1 + x_1^2 x_2 - x_1 X + x_1 Y}{1 + x_1 y_2}, & y_1 &\mapsto v_1 = x_1, \\ x_2 &\mapsto u_2 = y_2, & y_2 &\mapsto v_2 = \frac{x_2 + y_1 y_2^2 + y_2 X - y_2 Y}{1 + x_1 y_2}, \\ X &\mapsto U = \frac{y_1 y_2 - x_1 x_2 + X + x_1 y_2 Y}{1 + x_1 y_2}, & Y &\mapsto V = \frac{x_1 x_2 - y_1 y_2 + x_1 y_2 X + Y}{1 + x_1 y_2}, \end{aligned}$$

which statisfies the YB equation.

- Taking into account the first integral

$$f - pq_{10} = a \quad \Rightarrow \quad X - x_1 x_2 = a,$$

$$M(\mathbf{x}; a, \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a + x_1 x_2 & x_1 \\ x_2 & 1 \end{pmatrix}.$$

- The Lax-equation is uniquely solvable and it's equivalent to the Adler-Yamilov map

$$(\mathbf{x}, \mathbf{y}) \xrightarrow{Y_{a,b}} \left(y_1 - \frac{a-b}{1+x_1 y_2} x_1, y_2, x_1, x_2 + \frac{a-b}{1+x_1 y_2} y_2 \right). \qquad (2)$$

- Map (2) is completely integrable: Its invariants

$$I_1 = x_1 x_2 + y_1 y_2 + a + b,$$
$$I_2 = (a + x_1 x_2)(b + y_1 y_2) + x_1 y_2 + x_2 y_1 + 1,$$

which are in involution w.r.t.

$$\{x_1, x_2\} = \{y_1, y_2\} = 1, \qquad \text{and all the rest} \qquad \{x_i, y_j\} = 0.$$

- Invariant leaves

$$A_a = \{(x_1, x_2, X) \in \mathbb{R}^3; X = a + x_1 x_2\}, \quad B_b = \{(y_1, y_2, Y) \in \mathbb{R}^3; Y = b + y_1 y_2\}.$$

DNLS equation

- Recall: Darboux matrix for DNLS Lax operator

$$M := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ fq_{10} & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}. \qquad (3)$$

  Its entries obey the system

$$\begin{aligned} \partial_x p &= 2p(p_{10}q_{10} - pq) - \frac{2}{f}(p_{10} - cp), \\ \partial_x q_{10} &= 2q_{10}(p_{10}q_{10} - pq) - \frac{2}{f}(cq_{10} - q), \\ \partial_x f &= 2f(pq - p_{10}q_{10}), \end{aligned}$$

  which admits the first integral

$$\partial_x(f^2 pq_{10} - f) = 0.$$

- According to (3) we define the following matrix

$$M(\mathbf{x}, X; \lambda) = \lambda^2 \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the above matrix the refactorisation problem,

$$M(\mathbf{u},U;\lambda)M(\mathbf{v},V;\lambda)=M(\mathbf{y},Y;\lambda)M(\mathbf{x},X;\lambda),$$

is not uniquely solvable.

- One solution is the trivial map.
- A non-trivial solution is given by

$$\begin{aligned} x_1 &\mapsto u_1 = f_1(\mathbf{x},\mathbf{y},X,Y), & y_1 &\mapsto v_1 = f_2(\pi\mathbf{y},\pi\mathbf{x},Y,X),\\ x_2 &\mapsto u_2 = f_2(\mathbf{x},\mathbf{y},X,Y), & y_2 &\mapsto v_2 = f_1(\pi\mathbf{y},\pi\mathbf{x},Y,X),\\ X &\mapsto U = f_3(\mathbf{x},\mathbf{y},X,Y), & Y &\mapsto V = f_3(\pi\mathbf{y},\pi\mathbf{x},Y,X), \end{aligned}$$

where π is the *permutation function*, $\pi(x_1,x_2)=(x_2,x_1)$, and f_i, $i=1\ldots3$ are given by

$$f_1(\mathbf{x},\mathbf{y},X,Y) = \frac{(x_1+y_1)X - x_1Y - x_1x_2(x_1+y_1)}{X - x_1(x_2+y_2)},$$
$$f_2(\mathbf{x},\mathbf{y},X,Y) = \frac{X - x_1(x_2+y_2)}{Y - y_2(x_1+y_1)}y_2,$$
$$f_3(\mathbf{x},\mathbf{y},X,Y) = \frac{X - x_1(x_2+y_2)}{Y - y_2(x_1+y_1)}Y.$$

- We now restrict on the invariant leaves, using the first integral

$$f - f^2pq_{10} = a.$$

- Remark: We chose the variables of the Darboux matrix as $x_1 := fp$ and $x_2 := fq_{10}$, in order to avoid the branches when solving for f. Now,

$$f = a + (fp)(fq_{10}) \quad \Rightarrow \quad X = a + x_1x_2.$$

- The Darboux matrix now becomes

$$M(\mathbf{x};k;\lambda)=\lambda^2\begin{pmatrix}k+x_1x_2 & 0\\ 0 & 0\end{pmatrix}+\lambda\begin{pmatrix}0 & x_1\\ x_2 & 0\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

- For this matrix the refactorisation is unique and

$$M(\mathbf{x};\lambda)M(\mathbf{y};\lambda)=M(\mathbf{v};\lambda)M(\mathbf{u};\lambda),$$

  is equivalent to the following map

$$(\mathbf{x},\mathbf{y})\xrightarrow{Y_{a,b}}\left(y_1+\frac{a-b}{a-x_1y_2}x_1,\frac{a-x_1y_2}{b-x_1y_2}y_2,\frac{b-x_1y_2}{a-x_1y_2}x_1,x_2+\frac{b-a}{b-x_1y_2}y_2\right).$$

  Therefore, the above map is a reversible parametric YB map.

- The invariants of the YB map are given by

$$I_1(\mathbf{x},\mathbf{y})=(a+x_1x_2)(b+y_1y_2),\qquad I_2(\mathbf{x},\mathbf{y})=(x_1+y_1)(x_2+y_2)+a+b.$$

- The quantities $x_1 + y_1$ and $x_2 + y_2$ in I_2 are invariants themselves. The Poisson bracket in this case is given by

$$\{x_1, x_2\} = \{y_1, y_2\} = \{x_2, y_1\} = \{y_2, x_1\} = 1, \quad \text{and all the rest } 0.$$

  The rank of the Poisson matrix is 2, I_1 is one invariant and $C_1 := x_1 + y_1$ and $C_2 := x_2 + y_2$ are Casimir functions. The latter are preserved by the YB map, namely $C_i \circ Y_{a,b} = C_i$, $i = 1, 2$. Therefore, it is completely integrable.

- The 6−dimensional map is a restriction of the 4−dimensional YB map on the invariant leaves

$$A_a = \{(x_1, x_2, X) \in \mathbb{R}^3; X = a + x_1x_2\}, \quad B_b = \{(y_1, y_2, Y) \in \mathbb{R}^3; Y = b + y_1y_2\}.$$

Deformation of DNLS equation

- Recall: A Darboux transformation in this case

$$M = f\left(\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^{-2}\end{pmatrix} + \lambda\begin{pmatrix}0 & p\\ q_{10} & 0\end{pmatrix} + g\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix}0 & q_{10}\\ p & 0\end{pmatrix}\right).$$

Its entries obey

$$\begin{aligned}\partial_x p &= 2((p_{10}q_{10} - pq)p + (p - p_{10})g + q - q_{10}),\\ \partial_x q_{10} &= 2((p_{10}q_{10} - pq)q_{10} + p - p_{10} + (q - q_{10})g),\\ \partial_x g &= 2((p_{10}q_{10} - pq)g + (p - p_{10})p + (q - q_{10})q_{10}),\\ \partial_x f &= -2(p_{10}q_{10} - pq))f.\end{aligned}$$

- The above system admits two first integrals

$$\Phi_1 := f^2(g - pq_{10}) \qquad \Phi_2 := f^2(g^2 + 1 - p^2 - q_{10}^2).$$

- Here we cannot avoid the branches.
- We consider the matrix $N := fM$. Setting $x_1 := p, x_2 := q_{10}$ and $X := f^2$, and using Φ_1

$$N(\mathbf{x},X;c_1,\lambda)=\begin{pmatrix}\lambda^2X+x_1x_2X+c_1 & \lambda x_1X+\lambda^{-1}x_2X\\ \lambda x_2X+\lambda^{-1}x_1X & \lambda^{-2}X+x_1x_2X+c_1\end{pmatrix},$$

- The refactorisation is not unique. The Lax equation admits the trivial solution, and a non-trivial solution given by

$$x_1\mapsto u_1=\frac{f_1(\mathbf{x},\mathbf{y},X,Y;a,b)}{f_2(\mathbf{x},\mathbf{y},X,Y;a,b)},\qquad y_1\mapsto v_1=x_1$$

$$x_2\mapsto u_2=y_2,\qquad y_2\mapsto v_2=\frac{f_1(\pi\mathbf{y},\pi\mathbf{x},Y,X;b,a)}{f_2(\pi\mathbf{y},\pi\mathbf{x},Y,X;b,a)}$$

$$X\mapsto U=\frac{f_2(\mathbf{x},\mathbf{y},X,Y;a,b)}{f_3(\mathbf{x},\mathbf{y},X,Y;a,b)},\qquad Y\mapsto V=\frac{f_2(\pi\mathbf{y},\pi\mathbf{x},Y,X;b,a)}{f_3(\pi\mathbf{y},\pi\mathbf{x},Y,X;b,a)},$$

where f_1, f_2 and f_3 are given by

$$\begin{aligned} f_1(\mathbf{x},\mathbf{y},X,Y;a,b) = {} & a^2b^2x_1X + a^2b[x_2 - y_2 + 2x_1x_2y_1 + x_1^2(y_2 - 3x_2)]XY + \\ & a^2(y_2^2-1)[y_1(1+x_1^2) - x_1(1+y_1^2)]XY^2 - ab^2(x_1^2-1)(y_2-x_2)X^2 - \\ & ab(x_1^2-1)[x_2^2(3x_1-y_1) - x_1 - y_1 + 2y_2(y_1y_2 - x_1x_2)]X^2Y - \\ & a(x_1^2-1)(y_2^2-1)[y_2(y_1^2-1) + x_2(y_1^2 - 2x_1y_1 + 1)]X^2Y^2 + \\ & y_1(x_1^2-1)^2(x_2^2-1)(y_2^2-1)X^3Y^2 + b(x_1^2-1)^2(x_2^2-1)(y_2-x_2)X^3Y + \\ & a^3b(y_1-x_1)Y, \end{aligned}$$

$$\begin{aligned} f_2(\mathbf{x},\mathbf{y},X,Y;a,b) = {} & a^2b^2X + 2a^2by_2(y_1-x_1)XY + a^2(y_2^2-1)(x_1-y_1)^2XY^2 + \\ & 2ab(x_1^2-1)(1-x_2y_2)X^2Y + 2ax_2(x_1^2-1)(y_2^2-1)(x_1-y_1)X^2Y^2 + \\ & (x_1^2-1)^2(x_2^2-1)(y_2^2-1)X^3Y^2, \end{aligned}$$

$$\begin{aligned} f_3(\mathbf{x},\mathbf{y},X,Y;a,b) = {} & a^2b^2 - 2ab^2x_1(y_2-x_2)X - 2ab(x_1y_1-1)(y_2^2-1)XY \\ & b^2(x_1^2-1)(x_2-y_2)^2X^2 - 2by_1(x_2-y_2)(x_1^2-1)(y_2^2-1)X^2Y + \\ & (x_1^2-1)(y_1^2-1)(y_2^2-1)^2X^2Y^2. \end{aligned}$$

Conclusions

Conclusions - Future work

- We use recent classification results on automorphic Lie algebras (Bury, Mikhailov). In the case of 2×2 matrices there are essentially 3 different reduction groups:
  - The trivial group corresponding to the NLS equation
  - The $\mathbb{Z}_2$ reduction group corresponding to DNLS equation
  - The $\mathbb{D}_2$ reduction group corresponding to a deformation of the DNLS equation
- We derive Darboux transformations related to the Lax operators of all the above cases.
- We use the corresponding Darboux matrices:
  - to derive discrete integrable systems,
  - to derive $6-$dimensional YB maps.
- $6-$dimensional maps can be reduced to $4-$dimensional completely integrable discrete maps on invariant leaves.
- The above results can be extended on Grassmann algebras (joint work with G. Grahovski and A. Mikhailov - in preparation)

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Thank you!