*XVI International Conference*
*Geometry, Integrability and Quantization*
*6 – 11 June, 2014, Varna, Sofia*

# Riemann-Hilbert Problems and new Soliton Equations

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*It is my pleasure to congratulate Professor Jan Slawianowski for his 70-th birthday!*

# PLAN

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# Based on:


- V. S. Gerdjikov, D. J. Kaup. *Reductions of* $3\times3$ *polynomial bundles and new types of integrable* 3*–wave interactions.* In Nonlinear evolution equations: integrability and spectral methods, Ed. A. P. Fordy, A. Degasperis, M. Lakshmanan, Manchester University Press, (1981), p. 373–380

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- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).

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*and* $\mathbb{D}_h$ *reductions: conservation laws and generating operators.* J. Geom. Symmetry Phys. **31**, 57–92 (2013).

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# RHP with canonical normalization

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^+(x,t,\lambda) = \mathbb{1},$$

$\xi^\pm(x,t,\lambda) \in \mathfrak{G}$

Consider particular type of dependence $G(x,t,\lambda)$:

$$i\frac{\partial G}{\partial x} - \lambda^k[J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k[K, G(x,t,\lambda)] = 0.$$

where $J \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP:

$$\xi^\pm(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}.$$

where all $Q_k(x,t) \in \mathfrak{g}$. However,

$$\mathcal{J}(x,t,\lambda) = \xi^\pm(x,t,\lambda)J\hat{\xi}^\pm(x,t,\lambda), \qquad \mathcal{K}(x,t,\lambda) = \xi^\pm(x,t,\lambda)K\hat{\xi}^\pm(x,t,\lambda),$$

belong to the algebra $\mathfrak{g}$ for any $J$ and $K$ from $\mathfrak{g}$. If in addition $K$ also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}(x,t,\lambda), \mathcal{K}(x,t,\lambda)] = 0.$$

Zakharov-Shabat theorem

**Theorem 1.** *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables $x$ and $t$ as above. Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$\begin{aligned} L\xi^{\pm} \equiv& i\frac{\partial \xi^{\pm}}{\partial x} + U_s(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [J, \xi^{\pm}(x,t,\lambda)] = 0, \\ M\xi^{\pm} \equiv& i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [K, \xi^{\pm}(x,t,\lambda)] = 0. \end{aligned}$$

*Proof.* Introduce the functions:

$$\begin{aligned} g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda), \\ p^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda), \end{aligned}$$

and using

$$i\frac{\partial G}{\partial x} - \lambda^k [J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k [K, G(x,t,\lambda)] = 0.$$

prove that

$$g^+(x,t,\lambda) = g^-(x,t,\lambda), \qquad p^+(x,t,\lambda) = p^-(x,t,\lambda),$$

which means that these functions are analytic functions of $\lambda$ in the whole complex $\lambda$-plane. Next we find that:

$$\lim_{\lambda\to\infty} g^+(x,t,\lambda) = \lambda^k J, \qquad \lim_{\lambda\to\infty} p^+(x,t,\lambda) = \lambda^k K.$$

and make use of Liouville theorem to get

$$g^+(x,t,\lambda) = g^-(x,t,\lambda) = \lambda^k J - \sum_{l=1}^{k} U_{s;l}(x,t)\lambda^{k-l},$$

$$p^+(x,t,\lambda) = p^-(x,t,\lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(x,t)\lambda^{k-l}.$$

We shall see below that the coefficients $U_l(x,t)$ and $V_l(x,t)$ can be expressed in terms of the asymptotic coefficients $Q_s$ of $\xi^{\pm}(x,t,\lambda)$.

Now remember the definition of $g^{+}(x,t,\lambda)$

$$g^{\pm}(x,t,\lambda) = i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda)$$
$$= \lambda^k J - \sum_{l=1}^{k} U_{s;l}(x,t)\lambda^{k-l},$$

Multiply both sides by $\xi^{\pm}(x,t,\lambda)$ and move all the terms to the left:

$$i\frac{\partial \xi^{\pm}}{\partial x} + \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}\xi^{\pm}(x,t,\lambda) - \lambda^k [J, \xi^{\pm}(x,t,\lambda)] = 0,$$

i.e. $L\xi^{\pm}(x,t,\lambda) = 0$. □

**Lemma 1.** *The operators $L$ and $M$ commute*

$$[L, M] = 0,$$

*i.e. the following set of equations hold:*

$$i\frac{\partial U}{\partial t} - i\frac{\partial V}{\partial x} + [U(x,t,\lambda) - \lambda^k J, V(x,t,\lambda) - \lambda^k K] = 0.$$

*where*

$$U(x,t,\lambda) = \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}, \qquad V(x,t,\lambda) = \sum_{l=0}^{k} V_l(x,t)\lambda^{k-l}.$$

# Jets of order $k$

How to parametrize $U_s(x,t,\lambda)$ and $V(x,t,\lambda)$?
Use:

$$\xi^{\pm}(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}.$$

and consider the jets of order $k$ of $\mathcal{J}(x,\lambda)$ and $\mathcal{K}(x,\lambda)$:

$$\begin{aligned}\mathcal{J}(x,t,\lambda) &\equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) J_l \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k J - U(x,t,\lambda),\\ \mathcal{K}(x,t,\lambda) &\equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k K - V(x,t,\lambda).\end{aligned}$$

Express $U(x)\in\mathfrak{g}$ in terms of $Q_s(x)$:

$$\mathcal{J}(x,t,\lambda) = J + \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{ad}_Q^k J, \qquad \mathcal{K}(x,t,\lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{ad}_Q^k K,$$

$$\operatorname{ad}_Q Z = [Q,Z], \qquad \operatorname{ad}_Q^2 Z = [Q,[Q,Z]], \quad \dots$$

and therefore for $U_l$ we get:

$$\begin{aligned} U_1(x,t) &= -\operatorname{ad}_{Q_1} J, \qquad U_2(x,t) = -\operatorname{ad}_{Q_2} J - \frac{1}{2}\operatorname{ad}_{Q_1}^2 J\\ U_3(x,t) &= -\operatorname{ad}_{Q_3} J - \frac{1}{2}\left(\operatorname{ad}_{Q_2}\operatorname{ad}_{Q_1} + \operatorname{ad}_{Q_1}\operatorname{ad}_{Q_2}\right) J - \frac{1}{6}\operatorname{ad}_{Q_1}^3 J.\end{aligned}$$

and similar expressions for $V_l(x,t)$ with $J$ replaced by $K$.

# Reductions of polynomial bundles

$$\text{a)} \qquad A\xi^{+,\dagger}(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^{-}(x,t,\lambda), \qquad AQ^{\dagger}(x,t,\epsilon\lambda^*)\hat{A} = -Q(x,t,\lambda),$$

$$\text{b)} \qquad B\xi^{+,*}(x,t,\epsilon\lambda^*)\hat{B} = \xi^{-}(x,t,\lambda), \qquad BQ^{*}(x,t,\epsilon\lambda^*)\hat{B} = Q(x,t,\lambda),$$

$$\text{c)} \qquad C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^{-}(x,t,\lambda), \qquad CQ^{\dagger}(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda),$$

where $\epsilon^2 = 1$ and $A$, $B$ and $C$ are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the $\mathbb{Z}_N$-reductions we may have:

$$D\xi^{\pm}(x,t,\omega\lambda)\hat{D} = \xi^{\pm}(x,t,\lambda), \qquad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda),$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

# On $N$-wave equations – $k = 1$

Lax representation involves two Lax operators linear in $\lambda$:

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(x,t,\lambda) - \lambda[J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q(x,t)]\xi^{\pm}(x,t,\lambda) - \lambda[K, \xi^{\pm}(x,t,\lambda)] = 0.$$

The corresponding equations take the form:

$$i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[K, \frac{\partial Q}{\partial x}\right] - [[J,Q],[K,Q(x,t)]] = 0$$

$$Q(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad \begin{array}{r} J = \operatorname{diag}(a_1, a_2, a_3), \\ K = \operatorname{diag}(b_1, b_2, b_3), \end{array}$$

Then the 3-wave equations take the form:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0, \\ &\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0, \\ &\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0, \end{aligned}$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

## New 3-wave equations – $k \geq 2$

Let $\mathfrak{g} = sl(3)$ and

$$Q_1(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad Q_2(x,t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix},$$

Fix up $k = 2$. Then the Lax pair becomes

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]K, \xi^{\pm}(x,t,\lambda)] = 0,$$

where

$$U \equiv U_2 + \lambda U_1 = \left( [J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)] \right) + \lambda [J, Q_1],$$
$$V \equiv V_2 + \lambda V_1 = \left( [K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)] \right) + \lambda [K, Q_1].$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \operatorname{diag}(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus $Q_1$ and $Q_2$ get reduced into:

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix},$$

and we obtain new type of integrable 3-wave equations:

$$
\begin{aligned}
&i(a_1-a_2)\frac{\partial u_1}{\partial t} - i(b_1-b_2)\frac{\partial u_1}{\partial x} + \epsilon\kappa u_2^* u_3 + \epsilon\frac{\kappa(a_1-a_2)}{(a_1-a_3)} u_1|u_2|^2 = 0,\\
&i(a_2-a_3)\frac{\partial u_2}{\partial t} - i(b_2-b_3)\frac{\partial u_2}{\partial x} + \epsilon\kappa u_1^* u_3 - \epsilon\frac{\kappa(a_2-a_3)}{(a_1-a_3)} |u_1|^2 u_2 = 0,\\
&i(a_1-a_3)\frac{\partial u_3}{\partial t} - i(b_1-b_3)\frac{\partial u_3}{\partial x} - \frac{i\kappa}{a_1-a_3}\frac{\partial (u_1u_2)}{\partial x}\\
&\quad + \epsilon\kappa\left(\frac{a_1-a_2}{a_1-a_3}|u_1|^2 + \frac{a_2-a_3}{a_1-a_3}|u_2|^2\right) u_1u_2 + \epsilon\kappa u_3(|u_1|^2 - |u_2|^2) = 0,
\end{aligned}
$$

where

$$
\kappa = a_1(b_2-b_3) - a_2(b_1-b_3) + a_3(b_1-b_2), \qquad u_3 = w_3 + \frac{2a_2-a_1-a_3}{2(a_1-a_3)} u_1u_2.
$$

The diagonal terms in the Lax representation are $\lambda$-independent.

Two of them read:

$$
\begin{aligned}
i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,\\
i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,
\end{aligned}
$$

These relations are satisfied identically as a consequence of the NLEE.

# New types of 4-wave interactions

The Lax pair for these new equations will be provided by:

$$
\begin{aligned}
L\psi &= i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,\\
M\psi &= i\frac{\partial \psi}{\partial t} + (V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K)\psi(x,t,\lambda) = 0,
\end{aligned}
$$

where $U_j(x,t)$ and $V_j(x,t)$ are fast decaying smooth functions taking values in the Lie algebra $so(5)$

$$U_1(x,t) = [J, Q_1(x,t)], \qquad U_2(x,t) = [J, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}\,_{Q_1}^2 J,$$
$$V_1(x,t) = [K, Q_1(x,t)], \qquad V_2(x,t) = [K, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}\,_{Q_1}^2 K.$$

Here $\mathrm{ad}\,_{Q_1} X \equiv [Q_1(x,t), X]$.

Assume $Q_1(x,t)$ and $Q_2(x,t)$ to be generic elements of $so(5)$:

$$Q_1(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha^1 E_\alpha + p_\alpha^1 E_{-\alpha}) + r_1^1 H_{e_1} + r_2^1 H_{e_2},$$
$$Q_2(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha^2 E_\alpha + p_\alpha^2 E_{-\alpha}) + r_1^2 H_{e_1} + r_2^2 H_{e_2},$$
$$J = a_1 H_{e_1} + a_2 H_{e_2} = \mathrm{diag}\,(a_1, a_2, 0, -a_2, -a_1),$$
$$K = b_1 H_{e_1} + b_2 H_{e_2} = \mathrm{diag}\,(b_1, b_2, 0, -b_2, -b_1),$$

Next we impose on $Q_1(x,t)$ and $Q_2(x,t)$ the natural reduction

$$B_0 U(x,t,\epsilon\lambda^*)^\dagger B_0^{-1} = U(x,t,\lambda), \qquad B_0 = \mathrm{diag}\,(1,\epsilon,1,\epsilon,1), \quad \epsilon^2 = 1.$$

As a result:

$$B_0(\chi^+(x,t,\epsilon\lambda^*))^\dagger B_0^{-1} = (\chi^-(x,t,\lambda))^{-1}, \qquad B_0(T(t,\epsilon\lambda^*))^\dagger B_0^{-1} = (T(t,\lambda))^{-1},$$

which provide $p^1_\alpha = \epsilon(q^1_\alpha)^*$, $p^2_\alpha = \epsilon(q^2_\alpha)^*$. Then the Lax representation will be a (rather complicated) system of 8 NLEE for the 8 independent matrix elements $q^1_\alpha$ and $q^2_\alpha$.

However we can impose additional $\mathbb{Z}_2$ reduction condition

$$D\xi^\pm(x,t,-\lambda)\hat{D} = \xi^\pm(x,t,\lambda), \qquad DQ(x,t,-\lambda)\hat{D} = Q(x,t,\lambda),$$
$$D = \mathrm{diag}\,(1,-1,1,-1,1)$$

$$
\begin{aligned}
Q_1(x,t) &= u_1 E_{e_1-e_2} + u_2 E_{e_2} + u_3 E_{e_1+e_2} + v_1 E_{-e_1+e_2} + v_2 E_{-e_2} + v_3 E_{-e_1-e_2} \\
&= \begin{pmatrix} 0 & u_1 & 0 & u_3 & 0 \\ v_1 & 0 & u_2 & 0 & u_3 \\ 0 & v_2 & 0 & u_2 & 0 \\ v_3 & 0 & v_2 & 0 & u_1 \\ 0 & v_3 & 0 & v_1 & 0 \end{pmatrix}, \\
Q_2(x,t) &= u_4 E_{e_1} + v_4 E_{-e_1} + w_1 H_{e_1} + w_2 H_{e_2} \\
&= \begin{pmatrix} w_1 & 0 & u_4 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ w_4 & 0 & 0 & 0 & u_4 \\ 0 & 0 & 0 & -w_2 & 0 \\ 0 & 0 & -v_4 & 0 & -w_1 \end{pmatrix}, \\
J &= a_1 H_{e_1} + a_2 H_{e_2} = \operatorname{diag}(a_1, a_2, 0, -a_2, -a_1), \\
K &= b_1 H_{e_1} + b_2 H_{e_2} = \operatorname{diag}(b_1, b_2, 0, -b_2, -b_1),
\end{aligned}
$$

Combining both reductions for the matrix elements of $Q_j(x,t)$ we have:

$$
v_1 = \epsilon u_1^*, \qquad v_2 = \epsilon u_2^*, \qquad v_3 = \epsilon u_3^*, \qquad v_4 = u_4^*,
$$

The commutativity condition for the Lax pair

$$i\left(\frac{\partial V_2}{\partial x}+\lambda\frac{\partial V_1}{\partial x}\right)-i\left(\frac{\partial U_2}{\partial t}+\lambda\frac{\partial U_1}{\partial t}\right)+[U_2+\lambda U_1-\lambda^2 J, V_2+\lambda V_1-\lambda^2 K]=0$$

must hold identically with respect to $\lambda$. The terms proportional to $\lambda^4$, $\lambda^3$ and $\lambda^2$ vanish identically. The term proportional to $\lambda$ and the $\lambda$-independent term vanish provided $Q_i$ satisfy the NLEE:

$$\begin{aligned} &i\frac{\partial V_1}{\partial x}-i\frac{\partial U_1}{\partial t}+[U_2,V_1]+[U_1,V_1]=0,\\ &i\frac{\partial V_2}{\partial x}-i\frac{\partial U_2}{\partial t}+[U_2,V_2]=0. \end{aligned}$$

In components the corresponding NLEE:

$$
\begin{aligned}
&-2i(a_1-a_2)\frac{\partial u_1}{\partial t}+2i(b_1-b_2)\frac{\partial u_1}{\partial x}+\kappa\epsilon u_2^*(\epsilon u_2^* u_3-u_1u_2-2u_4)=0,\\
&-2ia_2\frac{\partial u_2}{\partial t}+2ib_2\frac{\partial u_2}{\partial x}-\kappa(u_2\epsilon(|u_3|^2-|u_1|^2)+2u_3u_4^*+2\epsilon u_1^*u_4)=0,\\
&-2i(a_1+a_2)\frac{\partial u_3}{\partial t}+2i(b_1+b_2)\frac{\partial u_3}{\partial x}+\kappa u_2(\epsilon u_2^* u_3-u_1u_2+2u_4)=0,\\
&-2ia_1\frac{\partial u_4}{\partial t}+2ib_1\frac{\partial u_4}{\partial x}+i\frac{\partial}{\partial t}\left(-(2a_2-a_1)u_1u_2+(2a_2+a_1)\epsilon u_2^*u_3\right)\\
&+i(2b_2-b_1)\frac{\partial(u_1u_2)}{\partial x}-i(2b_2+b_1)\epsilon\frac{\partial(u_2^*u_3)}{\partial x}-\kappa\Big(2\epsilon u_4(|u_1|^2-|u_3|^2)\\
&+\epsilon u_1u_2(|u_1|^2+3|u_3|^2)-u_3u_2^*(3|u_1|^2+|u_3|^2)\Big)=0.
\end{aligned}
$$

Let us now introduce

$$
U_4=u_4-\frac{1}{2a_1}((a_1-a_2)u_1u_2+(a_1+a_2)\epsilon u_3u_2^*).
$$

As a result we get:

$$
\begin{aligned}
&-2i(a_1-a_2)\frac{\partial u_1}{\partial t}+2i(b_1-b_2)\frac{\partial u_1}{\partial x}-\frac{\kappa\epsilon}{a_1}u_2^*(2a_1U_4+\epsilon a_2u_2^*u_3+(2a_1-a_2)u_1u_2)=0,\\
&-2ia_2\frac{\partial u_2}{\partial t}+2ib_2\frac{\partial u_2}{\partial x}-\frac{\kappa\epsilon}{a_1}u_2((2a_1+a_2)|u_3|^2-a_2|u_1|^2)\\
&\qquad -2\kappa(u_3U_4^*+\epsilon u_1^*U_4+u_1^*u_2^*u_3)=0,\\
&-2i(a_1+a_2)\frac{\partial u_3}{\partial t}+2i(b_1+b_2)\frac{\partial u_3}{\partial x}+\frac{\kappa}{a_1}u_2(\epsilon(2a_1+a_2)u_2^*u_3-a_2u_1u_2+2a_1U_4)=0,\\
&-2ia_1\frac{\partial U_4}{\partial t}+2ib_1\frac{\partial U_4}{\partial x}+\frac{i\kappa}{a_1}\frac{\partial u_1u_2}{\partial x}-\frac{i\kappa\epsilon}{a_1}\frac{\partial u_2^*u_3}{\partial x}\\
&-\frac{\kappa}{a_1}\Big(2\epsilon U_4(|u_1|^2-|u_3|^2)+(\epsilon u_1u_2-u_3u_2^*)((2a_1-a_2)|u_1|^2+(2a_1+a_2)|u_3|^2)\Big)=0,
\end{aligned}
$$

# Soliton equations with $sl(n)$-series

$$L\psi \equiv i\frac{\partial \psi}{\partial x} + U(x,t,\lambda)\psi = 0,$$
$$M\psi \equiv i\frac{\partial \psi}{\partial t} + V(x,t,\lambda)\psi = \psi C(\lambda),$$

For the case of $\mathbb{Z}_N$-reduction (Mikhailov (1981)):

$$C_1 U(x,t,\lambda) C_1^{-1} = U(x,t,\omega\lambda), \quad C_1 V(x,t,\lambda) C_1^{-1} = V(x,t,\omega\lambda),$$

where $C_1^N = \mathbb{1}$ is a Coxeter automorphism of the algebra $\mathfrak{sl}(N,\mathbb{C})$ and $\omega = \exp(2\pi i/N)$.

Let $\mathfrak{g} \simeq \mathfrak{sl}(N,\mathbb{C})$ and the group of reduction is $\mathbb{Z}_N$. The class of relevant NLEE may be considered as generalizations of the derivative NLS equations

$$i\frac{\partial \psi_k}{\partial t} + \gamma \frac{\partial}{\partial x}\left( \cot\left(\frac{\pi k}{N}\right) \cdot \psi_{k,x} + i\sum_{p=1}^{N-1} \psi_p \psi_{k-p} \right) = 0,$$

$k = 1, 2, \ldots, N-1$, where $\gamma$ is a constant and the index $k-p$ should be understood modulus $N$ and $\psi_0 = \psi_N = 0$.

The automorphism $\mathrm{Ad}\,_{C_1}$ ($\mathrm{Ad}\,_{C_1}(Y) \equiv C_1 Y C_1^{-1}$ for every $Y$ from $\mathfrak{g}$) defines a grading in the Lie algebra

$$\mathfrak{sl}(N, \mathbb{C}) = \mathop{\oplus}_{k=0}^{N-1} \mathfrak{g}^{(k)} \,,$$

$$J^{(k)} = \sum_{j=1}^{N} \omega^{kj} E_{j,j+s}, \qquad C^{-1} J^{(k)} C = \omega^{-k} J^{(k)}.$$

where $(E_{j,s})_{q,r} = \delta_{jq}\delta_{sr}$. Obviously

$$\left[ J^{(k)}, J_l^{(m)} \right] = \left( \omega^{ms} - \omega^{kl} \right) J_{s+l}^{(k+m)}.$$

Next choose $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as follows:

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \quad Q(x,t) = \sum_{j=1}^{N-1} \psi_j(x,t) J_j^{(0)}, \quad J = a J_0^{(1)}$$

$$V(x,t,\lambda) = V_3(x,t) + \lambda V_2(x,t) + \lambda^2 V_1(x,t) - \lambda^3 K,$$

where

$$V_1(x,t) = \sum_{k=1}^{N} v_k^1(x,t) J_k^{(2)}, \qquad V_2(x,t) = \sum_{l=1}^{N} v_l^2(x,t) J_l^{(1)},$$

$$V_3(x,t) = \sum_{j=1}^{N-1} v_j^3(x,t) J_j^{(0)}, \qquad K = bJ_0^{(3)}.$$

The constants $a$ and $b$ determine the dispersion law of the MKdV eqs.

The next step is to request that $[L, M] = 0$ identically with respect to $\lambda$.

$$v_k^1(x,t) = \frac{b}{a}(\omega^{2k} + \omega^k + 1)\psi_k, \qquad k = 1, \dots, N-1,$$

and $v_N^1 = C(t)$ with $C(t)$ - arbitrary function of time. For

$$v_l^2(x,t) = \frac{b}{a^2} \sum_{j+k=l}^{N-1} \frac{\omega^{2l} + \omega^{2j+k} - \omega^k - 1}{1-\omega^l} \psi_j \psi_k$$
$$+ i \frac{b}{a^2} \left( \frac{\omega^{2l} + \omega^l + 1}{1-\omega^l} \right) \frac{\partial \psi_l}{\partial x} - \frac{C}{a} (\omega^l + 1) \psi_l,$$

for $l = 1, \ldots, N-1$ and

$$v_N^2 = -\frac{b}{a^2} \sum_{j+l=0}^{N-1} \left( \cos \frac{2\pi j}{N} + \frac{1}{2} \right) \psi_j \psi_l + D(t),$$

with D(t) - another arbitrary function of time. And for

$$v_j^3 = \frac{b}{a^3} \cot\left(\frac{\pi j}{N}\right) \sum_{k+l=j}^{N-1} \frac{\partial}{\partial x} (\psi_k \psi_l) + \frac{C}{a^2} \sum_{m+l=j}^{N-1} (\psi_m \psi_l)$$
$$+ \frac{b}{2a^3} \sum_{k+l=j}^{N-1} \frac{\cos \frac{\pi(k-l)}{N}}{\sin \frac{\pi j}{N}} \frac{\partial}{\partial x} (\psi_k \psi_l) - \frac{D}{a} \psi_j$$

$$
\begin{aligned}
&+\frac{b}{a^3}\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}(\psi_i\psi_k\psi_m)+\frac{3b}{2a^3}\sum_{l+m=j}^{N-1}\cot\left(\frac{\pi l}{N}\right)\frac{\partial\psi_l}{\partial x}\psi_m\\
&+\frac{b}{a^3}\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}\frac{\sin\frac{\pi(j-2k)}{N}-\sin\frac{\pi(j-2m)}{N}}{\sin\frac{\pi j}{N}}(\psi_i\psi_k\psi_m)\\
&-\frac{b}{4a^3}\cot\left(\frac{\pi j}{N}\right)\sum_{l+m=j}^{N-1}\frac{\partial}{\partial x}(\psi_l\psi_m)+\frac{C}{a^2}\cot\left(\frac{\pi j}{N}\right)\frac{\partial\psi_j}{\partial x}\\
&-\frac{b}{2a^3}\sum_{l+m=j}^{N-1}\frac{\cos\frac{\pi(l-m)}{N}}{\sin\frac{\pi j}{N}}\frac{\partial}{\partial x}(\psi_l\psi_m)+\frac{b}{a^3}\left(\cot^2\frac{\pi j}{N}-\frac{1}{4\sin^2\frac{\pi j}{N}}\right)\frac{\partial^2\psi_j}{\partial x^2}\\
&+\frac{b}{a^3}\sum_{k=1}^{N-1}\left(\cos\frac{2\pi k}{N}+\frac{1}{2}\right)(\psi_k\psi_{N-k}\psi_j)
\end{aligned}
$$

where $j$ is running from 1 to N-1. We choose $C(t)=0$ and $D(t)=0$.

In the end we get the following system of mKdV equations:

$$\alpha\frac{\partial\psi_j}{\partial t}=\left(\cot^2\frac{\pi j}{N}-\frac{1}{4\sin^2\frac{\pi j}{N}}\right)\frac{\partial^3\psi_j}{\partial x^3}+\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}\frac{\partial}{\partial x}(\psi_i\psi_k\psi_m)$$

$$+\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}\frac{\sin\frac{\pi(j-2k)}{N}-\sin\frac{\pi(j-2m)}{N}}{\sin\frac{\pi j}{N}}\frac{\partial}{\partial x}(\psi_i\psi_k\psi_m)$$

$$+\sum_{k=1}^{N-1}\left(\cos\frac{2\pi k}{N}+\frac{1}{2}\right)\frac{\partial}{\partial x}(\psi_k\psi_{N-k}\psi_j)+\frac{3}{4}\cot\left(\frac{\pi j}{N}\right)\sum_{k+l=j}^{N-1}\frac{\partial^2}{\partial x^2}(\psi_k\psi_l)$$

$$+\frac{3}{4}\sum_{k+l=j}^{N-1}\frac{\partial}{\partial x}\left(\cot\left(\frac{\pi l}{N}\right)\frac{\partial\psi_l}{\partial x}\psi_k+\cot\left(\frac{\pi k}{N}\right)\frac{\partial\psi_k}{\partial x}\psi_l\right)$$

where $\alpha=a^3/b$.

# Additional Involutions and Examples

Along with the $\mathbb{Z}_N$-reduction we can introduce one of the following involutions ($\mathbb{Z}_2$-reductions):

$$
\begin{aligned}
&\text{a)} \qquad K_0^{-1} U^\dagger(x,t,\kappa_1(\lambda)) K_0 = U(x,t,\lambda), && \kappa_1(\lambda) = -\omega^{-1}\lambda^* \\
&\text{b)} \qquad K_0^{-1} U^*(x,t,\kappa_1(\lambda)) K_0 = -U(x,t,\lambda), && \kappa_1(\lambda) = \omega^{-1}\lambda^* \\
&\text{c)} \qquad U^T(x,t,-\lambda) = -U(x,t,\lambda), &&
\end{aligned}
$$

where $K_0^2 = \mathbb{1}$. We choose

$$K_0 = \sum_{k=1}^{N} E_{k,N-k+1}.$$

The action of $K_0$ on the basis is as follows:

$$K_0 \left(J^{(k)}\right)^\dagger K_0 = \omega^{k(s-1)} J^{(k)}, \qquad K_0 \left(J^{(k)}\right)^* K_0 = \omega^{-k} J^{(k)}_{-s},$$

from which there follow the reductions below.

An immediate consequences are the constraints on the potentials:

$$
\begin{array}{lll}
\text{a)} & K_0^{-1} Q^\dagger(x,t) K_0 = Q(x,t), & K_0^{-1} (J_0^{(1)})^\dagger K_0 = \omega^{-1} J_0^{(1)}, \\
\text{b)} & K_0^{-1} Q^*(x,t) K_0 = -Q(x,t), & K_0^{-1} (J_0^{(1)})^* K_0 = \omega^{-1} J_0^{(1)}, \\
\text{c)} & Q^T(x,t) = -Q(x,t), & (J_0^{(1)})^T = J_0^{(1)}.
\end{array}
$$

More specifically there follows that each of the algebraic relations below:

$$
\begin{array}{lll}
\text{a)} & \psi_j^*(x,t) = \psi_j(x,t), & \alpha = \alpha^*; \\
\text{b)} & \psi_j^*(x,t) = -\psi_{N-j}(x,t), & \alpha = \alpha^*; , \\
\text{c)} & \psi_j(x,t) = -\psi_{N-j}(x,t). &
\end{array}
$$

where $j = 1, \dots, N-1$, are compatible with the evolution of the MKdV eqs.

# Some particular cases

Special examples of DNLS systems.

In the case of $\mathfrak{sl}(2,\mathbb{C})$ algebra we obtain the well-known MKdV equation

$$\alpha\frac{\partial\psi_1}{\partial t}=-\frac{1}{4}\frac{\partial^3\psi_1}{\partial x^3}-\frac{1}{2}\frac{\partial}{\partial x}(\psi_1^3).$$

In the case of $\mathfrak{sl}(3,\mathbb{C})$ algebra we have the system of trivial equations $\partial_t\psi_1=0$ and $\partial_t\psi_2=0$. In the case of $\mathfrak{sl}(4,\mathbb{C})$ algebra we find:

$$\alpha\frac{\partial\psi_1}{\partial t}=\frac{1}{2}\frac{\partial^3\psi_1}{\partial x^3}+\frac{3}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi_2}{\partial x}\psi_3\right)+\frac{3}{2}\frac{\partial}{\partial x}(\psi_1\psi_2^2)+\frac{\partial}{\partial x}(\psi_3^3),$$

$$\begin{aligned}\alpha\frac{\partial\psi_2}{\partial t}=&-\frac{1}{4}\frac{\partial^3\psi_2}{\partial x^3}+\frac{3}{4}\frac{\partial^2}{\partial x^2}\left(\psi_1^2\right)-\frac{3}{4}\frac{\partial^2}{\partial x^2}\left(\psi_3^2\right)\\&+3\frac{\partial}{\partial x}(\psi_1\psi_2\psi_3)-\frac{1}{2}\frac{\partial}{\partial x}(\psi_2^3),\end{aligned}\qquad(1)$$

$$\alpha\frac{\partial\psi_3}{\partial t}=\frac{1}{2}\frac{\partial^3\psi_3}{\partial x^3}-\frac{3}{2}\frac{\partial}{\partial x}\left(\psi_1\frac{\partial\psi_2}{\partial x}\right)+\frac{3}{2}\frac{\partial}{\partial x}(\psi_2^2\psi_3)+\frac{\partial}{\partial x}(\psi_1^3).$$

If we apply case a) we get the same set of MKdV equations with $\psi_1,\psi_2$ and $\psi_3$ purely real functions.

In the case b) we put $\psi_1 = -\psi_3^* = u$ and $\psi_2 = -\psi_2^* = iv$ and get:

$$\alpha \frac{\partial v}{\partial t} = -\frac{1}{4}\frac{\partial^3 v}{\partial x^3} + \frac{3}{4i}\frac{\partial^2}{\partial x^2}\left(u^2 - u^{*,2}\right) - 3\frac{\partial}{\partial x}(|u|^2 v) + \frac{1}{2}\frac{\partial}{\partial x}(v^3),$$
$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - i\frac{3}{2}\frac{\partial}{\partial x}\left(u^* \frac{\partial v}{\partial x}\right) - \frac{3}{2}\frac{\partial}{\partial x}(uv^2) - \frac{\partial}{\partial x}((u^*)^3),$$

where $u$ is a complex function, but $v$ is a purely real function.
In the case c):

$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x}(u^3),$$

where $u$ is a complex function, we recover the well known MKdV equation. And finally in the case of $\mathfrak{sl}(6,\mathbb{C})$ algebra with $\mathbb{D}_6$-reduction in the case c) we find

$$\alpha \frac{\partial u}{\partial t} = 2\frac{\partial^3 u}{\partial x^3} - 2\sqrt{3}\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) - 6\frac{\partial}{\partial x}(uv^2),$$
$$\alpha \frac{\partial v}{\partial t} = \sqrt{3}\frac{\partial^2}{\partial x^2}\left(u^2\right) - 6\frac{\partial}{\partial x}(u^2 v),$$

where $u$ and $v$ are complex functions.

# MKdV and $so(8)$

Normally with each simple Lie algebra one can associate just one MKdV eq.

The only exception is $s0(8)$ which allows a one-parameter family of MKdV equations. The reason is that only $so(8)$ has 3 as a double exponent!

$$\partial_t q_1 = 2a\left[\partial_x^3 q_1 - \sqrt{3}\partial_x(q_1\partial_x q_2)\right] - \sqrt{3}\left[(3a+b)\partial_x(q_4\partial_x q_3) + (3a-b)\partial_x(q_3\partial_x q_4)\right]$$
$$- 3\partial_x\left[q_1\left(2aq_2^2 + (a-b)q_3^2 + (a+b)q_4^2\right)\right],$$

$$
\begin{aligned}
\partial_t q_2 = \sqrt{3} a \partial_x^2 q_1^2 + \frac{\sqrt{3}}{2}(a+b)\partial_x^2 q_3^2 + \frac{\sqrt{3}}{2}(a-b)\partial_x^2 q_4^2 \\
- 3\partial_x \left[ q_2 \left( 2a q_1^2 + (a+b) q_3^2 + (a-b) q_4^2 \right) \right],
\end{aligned}
$$

$$
\begin{aligned}
\partial_t q_3 = -(a+b)\left[ \partial_x^3 q_3 - \sqrt{3}\partial(q_3 \partial_x q_2) \right] - \sqrt{3}\left[ (3a+b)\partial_x(q_4 \partial_x q_1) + 2b\partial_x(q_1 \partial_x q_4) \right] \\
+ 3\partial_x \left[ q_3 \left( 2a q_4^2 + (a-b) q_1^2 + (a+b) q_2^2 \right) \right],
\end{aligned}
$$

$$
\begin{aligned}
\partial_t q_4 = -(a-b)\left[ \partial_x^3 q_4 - \sqrt{3}\partial_x(q_4 \partial_x q_2) \right] - \sqrt{3}\left[ (3a-b)\partial_x(q_3 \partial_x q_1) - 2b\partial_x(q_1 \partial_x q_3) \right] \\
+ 3\partial_x \left[ q_4 \left( 2a q_3^2 + (a-b) q_2^2 + (a+b) q_1^2 \right) \right].
\end{aligned}
$$

# Conclusions and open questions

- More classes of new integrable equations: i) higher rank simple Lie algebras; ii) different types of grading; iii) different power $k$ of the polynomials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ and iv) different reductions of $U$ and $V$.

- These new NLEE must be Hamiltonian. View the jets $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.

- Apply Zakharov-Shabat dressing method for constructing their $N$-soliton solutions and study their interactions.

- Apply the above methods to twisted Kac-Moody algebras – work in progress

Thank you for your attention!