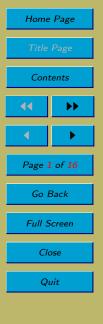
On New Ideas of Nonlinearity in Quantum Mechanics

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 [1] V. Kovalchuk and J. J. Sławianowski: *Hamiltonian systems inspired by the Schrödinger equation*, Symmetry, Integrability and Geometry: Methods and Applications 4, 046, 9 pages, 2008.

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[2] J. J. Sławianowski and V. Kovalchuk:

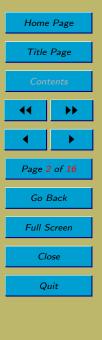
Schrödinger and related equations as Hamiltonian systems, manifolds of second-order tensors and new ideas of nonlinearity in quantum mechanics, Reports on Mathematical Physics

65, 1, pp. 29–76, 2010.

* * *

[3] J. J. Sławianowski and V. Kovalchuk:
Schrödinger equation as a hamiltonian system, essential nonlinearity, dynamical scalar product and some ideas of decoherence, in: Advances in Quantum Mechanics, Prof. Paul Bracken (Ed.), ISBN: 978-953-51-1089-7, InTech, Rijeka, pp. 81-103, 2013.





Quantum mechanics is plagued by paradoxes:

- decoherence,
- measurement process,
- reduction of the state vector.

Main concern is the linearity of the Schrödinger equation, which seems to be drastically incompatible with above-mentioned problems.

At the same time linearity works beautifully when:

- describing the unobserved unitary quantum evolution,
- finding the energy levels,
- in all statistical predictions.

Perhaps we deal here with a very sophisticated and delicate nonlinearity which becomes active and remarkable just in the process of interaction between quantum systems and "large" classical objects.





The main idea is:

• to analyze the Schrödinger equation and corresponding relativistic linear wave equations as usual self-adjoint equations of mathematical physics derivable from variational principles.

Lagrangian \Rightarrow Hamiltonian:

• Legendre transformation for the Schrödinger and Dirac equations is uninvertible and leads to constraints in the phase space. Dirac formalism is the solution.

Home Page

Title Page

Contents

Page 4 of 16

Go Back

Full Screen

Close

Quit

Incidentally, introducing the second-order time derivatives to dynamical equations, even as small corrections, regularizes Legendre transformation.

* * *

In non-relativistic quantum mechanics there are certain hints suggesting just such a modification in the nano-scale physics.

[Kozlowski M., Marciak-Kozlowska J., From quarks to bulk matter, Hadronic Press, USA, 2001.] [Marciak-Kozlowska J., Kozlowski M. Schrödinger equation for nanoscience, arXiv.org:cond-mat/0306699.]

Step 1:

The quantum Fourier equation which describes the heat (mass) diffusion on the atomic level has the following form:

$$\frac{\partial T}{\partial t} = \frac{\hbar}{m} \nabla^2 T$$

If we take the substitution $t \to it/2$ and $T \to \psi$, then we end up with the free Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi.$$

Step 2:

The complete Schrödinger equation with the potential term V after the reverse substitutions $t \to -2it$ and $\psi \to T$ gives us the parabolic quantum Fokker-Planck equation, which describes the quantum heat transport for $\Delta t > \tau$, where $\tau = \hbar/m\alpha^2 c^2 \sim 10^{-17}$ sec and $c\tau \sim 1$ nm, i.e.,

$$\frac{\partial T}{\partial t} = \frac{\hbar}{m} \nabla^2 T - \frac{2V}{\hbar} T$$



Home Page

Title Page

Contents

Page 5 of 16

Go Back

Full Screen

Close

Step 3:

For ultrashort time processes when $\Delta t < \tau$ one obtains the generalized quantum hyperbolic heat transport equation:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\hbar}{m} \nabla^2 T - \frac{2V}{\hbar} T$$

Step 4:

This leads us to the second-order modified Schrödinger equation

$$2\tau\hbar\frac{\partial^2\psi}{\partial t^2}+i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi$$

in which the additional term describes the interaction of electrons with surrounding spacetime filled with virtual positron-electron pairs.

* * *

Analogy to superposition of Dirac and d'Alembert operators (KGD equation).

[Sławianowski J.J., Kovalchuk V. Klein-Gordon-Dirac equation: physical justification and quantization attempts, Rep. Math. Phys. **49** (2002), 249–257.]





The conceptual transition from special to general theory of relativity:

• the metric tensor looses its status of the absolute geometric object and becomes included into degrees of freedom (gravitational field).

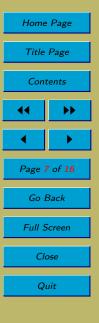
In our treatment:

• the Hilbert-space scalar product becomes a dynamical quantity which satisfies together with the state vector the system of differential equations.

The main idea:

- there is no fixed scalar product metric!
- the dynamical term of Lagrangian describing the self-interaction of the metric is invariant under the total group $\operatorname{GL}(n, \mathbb{C})$.





The natural metric of this kind:

• introducing to the theory a very strong nonlinearity which induces also the effective nonlinearity of the wave equation even if there is no "direct nonlinearity" in it.

Strong nonlinearity prevents us from finding a rigorous solution.

But some partial results are possible:

• if we fix the behaviour of wave function to some simple form, then for the scalar product behaviour there are rigorous exponential solutions (including infinitely growing/exponentially decaying in the future — some decay/reduction phenomena).

Two kinds of degrees of freedom (dynamical variables):

- wave function,
- scalar product.

They are mutually interacting.



Home Page

Title Page

Contents

Page 8 of 16

Go Back

Full Screen

Close

N-level quantum system:

We can define the "wave function" of the *n*-level quantum system as a following *n*-vector: Let us take a set of *n* elements and some function ψ defined on it, i.e.,

$$\psi = \begin{bmatrix} \psi^1 \\ \vdots \\ \psi^n \end{bmatrix}, \qquad \psi^a = \psi(a) \in \mathbb{C}.$$

Let H be a unitary space (n-dimensional "Hilbert space" \mathbb{C}^n) with the scalar product

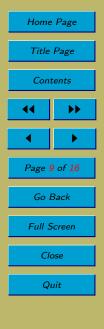
 $G: H \times H \to \mathbb{C},$

which is a sesquilinear hermitian form.

The general Lagrangian:

$$\begin{split} L &= \alpha_1 i G_{\bar{a}b} \left(\overline{\psi}^{\bar{a}} \dot{\psi}^{b} - \dot{\overline{\psi}}^{\bar{a}} \psi^{b} \right) + \alpha_2 G_{\bar{a}b} \dot{\overline{\psi}}^{\bar{a}} \dot{\psi}^{b} + \left[\alpha_4 G_{\bar{a}b} + \alpha_5 H_{\bar{a}b} \right] \overline{\psi}^{\bar{a}} \psi^{b} \\ &+ \alpha_3 \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^{b} \right] \dot{G}_{\bar{a}b} + \Omega[\psi, G]^{d\bar{c}b\bar{a}} \dot{G}_{\bar{a}b} \dot{G}_{\bar{c}d} - \mathcal{V}(\psi, G) \,, \end{split}$$





where

$$\begin{split} \Omega[\psi,G]^{d\bar{c}b\bar{a}} &= \alpha_6 \left[G^{d\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^d \right] \left[G^{b\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^b \right] \\ &+ \alpha_7 \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right] \left[G^{d\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^d \right] + \alpha_8 \overline{\psi}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d, \\ \Omega[\psi,G]^{d\bar{c}b\bar{a}} &= \Omega[\psi,G]^{b\bar{a}d\bar{c}}, \end{split}$$

and the potential \mathcal{V} can be taken, for instance, in the following quartic form:

$$\mathcal{V}(\psi,G) = \varkappa \left(G_{\bar{a}b}\overline{\psi}^{\bar{a}}\psi^{b}\right)^{2}.$$

The first and second terms (those with α_1 and α_2) describe the free evolution of wave function ψ while G is fixed. The Lagrangian for trivial part of the linear dynamics (those with α_4) can be also taken in the more general form $f(G_{\bar{a}b}\overline{\psi}^{\bar{a}}\psi^{\bar{b}})$, where $f: \mathbb{R} \to \mathbb{R}$. The term with α_5 corresponds to the Schrödinger dynamics while G is fixed and then

$$H^a{}_b = G^{a\bar{c}} H_{\bar{c}b}$$

is the usual Hamilton operator. If we properly choose the constants α_1 and α_5 , then we obtain precisely the Schrödinger equation. The dynamics of the scalar product G is described by the terms linear and quadratic in the time derivative of G. In the above formulae $\overline{\psi}^{\bar{a}} = \overline{\psi}^{\bar{a}}$ denotes the usual complex conjugation and α_i , $i = \overline{1,9}$, and \varkappa are some constants.



 Home Page

 Title Page

 Control

 Image: Control

 Image: Page: U of 16

 Go Back

 Full Screen

 Close

 Quit

$$\begin{aligned} & \text{The equations of motion:} \\ \frac{\delta L}{\delta \overline{\psi}^{\overline{a}}} &= \alpha_2 G_{\overline{a}b} \ddot{\psi}^b + \left(\alpha_2 \dot{G}_{\overline{a}b} - 2\alpha_1 i G_{\overline{a}b} \right) \dot{\psi}^b - 2\alpha_8 \dot{G}_{\overline{a}b} \psi^b \dot{G}_{\overline{c}d} \overline{\psi}^{\overline{c}} \psi^d \\ &- 2\alpha_9 \left(\alpha_6 \dot{G}_{\overline{a}d} \dot{G}_{\overline{c}b} + \alpha_7 \dot{G}_{\overline{a}b} \dot{G}_{\overline{c}d} \right) \psi^b \left(G^{d\overline{c}} + \alpha_9 \overline{\psi}^{\overline{c}} \psi^d \right) \\ &+ \left[\left(2\varkappa G_{\overline{c}d} \overline{\psi}^{\overline{c}} \psi^d - \alpha_4 \right) G_{\overline{a}b} - \alpha_5 H_{\overline{a}b} - \left[\alpha_3 \alpha_9 + \alpha_1 i \right] \dot{G}_{\overline{a}b} \right] \psi^b = 0, \\ \frac{\delta L}{\delta G_{\overline{a}b}} &= 2\Omega [\psi, G]^{b\overline{a}d\overline{c}} \ddot{G}_{\overline{c}d} + 2\dot{\Omega} [\psi, G]^{b\overline{a}d\overline{c}} \dot{G}_{\overline{c}d} + \left(2\varkappa G_{\overline{c}d} \overline{\psi}^{\overline{c}} \psi^d - \alpha_4 \right) \overline{\psi}^{\overline{a}} \psi^b \\ &+ 2G^{d\overline{a}} \left[\alpha_6 G^{b\overline{e}} \left(G^{f\overline{c}} + \alpha_9 \overline{\psi}^{\overline{c}} \psi^f \right) + \alpha_7 G^{b\overline{c}} \left(G^{f\overline{e}} + \alpha_9 \overline{\psi}^{\overline{e}} \psi^f \right) \right] \dot{G}_{\overline{c}d} \dot{G}_{\overline{e}f} \\ &- \alpha_2 \overline{\psi}^{\overline{a}} \dot{\psi}^b + \left[\alpha_3 \alpha_9 + \alpha_1 i \right] \overline{\psi}^{\overline{a}} \psi^b + \left[\alpha_3 \alpha_9 - \alpha_1 i \right] \overline{\psi}^{\overline{a}} \dot{\psi}^b = 0, \end{aligned}$$

. .

where

$$\begin{split} \dot{\Omega}[\psi,G]^{b\bar{a}d\bar{c}} &= \alpha_8 \left(\dot{\overline{\psi}}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d + \overline{\psi}^{\bar{a}} \dot{\psi}^b \overline{\psi}^{\bar{c}} \psi^d + \overline{\psi}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d + \overline{\psi}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d + \overline{\psi}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d \right) \\ &+ \alpha_6 \alpha_9 \left(\left[\dot{\overline{\psi}}^{\bar{a}} \psi^d + \overline{\psi}^{\bar{a}} \dot{\psi}^d \right] \left[G^{b\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^b \right] + \left[\dot{\overline{\psi}}^{\bar{c}} \psi^b + \overline{\psi}^{\bar{c}} \dot{\psi}^b \right] \left[G^{d\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^d \right] \right) \\ &+ \alpha_7 \alpha_9 \left(\left[\dot{\overline{\psi}}^{\bar{a}} \psi^b + \overline{\psi}^{\bar{a}} \dot{\psi}^b \right] \left[G^{d\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^d \right] + \left[\dot{\overline{\psi}}^{\bar{c}} \psi^d + \overline{\psi}^{\bar{c}} \dot{\psi}^d \right] \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right] \right) \\ &- \alpha_6 \left[G^{d\bar{e}} G^{f\bar{a}} \left(G^{b\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^b \right) + G^{b\bar{e}} G^{f\bar{c}} \left(G^{d\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^d \right) \right] \dot{G}_{\bar{e}f} \\ &- \alpha_7 \left[G^{b\bar{e}} G^{f\bar{a}} \left(G^{d\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^d \right) + G^{d\bar{e}} G^{f\bar{c}} \left(G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b \right) \right] \dot{G}_{\bar{e}f}. \end{split}$$



Home Page Title Page Contents **.** ► ◀ Page 11 of 16 Go Back Full Screen Close Quit

Pure dynamics for G:

The equations of motion for the pure dynamics of scalar product G while the wave function ψ is fixed are as follows:

$$\Omega[\psi,G]^{b\bar{a}d\bar{c}}\ddot{G}_{\bar{c}d} = \left(\frac{\alpha_4}{2} - \varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d\right)\overline{\psi}^{\bar{a}}\psi^b + \alpha_7\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f}\gamma[\psi,G]^{d\bar{e}f\bar{c}b\bar{a}} + \alpha_6\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f}\left(\gamma[\psi,G]^{b\bar{e}f\bar{c}d\bar{a}} + \gamma[\psi,G]^{f\bar{a}d\bar{e}b\bar{c}} - \gamma[\psi,G]^{b\bar{e}d\bar{a}f\bar{c}}\right)$$

where

$$\gamma[\psi,G]^{f\bar{e}d\bar{c}b\bar{a}} = G^{f\bar{e}}G^{d\bar{c}}\left(G^{b\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^{b}\right).$$

If we additionally suppose that $\alpha_4 = \alpha_8 = \alpha_9 = \varkappa = 0$, then the above expression simplifies significantly:

$$\left(\alpha_6 G^{b\bar{c}} G^{d\bar{a}} + \alpha_7 G^{b\bar{a}} G^{d\bar{c}}\right) \left(\ddot{G}_{\bar{c}d} - \dot{G}_{\bar{c}f} G^{f\bar{e}} \dot{G}_{\bar{e}d}\right) = 0$$

Hence, the pure dynamics of the scalar product is described by the following equations:

$$\ddot{G}_{\bar{a}b} - \dot{G}_{\bar{a}d}G^{d\bar{c}}\dot{G}_{\bar{c}b} = 0.$$



Home Page

Title Page

Contents

Page 12 of 16

Go Back

Full Screen

Close

Let us now demand that $\dot{G}G^{-1}$ is equal to some constant value E, i.e., $\dot{G} = EG$, then

 $\ddot{G} = E\dot{G} = E^2G$

and

$$\dot{G}G^{-1}\dot{G} = EGG^{-1}EG = E^2G,$$

therefore our equations of motion are fulfilled automatically and the solution is as follows:

$$G(t)_{\bar{a}b} = (\exp(Et))^{\bar{c}}{}_{\bar{a}}G_{0\bar{c}b}.$$

* * *

Similarly if we demand that $G^{-1}\dot{G}$ is equal to some other constant E', i.e., $\dot{G} = GE'$,

$$\ddot{G} = \dot{G}E'^2 = GE'^2, \qquad \dot{G}G^{-1}\dot{G} = GE'G^{-1}GE' = GE'^2,$$

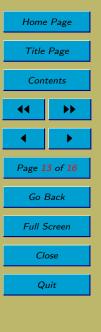
then the equations of motion are also fulfilled and the solution is as follows:

 $G(t)_{\bar{a}b} = G_{0\bar{a}d} \left(\exp(E't) \right)^d{}_b.$

The connection between these two different constants E and E' is written below:

$$\dot{G}(0) = \dot{G}_0 = G_0 E' = E G_0.$$





Usual and first-order modified Schrödinger equations:

The second interesting special case is obtained when we suppose that the scalar product G is fixed, i.e., the equations of motion are as follows:

$$\alpha_2 \ddot{\psi}^a - 2\alpha_1 i \dot{\psi}^a + \left(2\varkappa G_{\bar{c}d} \overline{\psi}^{\bar{c}} \psi^d \left[\psi, G\right] - \alpha_4\right) \psi^a - \alpha_5 H^a{}_b \psi^b = 0.$$

Then if we also take all constants of model to be equal to 0 except of the following ones:

$$\alpha_1 = \frac{h}{2}, \qquad \alpha_5 = -1,$$

we end up with the well-known usual Schrödinger equation:

$$i\hbar\dot{\psi}^a = H^a{}_b\psi^b.$$

Its first-order modified version is obtained when we suppose that G is a dynamical variable and α_2 is equal to 0, i.e.,

$$\begin{split} i\hbar\dot{\psi}^{a} &= H^{a}{}_{b}\psi^{b} - \left[\frac{i\hbar}{2} + \alpha_{3}\alpha_{9}\right]G^{a\bar{c}}\dot{G}_{\bar{c}b}\psi^{b} + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^{d} - \alpha_{4}\right)\psi^{a} \\ &- 2\alpha_{8}G^{a\bar{c}}\dot{G}_{\bar{c}b}\psi^{b}\dot{G}_{\bar{e}d}\overline{\psi}^{\bar{e}}\psi^{d} - 2\alpha_{9}G^{a\bar{c}}\left(\alpha_{6}\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}b} + \alpha_{7}\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\right)\psi^{b}\left(G^{d\bar{e}} + \alpha_{9}\overline{\psi}^{\bar{e}}\psi^{d}\right) \end{split}$$





$$2\Omega[\psi,G]^{b\bar{a}d\bar{c}}\ddot{G}_{\bar{c}d} = \left[\frac{i\hbar}{2} - \alpha_3\alpha_9\right]\overline{\psi}^{\bar{a}}\dot{\psi}^{b} - \left[\frac{i\hbar}{2} + \alpha_3\alpha_9\right]\dot{\overline{\psi}}^{\bar{a}}\psi^{b} - 2G^{d\bar{a}}\left[\alpha_6G^{b\bar{e}}\left(G^{f\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^{f}\right) + \alpha_7G^{b\bar{c}}\left(G^{f\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^{f}\right)\right]\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}} - \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^{d} - \alpha_4\right)\overline{\psi}^{\bar{a}}\psi^{b} - 2\dot{\Omega}[\psi,G]^{b\bar{a}d\bar{c}}\dot{G}_{\bar{c}d}.$$

We can rewrite the above equation of motion for ψ in the following form:

$$i\hbar\dot{\psi}^a = H_{\rm eff}{}^a{}_b\psi^b,$$

where the effective Hamilton operator is given as follows:

$$H_{\text{eff}}{}^{a}{}_{b} = H^{a}{}_{b} - \left[\frac{i\hbar}{2} + \alpha_{3}\alpha_{9}\right] G^{a\bar{c}}\dot{G}_{\bar{c}b} + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^{d} - \alpha_{4}\right)\delta^{a}{}_{b} - 2\alpha_{8}G^{a\bar{c}}\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\overline{\psi}^{\bar{e}}\psi^{d} - 2\alpha_{9}G^{a\bar{c}}\left(\alpha_{6}\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}b} + \alpha_{7}\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\right)\left(G^{d\bar{e}} + \alpha_{9}\overline{\psi}^{\bar{e}}\psi^{d}\right).$$

Future research:

• What if we admit "dissipative" models, where the Schrödinger equation does possess some "friction-like" term? ⇒ Some quantum models of dissipation.

Further investigation is required.



Home Page

Title Page

Contents

Page 15 of 16

Go Back

Full Screen

Close





