Applications of Lusternik-Schnirelmann Category and its Generalizations

John Oprea
Department of Mathematics
Cleveland State University

LECTURE 1: INTRODUCTION TO LS CATEGORY

What is Lusternik-Schnirelmann Category?

Goal of Algebraic Topology and Differential Geometry:
Define invariants (algebraic, topological, geometric) which describe the complexity of a space.

LS category is such an invariant originally defined in terms of open (or closed) covers of a space.

Motivations of Lusternik and Schnirelmann:

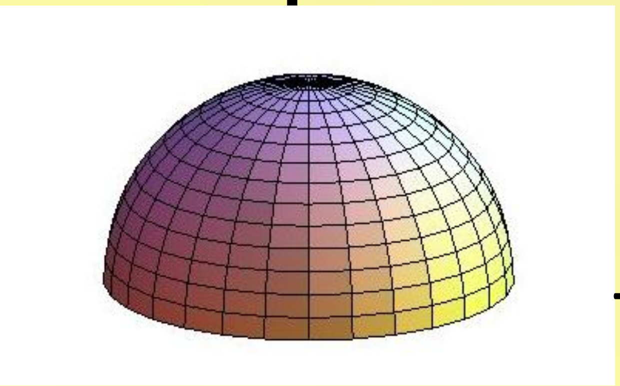
• Relate open covers and LS category to the existence of critical points for smooth functions on manifolds. This is a kind of “Morse Theory” in the degenerate case;

• Prove that the 2-sphere (with any metric) has at least 3 closed geodesics.

LS category has developed into an invariant that is useful not only in critical point theory, but in topology, differential and symplectic geometry and, most recently, robotics.

• The **LS category** of a space X, denoted $\text{cat}(X)$, is the least integer n so that X may be covered by open sets $U_1, \ldots, U_{n+1}$ having the property that each U_i is contractible to a point *in* X.

Example: The sphere is the union of two n-cells.



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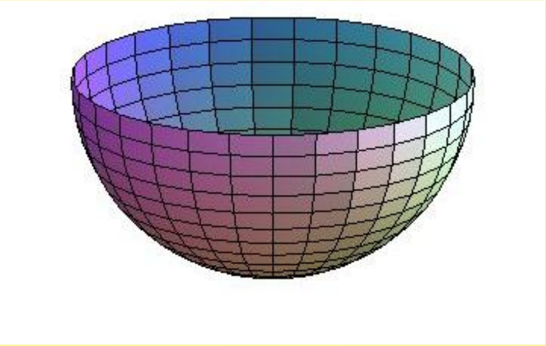

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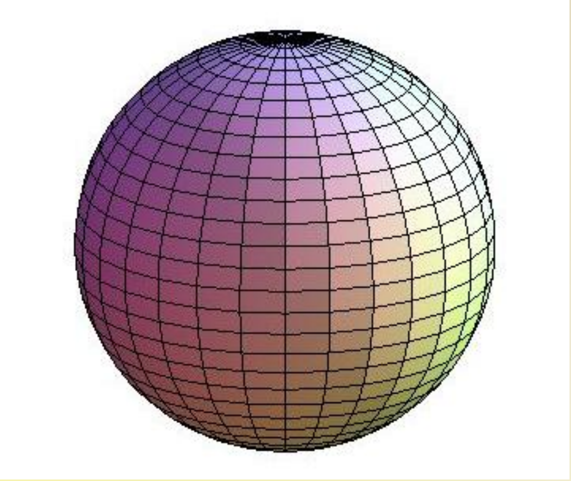


$\text{cat}(S^n) = 1.$

Theorem. LS category is a homotopy invariant.

Two maps f, $g\colon X \to Y$ are *homotopic*, denoted $f \simeq g$, if there is a map $H\colon X \times I \to Y$ with

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x).$$

$\mathrm{Maps}(X, Y)/\simeq \ \stackrel{\mathrm{def}}{=} \ [X, Y]$.

Two spaces X and Y are *homotopy equivalent*, denoted $X \simeq Y$, if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ with

$$f \circ g \simeq \mathrm{id}_Y \text{ and } g \circ f \simeq \mathrm{id}_X.$$

So, $X \simeq Y \Rightarrow \mathrm{cat}(X) = \mathrm{cat}(Y)$.

(Algebraic) Homotopy Invariants:

- Homotopy Groups: $\pi_k(X) = [S^k, X]$, the set of (based) homotopy classes of maps $S^k \to X$. A space X is n-*connected* if $\pi_k(X) = 0$ for $k = 1, \ldots, n$.

- $\pi_1(X)$ is called the *fundamental group* and it is the only possibly non-abelian homotopy group.

Example: $\pi_j(S^n) = 0$ for $j < n$ and $\pi_n(S^n) = \mathbb{Z}$. ($\pi_j(S^n)$, $j > n$ is the subject of a future lecture by someone who is not me!)

Example: The n-torus $T^n = S^1 \times \cdots \times S^1$ (n-times) has $\pi_1(T^n) = \bigoplus_1^n \mathbb{Z}$ and $\pi_j(T^n) = 0$ for $j > 1$.

• The Cohomology Algebra $H^*(X;\mathbb{F})$.

Example: $H^*(S^n;\mathbb{Z}) = \wedge(x)$, an exterior algebra on one generator in degree n.

Example: $H^*(T^n;\mathbb{Z}) = \wedge(x_1,\ldots,x_n)$, an exterior algebra on n generators all in degree one.

Example: $H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \dfrac{\mathbb{Z}_2[x_1]}{(x^{n+1})}$, a truncated polynomial algebra on a degree one generator.

Example: $H^*(\mathbb{C}P^n;\mathbb{Z}) = \dfrac{\mathbb{Z}[x_2]}{(x^{n+1})}$, a truncated polynomial algebra on a degree 2 generator.

The ***Basic Estimate*** for LS category is given by

$$\mathsf{cup}(M) \leq \mathsf{cat}(M) \leq \frac{\mathsf{dim}(M)}{n}$$

where, in the second inequality, M is (n-1)-connected (i.e. $\pi_j(M) = 0$ for all $j \leq n-1$).

Proof of First Inequality.

Recall that $\mathsf{cup}(X) = k$ if there are $x_i \in H^*(X;\mathcal{R})$, $i = 1, \ldots, k$ with $x_1 \smile x_2 \smile \cdots \smile x_k \neq 0$ and k is the largest such integer.

Also recall that cup product has the property:

$$H^*(X,A) \times H^*(X,B) \overset{\smile}{\to} H^*(X, A \cup B)$$

for any coefficients.

Now suppose $\text{cat}(X) = n$ with *categorical* cover $U_1, \ldots, U_{n+1}$ and, for each $i = 1, \ldots, n+1$, consider the long exact relative cohomology sequence

$$\ldots \to H^{s-1}(U_i) \to H^s(X, U_i) \to H^s(X) \to H^s(U_i) \to \ldots .$$

Since U_i contracts to a point in X, the maps $H^*(X) \to H^*(U_i)$ are all zero. Therefore, for each $x_i \in H^*(X)$, there exists a pre-image $\tilde{x}_i \in H^*(X, U_i)$.

Take any $(n+1)$ classes $x_1, \ldots, x_{n+1}$ in $H^*(X)$ with corresponding $\tilde{x}_j \in H^*(X, U_j)$.

Then, taking cup products, we get

$$\tilde{x}_1 \smile \cdots \smile \tilde{x}_{n+1} \mapsto x_1 \smile \cdots \smile x_{n+1}.$$

But $\tilde{x}_1 \smile \cdots \smile \tilde{x}_{n+1} \in H^*(X, \bigcup_j U_j) = H^*(X, X) = 0$, so we also get

$$x_1 \smile \cdots \smile x_{n+1} = 0.$$

Since this is true for all $x_i \in H^*(X)$, we have

$$\mathrm{cup}(X) \leq n = \mathrm{cat}(X).$$

Examples:

(0.) $\mathsf{cat}(S^n) = 1$ $(1 \leq \mathsf{cat}(S^n) \leq n/n = 1)$.

(1.) Let $T^n = S^1 \times \cdots \times S^1$ (n-times) be the n-torus. Then $\mathsf{cat}(T^n) = n$, since $H^*(T^n;\mathbb{Z}) = \wedge(x_1, \ldots, x_n)$.

(2.) $\mathsf{cat}(\mathbb{R}P^n) = n$, since $H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \dfrac{\mathbb{Z}_2[x_1]}{(x^{n+1})}$.

(3.) $\mathsf{cat}(\mathbb{C}P^n) = n$, since $H^*(\mathbb{C}P^n;\mathbb{Z}) = \dfrac{\mathbb{Z}[x_2]}{(x^{n+1})}$.

Let's actually prove a theorem.

Proposition. The following are equivalent:

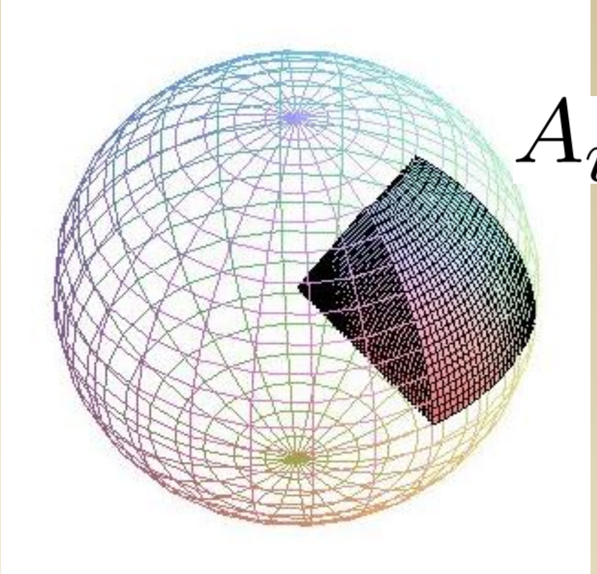
(1.) If S^n is covered by closed (or open) sets $C_1, \ldots, C_{n+1}$, then at least one C_i contains antipodal points.

(2.) Every continuous map $f\colon S^n \to \mathbb{R}^n$ takes some pair of antipodal points to the same value. *This is the famous Borsuk-Ulam Theorem*.

Theorem. If S^n is covered by closed (or open) sets $C_1, \ldots, C_{n+1}$, then at least one C_i contains antipodal points.

Proof. Assume no C_i contains antipodal points. Take $S^n \subset B^{n+1}$ and let $A_i \subset B^{n+1}$ be the closed set defined by connecting radial segments from the origin to each point of C_i. Note that A_i contracts to the origin.

$\mathbb{R}P^{n+1} = B^{n+1}/\sim$, where $\sim$ identifies points on the boundary S^n with their antipodes. Note that $A_i \hookrightarrow \mathbb{R}P^{n+1}$ by the hypothesis.



Also, A_i contracts to a point in $\mathbb{R}P^{n+1}$ as well since there are no identifications on A_i.

Since $\mathbb{R}P^{n+1}$ is covered by $A_1, \ldots, A_{n+1}$, then

$$\mathrm{cat}(\mathbb{R}P^{n+1}) \leq n.$$

This is a contradiction since $\mathrm{cat}(\mathbb{R}P^{n+1}) = n+1$.

The Lusternik-Schnirelmann Critical Point Theorem

Theorem. Let M be a smooth compact manifold and let $\mathrm{Crit}(M)$ denote the minimum number of critical points for any smooth function on M. Then

$$1 + \mathrm{cat}(M) \le \mathrm{Crit}(M).$$

Theorem. (F. Takens)

$$\mathrm{Crit}(M) \le 1 + \dim(M).$$

Basic Critical Point Estimate.

$$1 + \mathrm{cat}(M) \le \mathrm{Crit}(M) \le 1 + \dim(M).$$

Example. S^2

The height function on the sphere is a function with 2 critical points, so we have

$$2 = 1 + \mathrm{cat}(S^2) = \mathrm{Crit}(S^2) < 1 + \dim(S^2) = 3.$$

Theorem. If $\mathrm{Crit}(M^n) = 2$, then $M \cong_{\text{homeo}} S^n$.

This looks simple, but **BEWARE**!

Corollary. The validity of the equality

$$\mathrm{Crit}(S) = \mathrm{cat}(S) + 1$$

for homotopy spheres S is equivalent to the Poincaré conjecture.

Next time we will look at the critical point theorem in the context of symplectic geometry and a conjecture of V. Arnold.