Applications of Lusternik-Schnirelmann Category and its Generalizations

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Lecture 2: A Classical Application and Reformulations of LS Category

Recall in Lecture 1 that we proved the following:

Lusternik-Schnirelmann Theorem. If S^n is covered by closed (or open) sets $C_1, \ldots, C_{n+1}$, then at least one C_i contains antipodal points.

Let's use this to prove one of the most theorems in Mathematics.

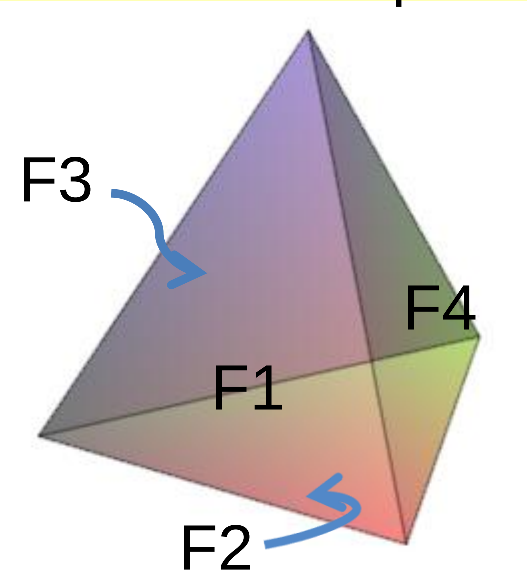
Proposition. The LS Theorem implies that there does not exist a map $f : S^n \to S^{n-1}$ such that $f(-x) = -f(x)$ for all x. (Such a map is called an antipodal map.)

Proof. Suppose an antipodal map $f : S^n \to S^{n-1}$ exists. Represent S^{n-1} as the boundary of an n-simplex and let the faces be denoted $F_1, F_2, \ldots, F_{n+1}$. Note that no F_j contains antipodal points.

Let $G_j = f^{-1}(F_j)$, $j = 1, \ldots, n+1$. The set $\{G_j\}$ covers S^n.

The LS theorem then says that there is some G_j containing antipodal points, x, $-x$. Then $f(x)$, $f(-x) = -f(x) \in F_j$ is a contradiction.

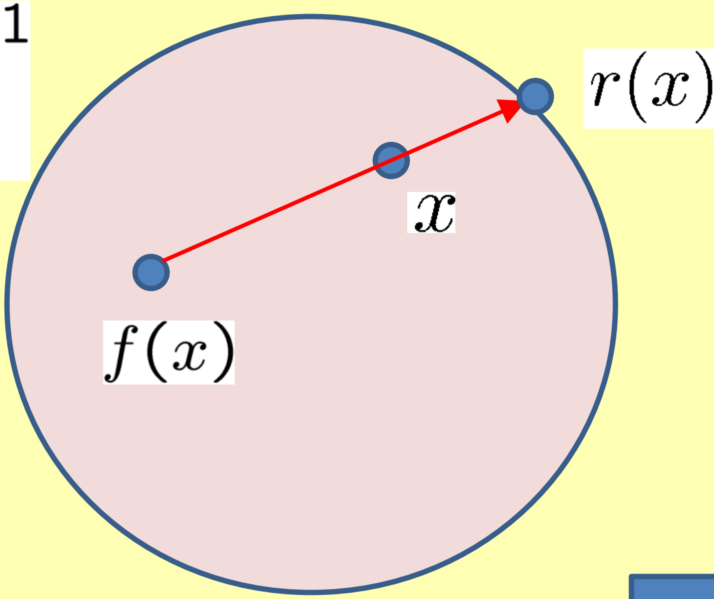
Ex: 3-Simplex



Brouwer Fixed Point Theorem. Every map $f \colon D^n \to D^n$ has a fixed point.

Lemma. If $f \colon D^n \to D^n$ does not have a fixed point, then there is a map $r \colon D^n \to S^{n-1}$ with $r \circ \mathrm{incl}_{S^{n-1}} = 1_{S^{n-1}}$ (i.e. a retraction).

Proof. The retraction $r \colon D^n \to S^{n-1}$ is depicted to the right.



Proof of BFPT. Suppose $f\colon D^n \to D^n$ does not have a fixed point. Let $r\colon D^n \to S^{n-1}$ be the consequent retraction from the Lemma.

Define a map $g\colon S^n \to S^{n-1}$ by

$$g(x_1, \ldots, x_{n+1}) = \begin{cases} r(x_1, \ldots, x_n) & \text{if } x_{n+1} \geq 0 \\ -r(-x_1, \ldots, -x_n) & \text{if } x_{n+1} \leq 0. \end{cases}$$

Note that g is antipodal. That is,

$$g(-x_1, \ldots, -x_{n+1}) = -g(x_1, \ldots, x_{n+1}).$$

Contradiction! ∎

While LS category proves some classical results, we can't get too far using only the open set definition.

Reformulation of LS Category

Let $PX = \{\gamma\colon I \to X | \gamma(0) = x_0\}$ be the contractible space of based paths. We construct

$$
\begin{array}{ccccccc}
\Omega X & \to & F_1(X) & \to & \cdots & F_k(X) & \cdots \\
\downarrow & & \downarrow & & \cdots & \downarrow & \\
PX & \to & G_1(X) & \to & \cdots & G_k(X) & \cdots \\
p_0 \downarrow & & p_1 \downarrow & & \cdots & p_k \downarrow & \\
X & \stackrel{1_X}{\to} & X & \stackrel{1_X}{\to} & \cdots & X & \cdots.
\end{array}
$$

where $G_{j+1}(X) = G_j(X) \cup C(F_j(X)) \simeq G_j(X)/F_j(X)$ is the mapping cone of the previous fibre inclusion.

Definition-Theorem. $\mathrm{cat}(X) \leq n$ if and only if there is a (homotopy) section $s\colon X \to G_n(X)$ (i.e. $p_n \circ s \simeq 1_X$).

$$\begin{array}{ccccccccc}
\Omega X & \to & F_1(X) & \to & \cdots & F_n(X) & \cdots \\
\downarrow & & \downarrow & & \cdots & \downarrow & \\
PX & \to & G_1(X) & \to & \cdots & G_n(X) & \cdots \\
p_0 \downarrow & & p_1 \downarrow & & \cdots & p_n \downarrow\uparrow s & \\
X & \stackrel{1_X}{\to} & X & \stackrel{1_X}{\to} & \cdots & X & \cdots .
\end{array}$$

Note that, in cohomology, we have $s^* \circ p_n^* = 1_{H^*}$, so p_n^* is injective.

Definition. For $f\colon Y \to X$, $\mathrm{cat}(f) \leq n$ if and only if there is a map $s\colon Y \to G_n(X)$ such that $p_n \circ s \simeq f$.

Note that $\mathrm{cat}(f) \leq \mathrm{cat}(X)$.

Things to Note.

1. $G_1(X) \simeq \Sigma\Omega(X)$. This follows since

$$G_1(X) = G_0(X) \cup C(\Omega X) \simeq * \cup C(\Omega X).$$

2. If $X = K(\pi, 1)$, then $G_1(X) \simeq \vee S^1$.

3. If $X = K(\pi, 1)$, then $G_k(X)$ is homotopy k-dimensional.

Definition. Let $u \in H^*(X;A)$. The *category weight* of u, denoted $\mathrm{wgt}(u)$, is the maximum k such that $p^*_{k-1}(u) = 0$, where $p^*_{k-1}\colon H^*(X;A) \to H^*(G_{k-1}(X);A)$ is the map induced on cohomology by $p_{k-1}\colon G_{k-1}(X) \to X$.

Properties: (1.) $\mathrm{wgt}(u) \leq \mathrm{cat}(X)$.

Proof. Suppose $\mathrm{cat}(X) = k$. Then we have

$$\begin{array}{ccccccc} \Omega X & \to & F_1(X) & \to & \cdots & F_k(X) & \cdots \\ \downarrow & & \downarrow & & \cdots & \downarrow & \\ PX & \to & G_1(X) & \to & \cdots & G_k(X) & \cdots \\ p_0 \downarrow & & p_1 \downarrow & & \cdots & p_k \downarrow\uparrow s & \\ X & \stackrel{1_X}{\to} & X & \stackrel{1_X}{\to} & \cdots & X & \cdots . \end{array}$$

Therefore, p^*_k is injective.

(2.) $\mathrm{wgt}(uv) \geq \mathrm{wgt}(u) + \mathrm{wgt}(v)$.

(3.) If $X = K(\pi, 1)$ and $u \in H^d(X)$, then $\mathrm{wgt}(u) \geq d$.

Proof of (3.) for d=2. $G_1(X) \simeq \Sigma\Omega X \simeq \vee S^1$ $\Rightarrow p_1^*(u) = 0$ (where $p_1\colon G_1(X) \to X$).

(4.) If $f\colon X \to Y$, $u \in H^*(Y)$ and $f^*(u) \neq 0$, then $\mathrm{wgt}(f^*(u)) \geq \mathrm{wgt}(u)$.

Here is a property to keep in mind!

(5.) If $f: X \to Y$ is a map and $f^*(u) \neq 0$, then $\mathrm{cat}(f) \geq \mathrm{wgt}(u)$.

Proof of (5.). Look at the diagram for $\mathrm{cat}(f) = k$.

$$\begin{array}{ccc} & & G_k(Y) \\ & s \nearrow & \downarrow p_k \\ X & \xrightarrow{f} & Y \end{array}$$

We assume that $f^*(u) \neq 0$, so the commutativity of the diagram then gives $p_k^*(u) \neq 0$. The definition of category weight then says that $\mathrm{wgt}(u) \leq k$.

Sectional Category.

Suppose $F \to E \xrightarrow{p} B$ is a fibration. Then the *sectional category* of p, denoted $\mathrm{secat}(p)$, is the least integer n such that there exists an open covering, $U_1, \ldots, U_{n+1}$, of B and, for each U_i, a map $s_i \colon U_i \to E$ having $p \circ s_i = \mathrm{incl}_{U_i}$. (That is, s_i is a local section of p).

Properties:

(1) $\mathrm{secat}(p) \leq \mathrm{cat}(B)$.

(2) If E is contractible, then $\mathrm{secat}(p) = \mathrm{cat}(B)$.

(3) If there are $x_1, \ldots, x_k \in \widetilde{H}^*(B; R)$ (any coefficient ring R) with

$$p^* x_1 = \ldots = p^* x_k = 0 \quad \text{and} \quad x_1 \cup \cdots \cup x_k \neq 0,$$

then $\mathrm{secat}(p) \geq k$.

(4) Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration arising as a pullback of a fibration $\widehat{p}\colon \widehat{E} \to \widehat{B}$ where $\widehat{E}$ is contractible (such as a principal bundle).

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & \widehat{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widehat{p}} \\ B & \xrightarrow{f} & \widehat{B} \end{array}$$

Then $\mathrm{secat}(p) = \mathrm{cat}(f)$.

Later we will need a refinement of our definition of LS category.

The *1-category* of a space X, denoted $\mathrm{cat}_1(X)$, is the least integer n so that X may be covered by open sets $U_0, \ldots, U_n$ having the property that, for each U_i, there is a partial section $s_i \colon U_i \to \widetilde{X}$, where $p \colon \widetilde{X} \to X$ is the universal cover (so $p \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow X$).

Note that this is just a specialization of sectional category to the universal covering.

$$\mathrm{cat}_1(X) = \mathrm{secat}(\widetilde{X} \to X).$$

A few properties of $\mathrm{cat}_1(X)$.

(1.) $\mathrm{cat}_1(X) = \mathrm{cat}(j_1 : X \to K(\pi_1 X, 1))$.

The category on the right is ***the category of a map***.

Theorem. If $\pi_1(X) = \pi$, $B\pi = K(\pi, 1)$ and k is the maximum degree for which $j_1^* : H^k(B\pi; \mathcal{A}) \to H^k(X; \mathcal{A})$ is non-trivial (for any local coefficients $\mathcal{A}$), then

$$k \leq \mathrm{cat}_1(X) \leq \mathrm{cat}(B\pi) = \dim(B\pi).$$

Moreover, if $X = K(\pi, 1)$, then $\mathrm{cat}_1(X) = \dim(B\pi)$ (for $\dim(B\pi) > 3$).

Examples: If X is simply connected, then $\mathrm{cat}_1(X) = 0$. Also, $\mathrm{cat}_1(T^n) = n$.

Theorem. (Eilenberg-Ganea) $\mathrm{cat}_1(X) \leq n$ if and only if there exists an n-dimensional complex L such that there is a map $f\colon X \to L$ inducing an isomorphism

$$f_*\colon \pi_1(X) \xrightarrow{\cong} \pi_1(L).$$

Corollary. $\pi_1(X)$ is free if and only if $\mathrm{cat}_1(X) = 1$.

The next two properties are more or less general properties of category-type invariants.

(2.) $\mathrm{cat}_1(X \times Y) \leq \mathrm{cat}_1(X) + \mathrm{cat}_1(Y)$.

(3.) If X is a CW complex and $p\colon \overline{X} \to X$ is a covering space, then $\mathrm{cat}_1(\overline{X}) \leq \mathrm{cat}_1(X)$.

The next lecture will focus on critical point theory and geometry!