Applications of Lusternik-Schnirelmann Category and its Generalizations

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LECTURE 2: LS CATEGORY, CRITICAL POINTS AND SYMPLECTIC GEOMETRY

Review of Critical Points and Manifold Structure.

Theorem. Let M be a smooth compact manifold and let $\mathsf{Crit}(M)$ denote the minimum number of critical points for any smooth function on M. Then

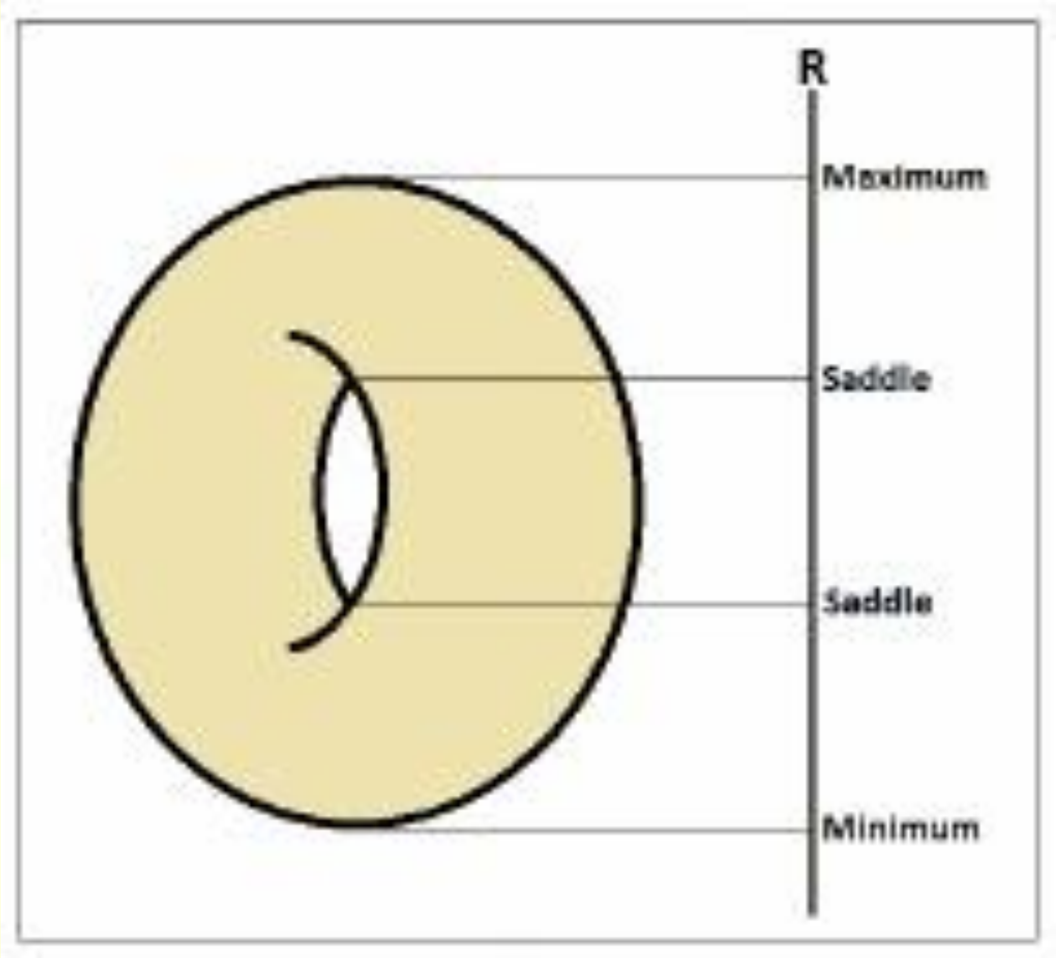
$$1 + \mathsf{cat}(M) \leq \mathsf{Crit}(M) \leq 1 + \dim(M).$$

Morse Theory versus LS Cat Theory

Example: The Torus

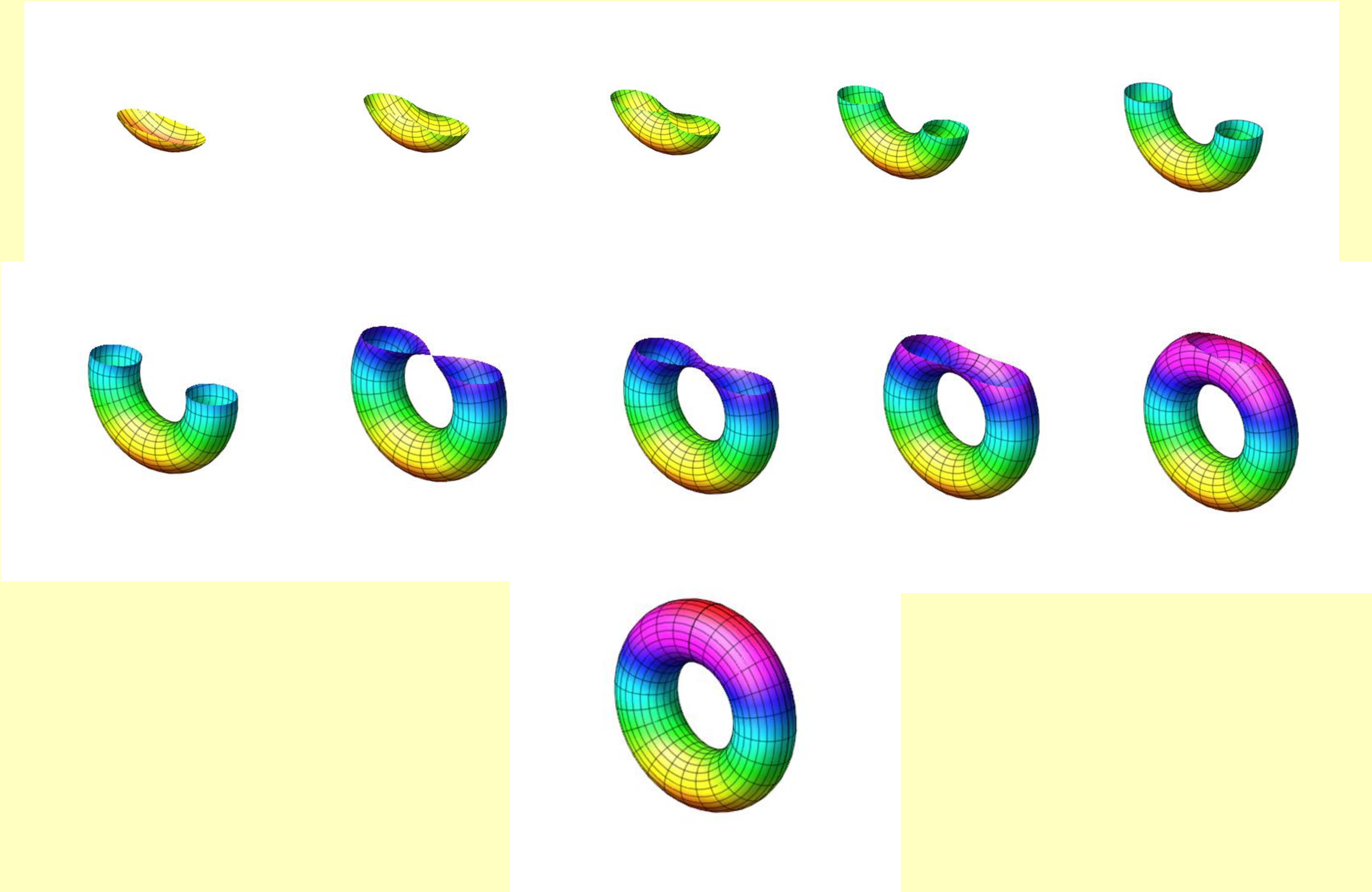
First, consider a height function on the Torus. There are 4 critical points: a minimum, 2 saddles and a maximum.

In local coords, each critical point has a non-singular Hessian matrix, so is non-degenerate.



Such a function is then called a **Morse function**.

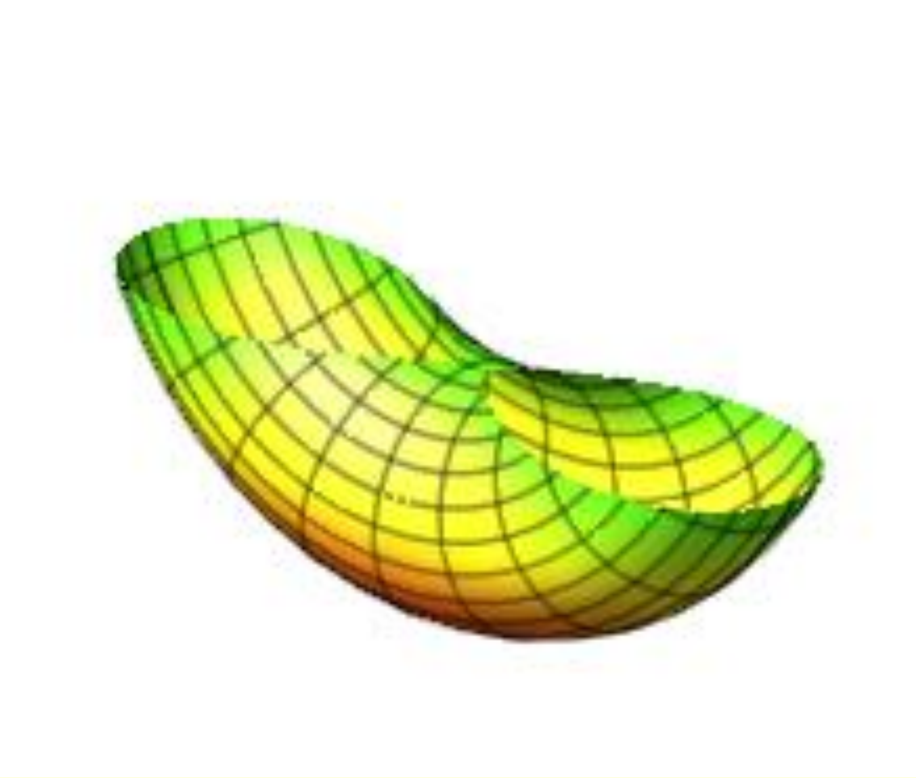
Sub-level sets of the height function on the torus.



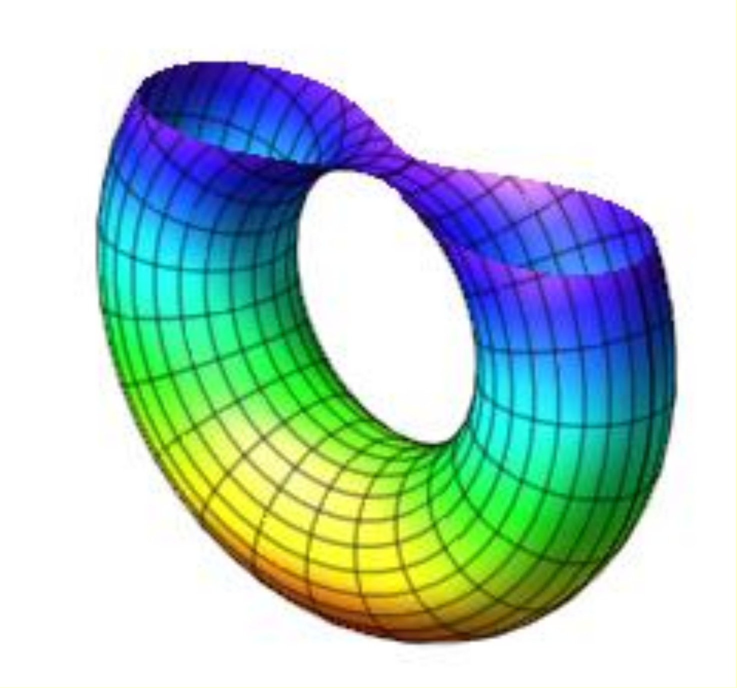
Let’s focus on the sub-level sets near critical points.

Near the Minimum:
A thickened 0-cell
(i.e. point)

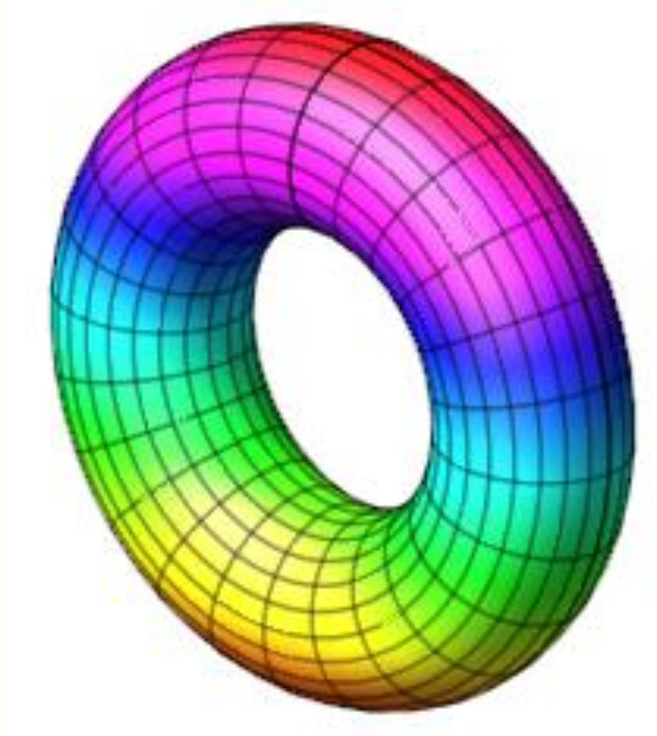
Near the first
saddle: Adding a
thickened 1-cell



Near the second saddle: Adding a thickened 1-cell



Near the Maximum: A 2-cell (i.e. disk)



Main Theorem of Morse Theory.

Theorem. If $f: M \to \mathbb{R}$ is a Morse function, then M has the homotopy type of a space constructed by attaching a cell of dimension k for each critical point of index k.

(The **index** of a critical point is the number of negative eigenvalues of the Hessian there.)

Corollary. If c_i is the number of critical points of index i, then for all $i \geq 0$, $\dim(H^i(M;\mathbb{R})) \leq c_i$.

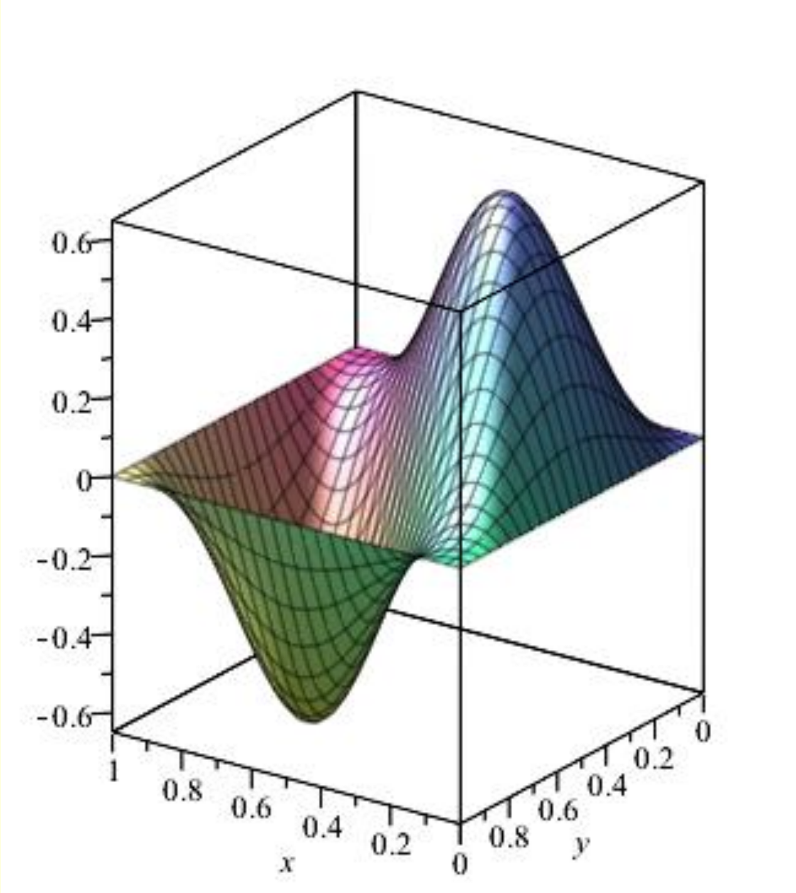
Corollary. A Morse function on the torus T^2 has at least 4 critical points.

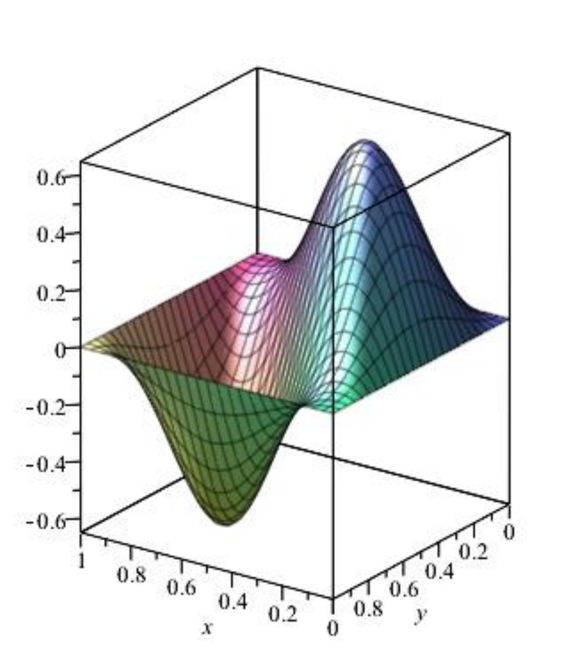
Proof. $\dim(H^0(T^2;\mathbb{R})) = 1$, $\dim(H^1(T^2;\mathbb{R})) = 2$, and $\dim(H^2(T^2;\mathbb{R})) = 1$.

Example. Let the torus T^2 be represented by the unit square $Sq = [0,1] \times [0,1]$ with $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$ $\forall x, y$. Define a function $G\colon Sq \to \mathbb{R}$:

$$G(x,y) = \sin(\pi x)\sin(\pi y)\sin(\pi(x+y)).$$

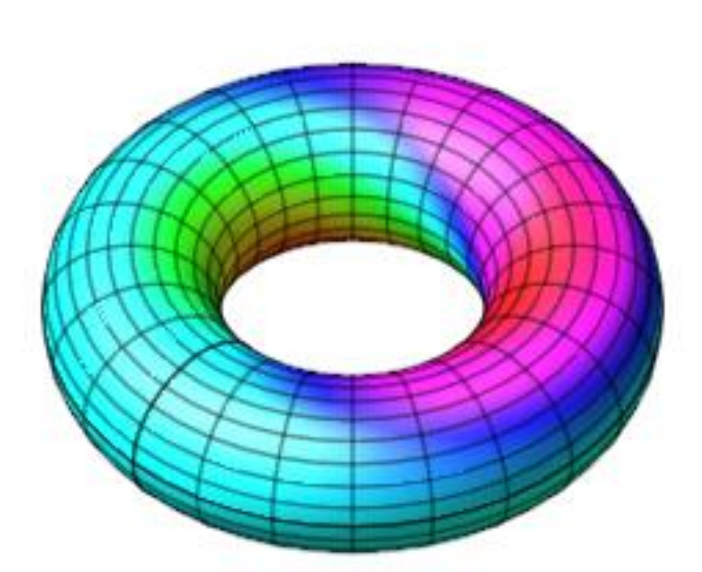
G has 3 critical points (recall $\mathrm{cat}(T^2) = 2$) at $(1/3, 1/3)$, $(2/3, 2/3)$ and $(0,0)$. The Hessian at $(0,0)$ is the zero matrix, so G is not a Morse function.

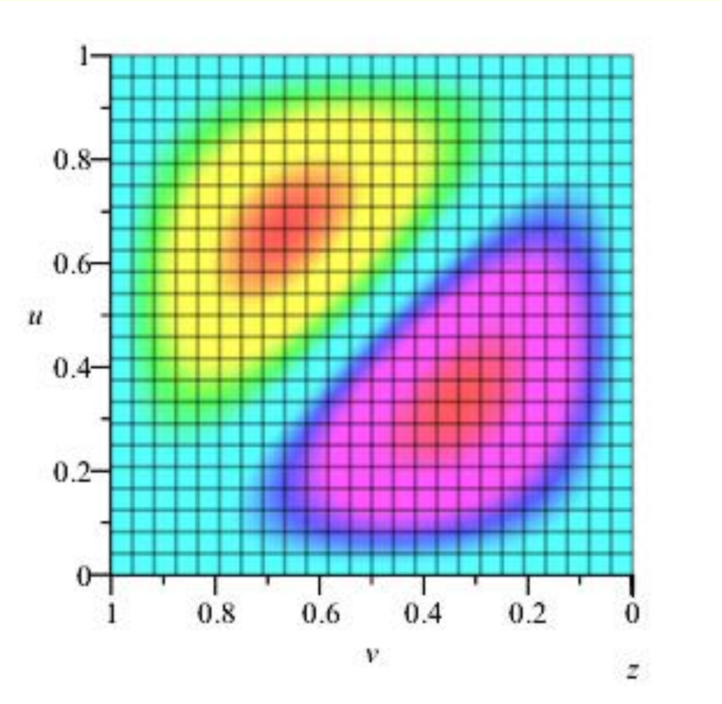




We can see how to reconstruct the torus from this picture too, but we don’t just attach one cell at a time. Start with the minimum, then go above the next critical point to get a wedge of two circles. Finally, attach a 2-cell to get the torus.

Here is G on the torus.





Applications of Category to Symplectic Topology

Definition. A manifold (M^{2n}, ω) is *symplectic* if ω is a closed 2-form such that ω^n is a volume form on M.

In particular, when M is closed, $\mathrm{cup}(M) \geq n$.

Examples:

(1.) $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

(2.) $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

(3.) T^{2n}, $T^*(M)$, Kähler manifolds, orientable surfaces.

Symplectic manifolds are the natural framework for Hamiltonian dynamics.

Take a time-dependent Hamiltonian $H\colon M \times \mathbb{R} \to \mathbb{R}$. In coordinate-free language, each function H_t has an associated vector field X_t with $i_{X_t}\omega = dH_t$. The time-dependent vector field X_t has integral curves $(q(t), p(t)$ with associated flow $\Phi\colon M \times \mathbb{R} \to M$; $\Phi((q,p),t) = (q(t),p(t))$, where $(q(0),p(0)) = (q,p)$.

The equality $i_{X_t}\omega = dH_t$ gives **Hamilton's Equations** for the flow,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Definition. The time-1 map $\phi = \Phi_1 = \Phi(-,1)$ of the flow determined by a time-dependent Hamiltonian is called a *Hamiltonian diffeomorphism*.

Example. Rotation of the torus about its axis is not a Hamiltonian diffeomorphism (because it does not have a fixed point).

The Arnold Conjecture.

If $\phi\colon M \to M$ is a Hamiltonian diffeomorphism on a closed symplectic manifold (M, ω) and $\mathsf{Fix}(\phi)$ stands for the number of fixed points of ϕ, then

$$\mathrm{Fix}(\phi) \geq \mathrm{Crit}(M).$$

The Hamiltonian diffeomorphism can be replaced by a 1-periodic flow and the fixed points can be replaced by the 1-periodic solutions of the associated Hamilton equations. These are critical points of the following *action functional* on (contractible) loops u in M (i.e. contractible u: $S^1 \to$ M):

$$A_H(u) = -\int_{D^2} \widetilde{u}^*\omega - \int_{S^1} H_t(u(t))\,dt$$

where $\widetilde{u}\colon D^2 \to M$ is an extension to D^2 of the loop $u\colon S^1 \to M$ using u's contractibility.

The important thing to notice is that the functional is *not well defined*! Different extensions to the disk can give different results --- unless

$$\omega|_{\pi_2(M)} = 0.$$

Geometrically, this is saying that

$$\int_{S^2} g^*\omega = 0$$

for all $g\colon S^2 \to M$. Pasting $\widetilde{u}$ and $\widehat{u}$ together along S^1 gives a map $g\colon S^2 \to M$ with

$$0 = \int_{S^2} g^*\omega = \int_{D^2} \widetilde{u}^*\omega - \int_{D^2} \widehat{u}^*\omega,$$

So A_H is well-defined.

Floer used this functional to prove a weak version of the Arnold Conjecture:

Theorem. If $\phi\colon M \to M$ is a Hamiltonian diffeomorphism, then

$$\mathrm{Fix}(\phi) \geq 1 + \mathrm{cup}_{\mathbb{Z}_2}(M).$$

Definition. (M^{2n}, ω) is *symplectically aspherical* if

$$\omega|_{\pi_2(M)} = 0.$$

Homotopy Interpretation:

We have $H^2(M;\mathbb{R}) \cong \mathrm{Hom}(H_2(M), \mathbb{R})$, so think of ω as a homomorphism $\omega\colon H_2(M) \to \mathbb{R}$.

The Hurewicz homomorphism (in degree 2), $h\colon \pi_2(M) \to H_2(M)$, is defined by $h(\alpha) = \alpha_*(\iota)$, where $\alpha\colon S^2 \to M$ is a representative of the homotopy class $\alpha \in \pi_2(M)$ and $\iota \in H_2(S^2) \cong \mathbb{Z}$ is a chosen (and fixed) generator.

The image of the Hurewicz homomorphism, $\mathrm{Im}(h)$, is then a subgroup of $H_2(M)$. The notation $\omega|_{\pi_2(M)} = 0$ then means that $\omega\colon H_2(M) \to \mathbb{R}$ vanishes on $\mathrm{Im}(h) \subseteq H_2(M)$.

Hopf's Theorem. The classifying map $f\colon M \to K(\pi_1(M), 1)$ induces isomorphisms with integral coefficients

$$H_1(M;\mathbb{Z}) \cong H_1(K(\pi_1(M),1);\mathbb{Z}),$$

$$\frac{H_2(M;\mathbb{Z})}{\mathrm{Im}(h_2)} \cong H_2(K(\pi_1(M),1);\mathbb{Z}).$$

Corollary. The condition $\omega|_{\pi_2(M)} = 0$ holds if and only if there exists $\omega_\pi \in H^2(K(\pi_1(M),1);\mathbb{R})$ with $f^*\omega_\pi = \omega$ where $f\colon M \to K(\pi_1(M), 1)$.

What does this say about LS category?

Recall the Reformulation of LS Category

Let $PX = \{\gamma\colon I \to X | \gamma(0) = x_0\}$. We construct

$$\begin{array}{ccccccc}
\Omega X & \to & F_1(X) & \to & \cdots & F_k(X) & \cdots \\
\downarrow & & \downarrow & & \cdots & \downarrow & \\
PX & \to & G_1(X) & \to & \cdots & G_k(X) & \cdots \\
p_0 \downarrow & & p_1 \downarrow & & \cdots & p_k \downarrow & \\
X & \xrightarrow{1_X} & X & \xrightarrow{1_X} & \cdots & X & \cdots .
\end{array}$$

where $G_{j+1}(X) = G_j(X) \cup C(F_j(X)) \simeq G_j(X)/F_j(X)$ is the mapping cone of the previous fibre inclusion.

Recall: $G_1(X) \simeq \Sigma\Omega X$ since $PX \simeq *$.

Also recall:

Definition-Theorem. $\mathrm{cat}(X) \leq n$ if and only if there is a (homotopy) section $s\colon X \to G_n(X)$ (i.e. $p_n \circ s \simeq 1_X$).

Definition. For $f\colon Y \to X$, $\mathrm{cat}(f) \leq n$ if and only if there is a map $s\colon Y \to G_n(X)$ such that $p_n \circ s \simeq f$.

LS Theorem for Flows. If X is a compact metric space with a gradient-like flow Ψ on it and $f\colon X \to Y$ is a map, then

$$1 + \mathrm{cat}(f) \leq \mathrm{Rest}(\Psi).$$

Definition. Let $u \in H^*(X; A)$. The *category weight* of u, denoted $\mathrm{wgt}(u)$, is the maximum k such that $p_{k-1}^*(u) = 0$, where $p_{k-1}^*: H^*(X; A) \to H^*(G_{k-1}(X); A)$ is the map induced on cohomology by $p_{k-1}: G_{k-1}(X) \to X$.

Properties:

(1.) $\mathrm{wgt}(u) \leq \mathrm{cat}(X)$.

(2.) $\mathrm{wgt}(uv) \geq \mathrm{wgt}(u) + \mathrm{wgt}(v)$.

(3.) If $X = K(\pi, 1)$ and $u \in H^d(X)$, then $\mathrm{wgt}(u) \geq d$.

(4.) If $f\colon X \to Y$, $u \in H^*(Y)$ and $f^*(u) \neq 0$, then $\mathrm{wgt}(f^*(u)) \geq \mathrm{wgt}(u)$.

(5.) If $f\colon X \to Y$ is a map and $f^*(u) \neq 0$, then $\mathrm{cat}(f) \geq \mathrm{wgt}(u)$.

The link between the hard analysis of the Arnold Conjecture and LS category is the following result used by Floer in his proof of the weakened form of the conjecture.

Floer-Hofer Theorem. Suppose (M^{2n}, ω) is a symplectically aspherical manifold and $H\colon M \times \mathbb{R} \to \mathbb{R}$ is a 1-periodic time-dependent Hamiltonian. Then

(1.) There is a gradient-like flow Ψ on a compact metric space X_∞ such that

$$\text{Rest}(\Psi) \leq \text{Number of contractible } 1-\text{periodic orbits of the flow } \Phi \text{ associated to } H.$$

(2.) There is a map $\tau\colon X_\infty \to M$ that induces an injection in cohomology (with any coefficients R),
$\tau^*\colon H^*(M; R) \to H^*(X_\infty; R)$.

Recalling that we replaced fixed points by 1-periodic orbits and using the FH theorem, we have

$$\mathrm{Fix}(\phi) \geq \text{Number of contractible } 1\text{−periodic orbits}$$

$$\geq \mathrm{Rest}(\Psi).$$

$$\geq 1 + \mathrm{cat}(\tau), \text{ by LS Thm for Flows.}$$

Also, since $\tau^*(\omega^n) \neq 0$, we have

$$2n + 1 = \mathrm{dim}(M) + 1 \geq \mathrm{Crit}(M) \geq \mathrm{cat}(M) + 1$$

$$\geq \mathrm{cat}(\tau) + 1 \geq \mathrm{wgt}(\omega^n) + 1 = 2n + 1$$

since $\omega = f^*(\omega_\pi)$ for $f\colon M \to K(\pi_1(M), 1)$.

Hence, $\mathrm{Crit}(M) = \mathrm{cat}(\tau) + 1$.

Original Arnold Conjecture Theorem (Rudyak-Oprea). Let (M, ω) be a closed symplectically aspherical manifold and let $\mathrm{Crit}(M)$ denote the minimum number of critical points for any smooth function $f\colon M \to \mathbb{R}$. If $\phi\colon M \to M$ is a Hamiltonian diffeomorphism and $\mathrm{Fix}(\phi)$ stands for the number of fixed points of ϕ, then

$$\mathrm{Fix}(\phi) \geq \mathrm{Crit}(M).$$

Proof.

$$\begin{aligned}\mathrm{Fix}(\phi) &\geq \text{Number of contractible } 1-\text{periodic orbits}\\ &\geq \mathrm{Rest}(\Psi)\\ &\geq 1 + \mathrm{cat}(\tau) = \mathrm{Crit}(M).\end{aligned}$$