The Mylar Balloon: Alternative Parametrizations and Mathematica®

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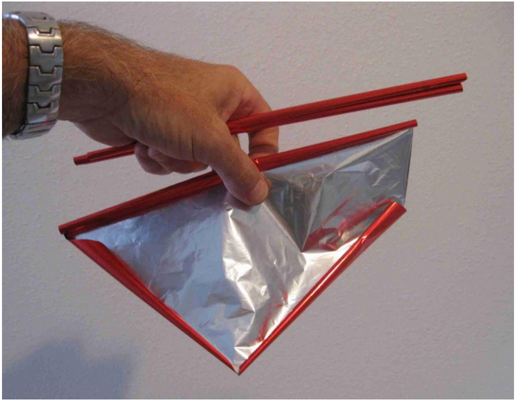
Outline

The Physical Prototype of the Mylar Balloon

Mylar is a Trademark

- Mylar is extremely thin polyester film.
- Mylar is flexible and inelastic material.
- Mylar is having a great tensile stress.

The Mylar Sheets



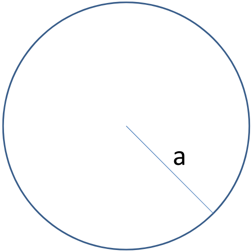
Mylar is a Geometrical Figure

- Mylar (or Mylar balloon) is the name of a surface of revolution that resembles a slightly flattened sphere.
- The term Mylar was coined by (Paulsen, 1994) who first investigated the shape.
- Mylar is a surface that encloses maximum volume for a given directrice arclength.

Constructing the Mylar Balloon

- Take two circular disks made of Mylar.
- Sew the disks together along their boundaries.
- Inflate with either air or helium.

The Deflated Mylar



First Geometrical Depiction (Paulsen, 1994)

- What is the shape of the inflated Mylar balloon?
- What is the radius of the inflated Mylar balloon?
- What is the thickness of the inflated Mylar balloon?
- What is the volume of the inflated Mylar balloon?

Mathematical Problem

Given a circular Mylar balloon what will be the shape of the balloon when it is fully inflated?

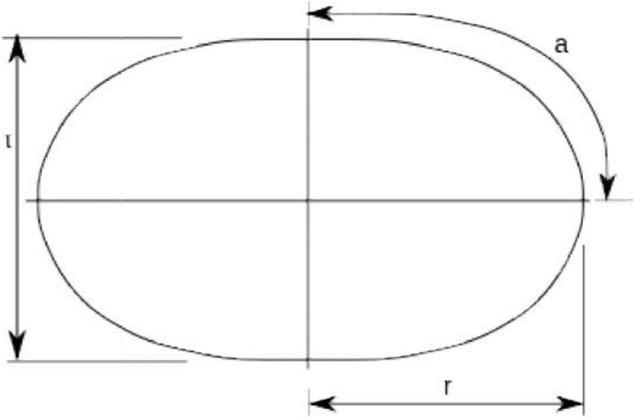
Preliminary Assumptions

- The deflated balloon lies in the xy-plane.
- The deflated balloon is centered at the origin.
- The deflated balloon has radius a.
- Oz is the axis of revolution.

The Profile Curve

- The profile curve lies in the first quadrant $z = z(x)$, $x \geq 0$.
- The axis of revolution is the z-axis.
- The bottom half of the Mylar is obtained by reflection of the upper half in the xy-plane.

The Supposed Profile of the Mylar



Calculus of Variations Problem

Find the profile curve $z = z(x), \quad z(r) = 0, \quad x \geq 0$

by maximizing the volume $V = 4\pi \int\limits_0^r xz(x)dx$

subject to the constraint $\int\limits_0^r \sqrt{1 + z'(x)^2}dx = a$

and the transversality condition $\lim\limits_{x \to r^-} z'(x) = -\infty$

The Euler-Lagrange Equation

$$\frac{dz}{dx} = -\frac{x^2}{\sqrt{r^4 - x^4}}, \qquad z(r) = 0, \qquad 0 \le x \le r$$

The Profile of the Mylar in Elliptic Integrals
(Mladenov and Oprea, 2003)

$$x(u) = r\cos u, \qquad z(u) = r\sqrt{2}\left[E(u, \frac{1}{\sqrt{2}}) - \frac{1}{2}F(u, \frac{1}{\sqrt{2}})\right], \qquad u \in [0\,,\ \frac{\pi}{2}],$$

The Euler-Lagrange Equation

$$\frac{dx}{du} = \sqrt{r^4 - x^4}$$

$$\frac{dz}{du} = -x^2, \qquad 0 \leq x \leq r$$

The function $x(u)$ is expressed by the Weierstrassian $\wp(u)$

$$x(u) = c + \frac{f'(c)}{4}\left(\wp(u + C_1) - \frac{f''(c)}{24}\right)^{-1}$$

where c is an arbitrary root of the polynomial

$$f(\tau) = -\tau^4 + r^4$$

with the invariants of $\wp(u)$

$$g_2 = -r^4, \qquad g_3 = 0$$

The function $z(u)$ is expressed by

$$z(u) = 2c^4 J_1(u + C_1) - c^6 J_2(u + C_1) - c^2 u + C_2$$

$$J_1(u) = \frac{1}{\wp'(\mathring{u})}\left(2\zeta(\mathring{u})u + \ln\frac{\sigma(u-\mathring{u})}{\sigma(u+\mathring{u})}\right)$$

$$J_2(u) = -\frac{1}{\wp'^2(\mathring{u})}\left(\wp''(\mathring{u})J_1(u) + 2\wp(\mathring{u})u + \zeta(u-\mathring{u}) + \zeta(u+\mathring{u})\right)$$

where $\wp(u)$, $\zeta(u)$, $\sigma(u)$ are the Weierstrassian functions
and $\mathring{u}$ denotes the argument of $\wp(\cdot)$ which produces $\frac{f''(c)}{24}$

Pseudo-Lemniscatic Weierstrassian Functions
$(g_2 = -1,\ g_3 = 0)$

$$
\begin{aligned}
\wp''(u;\ -r^4,\ 0) &= r^4\wp''(ru;\ -1,\ 0)\\
\wp'(u;\ -r^4,\ 0) &= r^3\wp'(ru;\ -1,\ 0)\\
\wp(u;\ -r^4,\ 0) &= r^2\wp(ru;\ -1,\ 0)\\
\zeta(u;\ -r^4,\ 0) &= r\zeta(ru;\ -1,\ 0)\\
\sigma(u;\ -r^4,\ 0) &= r^{-1}\sigma(ru;\ -1,\ 0)
\end{aligned}
$$

The Profile of the Mylar in Pseudo-Lemniscatic Weierstrassian Functions

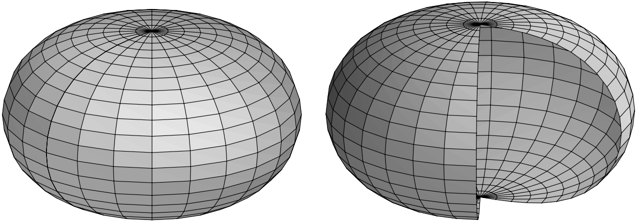
On taking $c = r$ the solution is transformed to

$$x(u) = \frac{r(2\wp(ru;\ -1,\, 0) - 1)}{2\wp(ru;\ -1,\, 0) + 1}$$

$$z(u) = 2r^4 J_1(u + C_1) - r^6 J_2(u + C_1) - r^2 u + C_2$$

where $J_1(u)$, $J_2(u)$ are expressed through the Pseudo-Lemniscatic Weierstrassian functions.

Mylar via Mathematica®



Radius $r = \frac{\sqrt{2}}{K(1/\sqrt{2})} a \approx 0.7627a$

Thickness $\tau = 2\sqrt{2}\left[E(1/\sqrt{2}) - \frac{1}{2}F(1/\sqrt{2})\right] a \approx 0.9139a$

Scale Invariance $\frac{\tau}{2r} \approx 0.599$

Surface Area $A(S) = \pi^2 r^2$

Volume $V = \frac{\pi\sqrt{2}}{3} K(\frac{1}{\sqrt{2}}) r^3$

Decrement of the Surface Area

$$\frac{S_{\text{defl}}}{S_{\text{infl}}} = \frac{2\pi a^2}{\pi^2 r^2} \approx 1.0942$$

Crimping Factor

$$C(x) = \frac{r^2}{x} \int_0^x \frac{dt}{\sqrt{r^4 - t^4}}, \qquad 0 \leq x \leq r$$

The Physical Crimping

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Thank You!