Non-local and local Non-linear SchrÖdinger Equation from Geometric Curve flows in Low dimensional Hermittian Symmetric spaces

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Outline

1. History of Geometric flow, De Rios flow and Hashimoto transformation and Schrödinger equations
2. Integrability(Bi-Hamiltonian equations)
3. Natural framing of Bishop, generalization to sphere
4. Cartan Structure equation on the flow in Symmetric spaces
5. definition of parallel frame in symmetric spaces, example: sphere
6. Local and Non-local Schrödinger Equation on Hermitian symmetric spaces,

Example of geometric curve flows

Motion of vortex filament(Hasimoto 1972, De Rios, 1918(Student of Levi Civita))

$$\vec{\gamma}_t = \vec{T} \wedge \vec{T}_x = \kappa \vec{B}$$

Then the dynamical variable

$$u = \kappa e^{i \int \tau dx},$$

satisfy

$$-iu_t = u_{xx} + \frac{1}{2}|u|^2 u, \quad NLSE.$$

Here $T := \gamma_x$ is the $x-$flow direction and γ_t is $t-$flow direction and γ is a flow of curves swapping a surface in $\mathbb{R}^3$.

Change of framing along curve

(Langer, Perline) and (Doliwa, Santini)

Hasimoto transformation given by $So(2)$ rotation of $\vec{N}, \vec{B}$ through angle $\theta = -\int \tau dx$ in normal plane, yields frame $\{\vec{T}, \vec{n}, \vec{b}\}$:

$$\begin{pmatrix} \vec{T} \\ \vec{n} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}$$

called Parallel frame or Natural frame and transport equation becomes

$$\begin{pmatrix} \vec{T} \\ \vec{n} \\ \vec{b} \end{pmatrix}_x = \begin{pmatrix} 0 & \kappa\cos(\theta) & \kappa\sin(\theta) \\ -\kappa\cos(\theta) & 0 & 0 \\ -\kappa\sin(\theta) & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{n} \\ \vec{b} \end{pmatrix}$$

The Natural frame satisfies Transport eq.

$$\begin{pmatrix} \vec{T} \\ \vec{n} \\ \vec{b} \end{pmatrix}_x = \begin{pmatrix} 0 & Re(u) & Im(u) \\ -Re(u) & 0 & 0 \\ -Im(u) & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{n} \\ \vec{b} \end{pmatrix},$$

with $u := \kappa e^{i\theta} = \kappa \cos(\theta) + i\kappa \sin(\theta)$

Drivation of NLS equation

Take the Lax pair

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi$$

in which $\Phi = \begin{pmatrix} \vec{T} & \vec{n} & \vec{b} \end{pmatrix}^T \in SO(3)$, is the natural frame. Let

$$V = \begin{pmatrix} 0 & Re(w^{\parallel}) & Im(w^{\parallel}) \\ -Re(w) & 0 & w_{\perp} \\ -Im(w^{\parallel}) & -w_{\perp} & 0 \end{pmatrix} = \omega(D_t), \qquad (1)$$

$$U = \begin{pmatrix} 0 & Re(u) & Im(u) \\ -Re(u) & 0 & 0 \\ -Im(u) & 0 & 0 \end{pmatrix} = \omega(D_x) \qquad (2)$$

in which ω is $\mathfrak{so}(3)-$Cartan connection applied to vector fields D_x and D_t.

Drivation of NLS equation(Hashimoto, 1972,Sanders-Wang-Beffa 2003)

Let also

$$\gamma_x = T,$$

$$\gamma_t = h_{\|}\vec{T} + Re(h_{\perp})\vec{n} + Im(h_{\perp})\vec{b}$$

compatibility condition

$$\gamma_{xt} = \gamma_{tx}$$

for flow direction then yield the following equations for $h_{\|}, w^{\perp}$ in terms of $h_{\perp}$:

$$h_{\|} = iD_x^{-1}Re(\bar{u}h_{\perp})$$

$$w^{\|} = \mathcal{I}(h_{\perp}), \quad \mathcal{I} = D_x + iuD_x^{-1}Re(\bar{u}.)$$

Continue:Drivation of NLS equation

Compatibility condition $\begin{pmatrix}\vec{T} & \vec{n} & \vec{b}\end{pmatrix}^T_{xt} = \begin{pmatrix}\vec{T} & \vec{n} & \vec{b}\end{pmatrix}^T_{tx}$ for the frame gives:

$$V_x - U_t + [U, V] = 0, \quad \text{Called Zero curvature equation}$$

which yields following equation for u_t in term of $w^{\parallel}$:

$$u_t = \mathcal{H}(w^{\parallel}), \quad \mathcal{H} = D_x - iuD_x^{-1} Im(\bar{u}.)$$

The Hamiltonian $\mathcal{H}$ and Symplectic operator $\mathcal{J}$ are compatible. So $\mathcal{R} = \mathcal{H}\mathcal{J}$ gives mKdV/NLS(To be found later on) recursion operator.

Continue:Drivation of mKdV and NLS equation

The Hamiltonian operator $\mathcal{H}$ and Symplectic operator are symmetric under $x-$translation and phase rotation on u. So if we take $h_\perp = iu$ then we get NLS equation:

$$u_t = iu_{xx} + 1/2|u|^2 u$$

Notice that in this case clearly $h_{||} = 0$ and so

$$\gamma_t = h_{||}\vec{T} + Re(h_\perp)\vec{n} + Im(h_\perp)\vec{b} = -Im(u)\vec{n} + Re(u)\vec{b} = \vec{T} \wedge T_x$$

and if $h_\perp = u_x$ then we get mKdV equation:

$$u_t = u_{xxx} + 3/2|u|^2 u.$$

and so the flow equation is given as

$$\begin{aligned}\gamma_t &= h_{||}\vec{T} + Re(h_\perp)\vec{n} + Im(h_\perp)\vec{b} \\ &= \nabla_{\gamma_x}\gamma_x + (1 + \frac{1}{2}i)|\nabla_{\gamma_x}\gamma_x|^2\gamma_x\end{aligned}$$

Algebraic characterization of parallel frames:Bishop(AMM)

Stabilizere group of $\vec{T}$ consists of $So(2)$ rotations in normal plane at each point along the curve γ. Indeed $So(2) \subset SO(3)$ is the gauge group of frame bundle and

$$\underbrace{\begin{pmatrix} 0 & \vec{a} \\ -\vec{a}^t & A \end{pmatrix}}_{\in \mathfrak{so}(3)} = \underbrace{\begin{pmatrix} 0 & \vec{0} \\ -\vec{0}^t & A \end{pmatrix}}_{\in \mathfrak{so}(2)} + \underbrace{\begin{pmatrix} 0 & \vec{a} \\ -\vec{a}^t & 0 \end{pmatrix}}_{\mathfrak{so}(2)^{\perp}}$$

The action of infinitesimal rotation $\mathfrak{so}(2)$ preserves class of frames $(\vec{T}, 0, 0)^T$ adapted to γ.

Properties of parallel connection matrix

The matrix belongs to the perp space $\mathfrak{so}(2)^{\perp}$ of infinitesimal stabilizer group $\mathfrak{so}(2) \subset \mathfrak{so}(3)$. and is preserved by infinitesimal rigid (x-independent) rotations $\mathfrak{so}(2)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

with θ constant and so

$$\begin{pmatrix} \vec{T} \\ \vec{n} \\ \vec{b} \end{pmatrix} \mapsto \begin{pmatrix} \vec{T} \\ \vec{n}_1 \\ \vec{b}_1 \end{pmatrix} = \begin{pmatrix} \vec{T} \\ \cos(\theta)\vec{n} + \sin(\theta)\vec{b} \\ -\sin(\theta)\vec{n} + \cos(\theta)\vec{b} \end{pmatrix} \tag{3}$$

$$u \overset{\text{here}}{\mapsto} e^{-i\theta} u \tag{4}$$

That is, u is $U(1)$ covariant.

Generalization

Here we have already considered $R^3 = Euc(3)/So(3)$ as flat Riemannian symmetric spaces in which $Euc(3)$ is isometry group and $So(3)$ is the gauge group of frame bundle or as is seen as isotropy group at origin.

Generalization to simplest compact Symmetric space: sphere

Take $M = S^n = SO(n+1)/SO(n)$ Then the Cartan Matrix $\omega(D_x) \in \mathfrak{so}(n) \subset \mathfrak{so}(n+1)$ will be given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{u} \\ 0 & -\mathbf{u}^T & 0 \end{pmatrix}, \quad \mathbf{u} \in \mathbb{R}^n - 1$$

In this case the Cartan subspace of this symmetric space which 1 dim is given by

$$e := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the Cartan matrix lies in complement of the

$$C_{\mathfrak{so}(n)}(ee) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathfrak{so}(n-1) \end{pmatrix} = \mathfrak{so}(n-1).$$

Klein geometries, Riemannian symmetric space

$$M = G/H, \qquad G = \text{semisimple Lie group}$$
$$H = \text{Lie subgroup invariant under an involutive automorphism}$$

View G as a principal $H-$bundle over M : Local trivialization:

$$\psi : U \subset M \to G \approx U \times H$$

is a section of bundle over coordinate chart.

$$\Omega = \text{Maurer-Cartan form on } G : \text{Flat connection } 1-\text{form}$$
$$\Omega_\psi = \text{pull back of } \Omega \text{ to } U \subset M \text{ by } \psi$$
$$d\Omega_\psi + \frac{1}{2}[\Omega_\psi, \Omega_\psi] = 0 : \quad \text{Zero curvature equation}$$

Frame bundle structure

Coframe and Connection

$$\Omega_\psi = e + \omega, \qquad \mathfrak{g} - \text{valued } 1\text{−form on } M$$
$$e : T_x M \to \mathfrak{m} = T_o M = \mathfrak{g}/\mathfrak{h}, \qquad \omega : T_x M \to \mathfrak{h} = Lie(H)$$

Zero-curvature equation

$d\Omega_\psi + \frac{1}{2}[\Omega_\psi, \Omega_\psi] = 0$: "Zero curvature equation" leads to "Cartan structure equations":

$$\boxed{\begin{aligned} de + [\omega, e] &= 0 \qquad (*) \\ d\omega + \tfrac{1}{2}[\omega, \omega] &= -\tfrac{1}{2}[e, e] \qquad (**) \end{aligned}}$$

Note here that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ in which the following bracket relation holds:

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$$

Cartan subspace in symmetric space and highly non-singular elelemnts of it

Take a Cartan subspace $\mathfrak{a}$ in $\mathfrak{m}$. Then take an element $e \in \mathfrak{a}$ such that the centralizer of it in $\mathfrak{h}$ is of maximal dimension. Notaion:

$$\mathfrak{h}_{\|} := C_{\mathfrak{h}}(e), \quad \mathfrak{m}_{\|} := C_{\mathfrak{m}}(e)$$

and consider their complements in $\mathfrak{g}$ w.r.t Killing form call them $\mathfrak{h}_{\perp}, \quad \mathfrak{m}_{\perp}$ so that $\mathfrak{m} = \mathfrak{m}_{\|} \oplus \mathfrak{m}_{\perp}, \quad \mathfrak{h} = \mathfrak{h}_{\|} \oplus \mathfrak{h}_{\perp}$ Important: $ad(e) : \mathfrak{m}_{\perp} \longmapsto \mathfrak{h}_{\perp}$ is invertible and Bracket relation:

$$[\mathfrak{m}_{\|}, \mathfrak{m}_{\|}] \subseteq \mathfrak{h}_{\|}, \quad [\mathfrak{m}_{\|}, \mathfrak{h}_{\|}] \subseteq \mathfrak{m}_{\|}, \quad [\mathfrak{h}_{\|}, \mathfrak{h}_{\|}] \subseteq \mathfrak{h}_{\|}, \tag{5}$$

$$[\mathfrak{h}_{\|}, \mathfrak{m}_{\perp}] \subseteq \mathfrak{m}_{\perp}, \quad [\mathfrak{h}_{\|}, \mathfrak{h}_{\perp}] \subseteq \mathfrak{h}_{\perp}, \tag{6}$$

$$[\mathfrak{m}_{\|}, \mathfrak{m}_{\perp}] \subseteq \mathfrak{h}_{\perp}, \quad [\mathfrak{m}_{\|}, \mathfrak{h}_{\perp}] \subseteq \mathfrak{m}_{\perp}, \tag{7}$$

while the remaining Lie brackets

$$[\mathfrak{m}_{\perp}, \mathfrak{m}_{\perp}], \quad [\mathfrak{h}_{\perp}, \mathfrak{h}_{\perp}], \quad [\mathfrak{m}_{\perp}, \mathfrak{h}_{\perp}] \tag{8}$$

obey the general relations .

H-Parallel frame and Curve flows

$\gamma : \mathbb{R} \longmapsto M = G/H$
$e_x = \gamma_x \rfloor e =$
frame components of tangent vector along γ, $\quad e_x \in \mathfrak{m}$.
$\omega_x = \gamma_x \rfloor \omega =$
components of connection in tangent direction $\quad \omega_x \in \mathfrak{h}$

Cartan subspace

fixed an element e_x in any maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}$
Choose $\omega_x \in \mathfrak{h}_\perp$,

above choices yields framing along γ via "transition equation"

$$\nabla_x = -\mathrm{ad}(\omega_x)e$$

generalizes parallel framing of space curves in
$\mathrm{Re}^3 = \mathrm{Euc}(3)/\mathrm{So}(3)$.

Example of Riemannian symplectic space of symplectic gauge groups: $Sp(n+1)/Sp(1)\times SP(n)$ and $SU(2n)/Sp(n)$(Anco-A.-J. Phys A.2012)

Matrix connection ω_x: $Sp(n+1)/Sp(1)\times SP(n)$–Quaternionic version!

$$\omega_x = \begin{pmatrix} \mathrm{u} & 0 & \mathbf{0} \\ 0 & -\mathrm{u} & \mathbf{u} \\ \mathbf{0} & -\overline{\mathbf{u}}^t & \mathbf{0} \end{pmatrix} \in \mathfrak{h}_\perp .$$

Covariants of $\gamma : \mathfrak{sp}(1)\times\mathfrak{sp}(n-1)$

Matrix connection for $SU(2n)/Sp(n)$

$$\omega_x = \begin{pmatrix} 0 & \mathbf{u} \\ -\overline{\mathbf{u}}^t & \mathbf{0} \end{pmatrix} \in \mathfrak{h}_\perp .$$

Covariants of $\gamma : \mathfrak{sp}(1)\times\mathfrak{sp}(n-1)$

Hermitian symmetric space $SU(n+1)/U(n)$

elements of $\mathfrak{h}, \mathfrak{m}$

$$\begin{pmatrix} -tr(\mathbf{C}) & 0 \\ 0 & \mathbf{C} \end{pmatrix} := (\mathbf{C}) \in \mathfrak{h} = \mathfrak{u}(n), \quad \mathbf{C} \in \mathfrak{u}(n)$$

$$\begin{pmatrix} 0 & \mathbf{a} \\ -\overline{\mathbf{a}}^{\mathrm{t}} & \end{pmatrix} := (\mathbf{a}) \in \mathfrak{m} = C^n, \quad \mathbf{a} \in C^n$$

choice of e_x of Cartan subspace $\mathfrak{a}$. the space is of rank 1. so there is only one choice:

$$e_x = (1, 0) \in \mathfrak{m} = \mathbb{C}^n, \quad 0 \in \mathbb{C}^{n-1}$$

The perp and parallel subspaces:

perp and parallel subspace

$$\left(\begin{pmatrix} -\frac{1}{2}tr(\mathbf{C}_{\|}) & 0 \\ 0 & \mathbf{C}_{\|} \end{pmatrix}\right) := (\mathbf{C}_{\|}) \in \mathfrak{h}_{\|} \subset \mathfrak{h}, \quad \mathbf{C}_{\|} \in \mathfrak{u}(n-1)$$

$$\left(\begin{pmatrix} \mathrm{ic}_{\perp} & \mathbf{c}_{\perp} \\ -\bar{\mathbf{c}}_{\perp} & 0 \end{pmatrix}\right) := (\mathrm{ic}_{\perp}, \mathbf{c}) \in \mathfrak{h}_{\perp} \subset \mathfrak{h}, \quad \mathbf{c} \in \mathbb{C}^{n-1}, \quad \mathrm{c}_{\perp} \in \mathbb{R}$$

$$(\mathrm{a}_{\|}) := ((\mathrm{a}_{\|}, 0)) \in \mathfrak{m}_{\|} \subset \mathfrak{m}, \quad \mathrm{a}_{\|} \in \mathbb{R}$$

$$(\mathrm{ia}_{\perp}, \mathbf{a}_{\perp}) := ((\mathrm{ia}_{\perp}, \mathbf{a}_{\perp})) \in \mathfrak{m}_{\perp} \subset \mathfrak{m}, \quad \mathbf{a} \in \mathbb{C}^{n-1}, \quad \mathrm{a}_{\perp} \in \mathbb{R}$$

$$e_x = (1,0) = (1) \in \mathfrak{m}_{\|}$$

Flow direction

Flow directions:

$$\mathrm{e} = \frac{1}{\sqrt{\chi}}(1,0) \in \mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{m}_{\|}, \quad \chi = const. \tag{9}$$

$$u = (\mathrm{iu}, \mathbf{u}) \in \mathrm{i}\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_{\perp}, \tag{10}$$

$$h_{\|} = (\mathrm{h}_{\|}) \in \mathbb{R} \simeq \mathfrak{m}_{\|}, \tag{11}$$

$$h_{\perp} = (\mathrm{ih}_{\perp}, \mathbf{h}_{\perp}) \in \mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{m}_{\perp}, \tag{12}$$

$$\varpi^{\|} = (\Theta) \in \mathfrak{u}(n-1) \simeq \mathfrak{h}_{\|}, \tag{13}$$

$$\varpi^{\perp} = (\mathrm{iw}, \mathbf{w}) \in \mathrm{i}\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_{\perp}, \tag{14}$$

as well as

$$h^{\perp} = (\mathrm{ih}^{\perp}, \mathbf{h}^{\perp}) = \mathrm{ad}(\mathrm{e})h_{\perp} = \frac{1}{\sqrt{\chi}}(-2\mathrm{ih}_{\perp}, -\mathbf{h}_{\perp}) \in \mathrm{i}\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_{\perp} \tag{15}$$

hence

$$\mathrm{h}_{\perp} = -\sqrt{\chi}\frac{1}{2}\mathrm{h}^{\perp}, \quad \mathbf{h}_{\perp} = -\sqrt{\chi}\mathbf{h}^{\perp}$$

Local NLS system:Let The $J_0 = (\mathrm{i}I) \in \mathfrak{h}_{\|}$ be $\mathfrak{u}_1$ subalegbras of $\mathfrak{h}_{\|} = \mathfrak{u}(n-1)$. then let

$$(\mathrm{ih}^{\perp}, \mathbf{h}^{\perp}) = h^{\perp} := \mathrm{ad}(J_0)u = (0, \mathrm{i}\lambda\mathbf{u}), \quad \lambda = const. = -\frac{1}{2}(n-1)-1$$

Hence putting

$$\mathrm{h}^{\perp} = 0, \quad \mathbf{h}^{\perp} = \lambda\mathrm{i}\mathbf{u}$$

in the adapted flow equations Then NLS type equation is derived as

$$\frac{1}{\lambda\chi}\mathrm{iu}_t = \mathrm{i}|\mathbf{u}|^2$$
$$\frac{1}{\lambda\chi}\mathbf{u}_t = \mathrm{i}\mathbf{u}_{xx} + \mathrm{i}\mathbf{u}[(\mathrm{u}^2 + |\mathbf{u}|^2) - \mathrm{iu}_x]$$

Hermittian symmetric space of $SP(2)/U(2)$

choose $e_x = \mathrm{i}I = \mathrm{i}(E_{11} + E_{22})$. The parallel and perp representation of $\mathfrak{m}$ and $\mathfrak{h}$ is given by

$$\begin{pmatrix} \mathrm{i}Im(a_{11}) & \mathrm{i}Im(a_{12}) \\ \mathrm{i}Im(a_{12}) & \mathrm{i}Im(a_{22}) \end{pmatrix} \in \mathfrak{m}_{\parallel}, \quad \begin{pmatrix} Re(a_{11}) & Re(a_{12}) \\ Re(a_{12}) & Re(a_{22}) \end{pmatrix} \in \mathfrak{m}_{\perp} \tag{16}$$

$$\begin{pmatrix} 0 & Re(c_{12}) \\ -Re(c_{12}) & 0 \end{pmatrix} \in \mathfrak{h}_{\parallel} = \mathfrak{u}(1), \quad \begin{pmatrix} c_{11} & \mathrm{i}Im(c_{12}) \\ \mathrm{i}Im(c_{12}) & c_{22} \end{pmatrix} \in \mathfrak{h}_{\perp} \tag{17}$$

in which $c_{11}, c_{22} \in \mathrm{i}\mathbb{R}$
Now take the matrix

$$J_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generator of $\mathfrak{h}_{\perp}$. Then we wanted to displite all parallel and perp subspace into invariant and complex variable.

flow direction

$$\mathrm{e} = \frac{1}{\sqrt{\chi}}(1,0) \in \mathbb{R} \oplus \mathbb{C} \simeq \mathfrak{m}_{\|}, \quad \chi = ??? \tag{18}$$

$$u = (\mathrm{u}, \mathbf{u}) \in \mathbb{R} \oplus \mathbb{C} \simeq \mathfrak{h}_{\perp}, \tag{19}$$

and

$$h_{\|} = (\mathrm{h}_{\|}, \mathbf{h}_{\|}) \in \mathbb{R} \oplus \mathbb{C} \simeq \mathfrak{m}_{\|}, \tag{20}$$

$$h_{\perp} = (\mathrm{h}_{\perp}, \mathbf{h}_{\perp}) \in \mathbb{R} \oplus \mathbb{C} \simeq \mathfrak{m}_{\perp}, \tag{21}$$

$$\varpi^{\|} = (\Theta) \in \mathfrak{u}(1) \simeq \mathfrak{h}_{\|}, \tag{22}$$

$$\varpi^{\perp} = (\mathrm{w}, \mathbf{w}) \in \mathbb{R} \oplus \mathbb{C} \simeq \mathfrak{h}_{\perp}, \tag{23}$$

as well as

$$h^{\perp} = (\mathrm{h}, \mathbf{h}) = \mathrm{ad}(\mathrm{e})h_{\perp} = \frac{1}{\sqrt{\chi}}(2\mathrm{h}_{\perp}, 2\mathrm{i}\mathbf{h}_{\perp}) \in \mathbb{R} \oplus \mathbb{C} \simeq \mathfrak{h}_{\perp} \tag{24}$$

Non-local NLS

$$\mathrm{u}_t = \mathrm{u}|\mathbf{u}|^2 + Re(\bar{\mathbf{u}}_x D_x^{-1}(\mathrm{u}\mathbf{u})) \tag{25}$$

$$\mathbf{u}_t = \mathrm{i}(\frac{1}{4}\mathbf{u}_{xx} + \frac{1}{2}\mathbf{u}|\mathbf{u}|^2 + +\mathrm{u}^2\mathbf{u} + \mathrm{u}_x D_x^{-1}(\mathrm{u}\mathbf{u}) + 2\mathbf{u}|D_x^{-1}(\mathrm{u}\mathbf{u})|^2) \tag{26}$$

Thank you

$U(1)$-covariant

$$\begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{n}_1 \\ \vec{b}_1 \end{pmatrix} = \begin{pmatrix} \vec{T} \\ \vec{n}_1 \\ \vec{b}_1 \end{pmatrix}_x = \begin{pmatrix} Re(u)\vec{n} + Im(u)\vec{b} \\ \left(\cos(\theta)Re(u) - \sin(\theta)Im(u)\right)\vec{T} \\ \left(\sin(\theta)Re(u) - \cos(\theta)Im(u)\right)\vec{T} \end{pmatrix}$$

Hence we can drive that

$$Re(u_1) := \kappa_1 = \cos(\theta)Re(u) + \sin(\theta)Im(u) = Re(e^{-i\theta}u) \quad (27)$$

$$Im(u_1) := \kappa_2 = -\sin(\theta)Re(u) + \cos(\theta)Im(u) = Im(e^{-i\theta}u) \quad (28)$$

Thus

$$u_1 = e^{-i\theta}u$$

Back to main.