Error Analysis in an Iterative Algorithm for the Solution of the Regulator Equations for Distributed Parameter Systems

Eugenio Aulisa

Hansameenu Wijenayaka and David Gilliam

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Outline

1. Statement of Problem and Dynamic Controller
2. β-Iterative Scheme
3. Error Estimates
4. Numerical Example

Statement of Problem

- Consider a MIMO plant modeled by a linear system
$$
\begin{aligned}
z_t(t) &= Az(t) + B_{\text{in}}u(t) + B_{\text{d}}d(t),\\
z(0) &= z_0,\\
y(t) &= Cz(t).
\end{aligned}
$$
We are given
  - y_r a signal to be tracked,
  - d a disturbance to be rejected.
- $z(t)$ is the state variable in the infinite dimensional Hilbert space $\mathcal{Z}$.
- Problem:
Find the time dependent controller $u(t)$ such that $y = Cz(t)$ satisfies
$$\lim_{t\to\infty} \|y_r(t) - y(t)\| = 0.$$

Statement of Problem Cont'd.

- A is an unbounded sectorial operator with dense domain $\mathcal{D}(A)$.
- A generates an exponentially stable analytic semigroup e^{At} in $\mathcal{Z}$

$$\|e^{At}\| < Me^{-at} \qquad a > 0 \text{ and } M > 1.$$

- $B_{\text{in}} \in \mathcal{L}(\mathbb{R}^k, \mathcal{Z})$, and $B_{\text{d}} \in \mathcal{L}(\mathbb{R}^m, \mathcal{Z})$.
- C is unbounded but is relatively bounded by a fractional power of $(-A)$. There exists $s \in (0, 2)$ s.t.

$$C \in \mathcal{L}(\mathcal{H}^s, \mathbb{R}^k), \quad \text{where} \quad \mathcal{H}^s = \mathcal{D}((-A)^{s/2}).$$

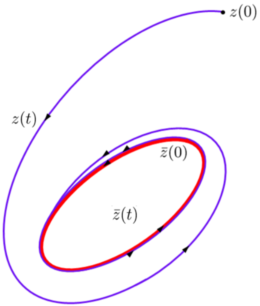
- Without loss of generality, by rescaling, we assume

$$-CA^{-1}B_{\text{in}} = I.$$

Dynamic Controller (DC)

- DC
  For given $y_r(t)$ and $d(t)$
  find $\overline{z}$, $\overline{\gamma}(t)$ and $\overline{z}_0$ satisfying
  $\overline{z}_t = A\overline{z} + B_{\text{in}}\overline{\gamma}(t) + B_{\text{d}}d(t),$
  $\overline{z}(0) = \overline{z}_0,$
  such that
  $C\overline{z}(t) = y_r(t) \ , \ \forall \ \ t \geq 0.$

- In the closed loop system, we set $u(t) = \overline{\gamma}(t)$, and expect
  $$\lim_{t\to\infty} \|Cz(t) - C\overline{z}(t)\| = 0,$$
  for any initial data z_0.



Why the Dynamic Controller?

- In the classical geometric method $y_r(t)$ and $d(t)$ are generated by a finite dimensional exogenous system (exosystem)
- In this case we obtain a system of two functional equations called the Regulation Equations
- Solvability of the Regulator Problem is equivalent to the solvability of the Regulator Equations
- Solving the Regulator Equations is not easy.
- The Exosystem might be non-linear.
- The Exosystem might be infinite dimensional.
- The plant could be non-linear.

Regularized Dynamic Controller RDC

- We can rewrite the DC as

$$(I + B_{\text{in}} C A^{-1}) \overline{z}_t = A\overline{z}(t) + B_{\text{in}} y_r(t) + (I + B_{\text{in}} C A^{-1}) B_{\text{d}} d(t).$$

- For $0 < \beta < 1$ we set,

$$(I + (1 - \beta) B_{\text{in}} C A^{-1}) \overline{z}^1_t = A\overline{z}^1(t) + B_{\text{in}} y_r(t) + (I + B_{\text{in}} C A^{-1}) B_{\text{d}} d(t).$$

Regularized Dynamic Controller RDC, The Equivalent Form

$$\overline{z}_t^1 = A_\beta \overline{z}^1(t) + \frac{1}{\beta} B_{\text{in}} y_r(t) + F\, d(t),$$

where $A_\beta = \left(A - \dfrac{(1-\beta)}{\beta} B_{\text{in}} C \right)$, $\quad F = (I + B_{\text{in}} C A^{-1}) B_{\text{d}}$.

The control can be aposteriori evaluated as

$$\overline{\gamma}^1(t) = y_r(t) - (1-\beta) C A^{-1} (\overline{z}_t^1) + C A^{-1} B_{\text{d}} d(t).$$

Corollary (Lassi Paunonen)

For $\delta = \frac{1-\beta}{\beta}$ sufficiently close to 0 (thus for β sufficently close to 1) $A_\beta = A - \delta B_{in} C$ generates an exponentially stable semigroup.

General Form of RDC

- We solve the following RDC in each iteration $i = 1, 2, ...,$ for given target $\mathcal{Y}_r(t)$ and disturbance $\mathcal{D}(t)$.

$$\overline{z}^i_t = A_\beta \overline{z}^i(t) + \frac{1}{\beta} B_{\text{in}} \mathcal{Y}_r(t) + F\mathcal{D}(t).$$

- Then $\overline{\gamma}^i(t)$ can be written in following explicit form.

$$\overline{\gamma}^i(t) = \mathcal{Y}_r(t) - (1-\beta)CA^{-1}(\overline{z}^i_t) + CA^{-1}B_{\text{in}}\mathcal{D}(t).$$

β-iterative scheme

Find approximate values for $\overline{z}(t)$ and $\overline{\gamma}(t)$ by seeking

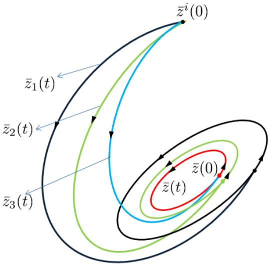
$$\overline{z}_n(t) = \sum_{j=1}^{n} \overline{z}^j(t), \quad \overline{\gamma}_n(t) = \sum_{j=1}^{n} \overline{\gamma}^j(t)$$

with $\overline{z}_n(t) \xrightarrow{n\to\infty} \overline{z}(t)$ and $\overline{\gamma}_n(t) \xrightarrow{n\to\infty} \overline{\gamma}(t)$.

$$E_1(t) = y_r(t) - C(\overline{z}^1(t)),$$

and for $i = 2, 3, ..., n$,

$$E_i(t) = E_{i-1}(t) - C(\overline{z}^i(t)).$$



β-Iterative Algorithm

- Iteration 0: Solve Set Point control problem for $\overline{z}^0(x)$ for tracking $y_r(0)$ and rejecting $d(0)$.

- Iteration 1: Solve RDC for $\overline{z}^1(x)$ by setting $\mathcal{Y}_r(t) = y_r(t)$ and $\mathcal{D}(t) = d(t)$ with the I.C. $\overline{z}^1(x,0) = \overline{z}^0(x)$.

- Iteration $i > 1$: Solve RDC for $\overline{z}^i(x)$ by setting $\mathcal{Y}_r(t) = E_{i-1}(t)$ and $\mathcal{D}(t) = 0$ with the I.C. $\overline{z}^i(x,0) = 0$.

Error as a convolution integral.

Let us define

$K(t) = -CA_\beta^{-1} e^{A_\beta t} \frac{1}{\beta} B_{\text{in}}$ and
$K_d(t) = -CA_\beta^{-1} e^{A_\beta t} (I + B_{\text{in}} CA^{-1}) B_{\text{d}}$,

- It can be shown that the first iteration error is given by

$$E_1(t) = y_r(t) - C(\overline{z}^1(t)) = K * y_r'(t) + K_d * d'(t).$$

- and the error at the ith-iteration is given by

$$E_i(t) = E_{i-1}(t) - C(\overline{z}^i) = K * E_{i-1}'(t) \text{ for } i = 2, \cdots, n.$$

Theorem

Assume $y_r, d \in C_b^n[0,\infty)$ and let β_0 s.t. A_β generates an exponentially stable analytic semigroup in $\mathcal{Z}$ for $\beta \in (\beta_0, 1)$. Then for any $T \geq 0$ we have

$$E_n(t) = \mathcal{E}_{1,T,n}(t) + \mathcal{E}_{2,T,n}(t)$$

where $\limsup_{t\to\infty} |\mathcal{E}_{1,T,n}(t)| = 0$ *and*

$$\sup_{t\in[T,\infty)} |\mathcal{E}_{2,T,n}(t)| \leq D^n\Big(\sup_{t\in[T,\infty)} |y_r^{(n)}(t)| + \beta\|B_{in}\|^{-1}\|B_d\| \sup_{t\in[T,\infty)} |d^{(n)}(t)|\Big),$$

where $D(A, B_{in}, C, \beta)$ is a constant.

Example 1: 1D Heat Equation with no disturbance

- We consider the control system defined on $0 \leq x \leq 1$ for $t \geq 0$ given by

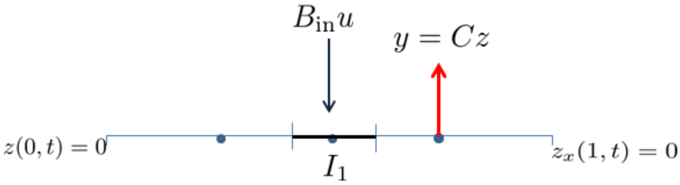
$$\begin{aligned}
&\frac{\partial z(t)}{\partial t} = Az(t) + B_{\text{in}} u(t), \\
&z(0,t) = 0, \frac{\partial z}{\partial x}(1,t) = 0, \\
&z(x,0) = 0, \\
&y(t) = Cz(t), y_r(t) = \cos(t).
\end{aligned}$$

- In this case we define the operator $A = \dfrac{d^2}{dx^2}$ with domain

$$\mathcal{D}(A) = \{\varphi \in H^2(0,1) \,:\, \varphi(0) = 0, \quad \varphi'(1) = 0\}$$

in the Hilbert state space $\mathcal{Z} = L^2(0,1)$.

- $Cz(t) = z(0.75, t)$.
- $B_{\text{in}} u(t) = \frac{1}{|I_1|} \chi_{I_1} u(t)$.
- $I_1 = (0.5 - \delta, 0.5 + \delta)$.



- We set $\delta = 0.05$.

Continuous dependence with respect to β for the eigenvalues of

$$A_\beta = A - \frac{1-\beta}{\beta} B_{\text{in}} C$$

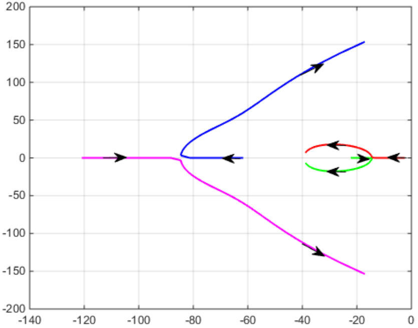


Figure 1 : Eigenvalues λ_1, λ_2, λ_3, and λ_4 for β varying from 1 to 0.01.

For $\beta = 0.27$, the plot of e_1(green),e_2(blue)and e_3(red).

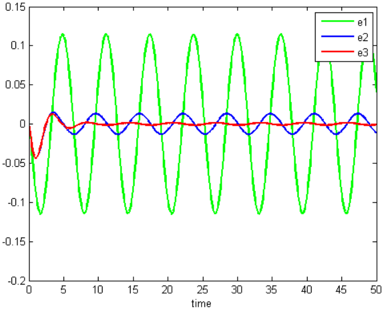


Figure 2 : The first three iteration errors.

$$D \sim 0.1$$

$0 \leq t \leq 20$	Error
$\|\|e_1\|\|_\infty$	1.15×10^{-1}
$\|\|e_2\|\|_\infty$	4.37×10^{-2}
$\|\|e_3\|\|_\infty$	4.36×10^{-2}
$20 \leq t \leq 35$	Error
$\|\|e_1\|\|_\infty$	1.15×10^{-1}
$\|\|e_2\|\|_\infty$	1.32×10^{-2}
$\|\|e_3\|\|_\infty$	1.52×10^{-3}
$35 \leq t \leq 50$	Error
$\|\|e_1\|\|_\infty$	1.15×10^{-1}
$\|\|e_2\|\|_\infty$	1.32×10^{-2}
$\|\|e_3\|\|_\infty$	1.52×10^{-3}

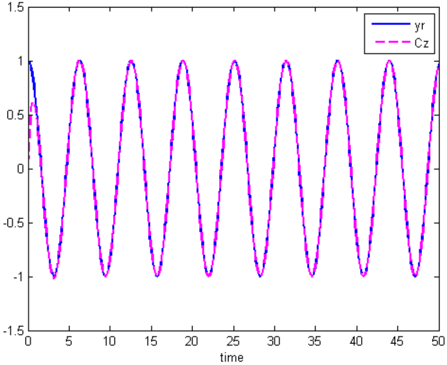


Figure 3 : y_r and $C(z)$.

Example 2: 1D Heat Equation with nonlinear Exosystem

We consider the control system defined on $0 \leq x \leq 1$ for $t \geq 0$ given by

$$\begin{aligned}
&\frac{\partial z(t)}{\partial t} = Az(t) + B_{\text{in}} u(t) + B_{\text{d}} d(t),\\
&z(0,t) = 0, \frac{\partial z}{\partial x}(1,t) = 0,\\
&z(x,0) = 0,\\
&y(t) = Cz(t) = z(1,t).
\end{aligned}$$

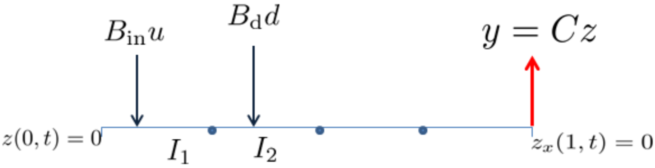
In this case we define the operator $A = \dfrac{d^2}{dx^2}$ with domain

$$\mathcal{D}(A) = \{\varphi \in H^2(0,1) \,:\, \varphi(0) = 0, \quad \varphi'(1) = 0\}$$

in the Hilbert state space $\mathcal{Z} = L^2(0,1)$.

In our specific numerical example we have set

$$I_1 = \{x : 0 \leq x < 1/4\},$$
$$I_2 = \{x : 1/4 \leq x \leq 1/2\},$$



The reference signal y_r and the disturbance d are given by the solution of

$$\ddot{\omega} + \dot{\omega} - \omega + \omega^3 = 0,$$

s.t

$$\text{for the I.C } \omega(0) = 0, \dot{\omega}(0) = 1.7, \quad y_r(t) = \omega(t) \longrightarrow 1 \text{ as } t \to \infty,$$

$$\text{for the I.C } \omega(0) = 1, \dot{\omega}(0) = 1, \quad d(t) = \omega(t), \longrightarrow -1 \text{ as } t \to \infty.$$

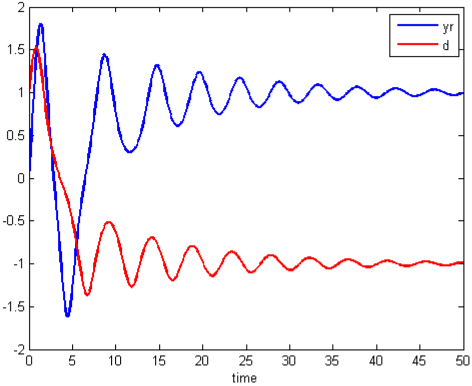


Figure 4 : The reference y_r and disturbance d.

For $\beta = 0.1$, the plot of e_1(green),e_2(blue)and e_3(red).

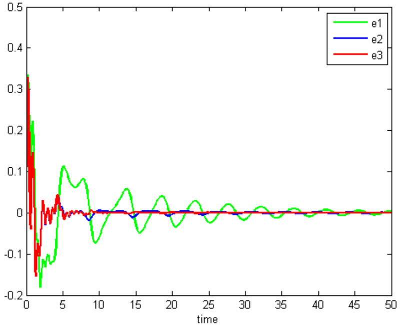


Figure 5 : The first three iteration errors.

$D \sim 0.14$

$0 \leq t \leq 20$	Error
$\|e_1\|_\infty$	3.36×10^{-1}
$\|e_2\|_\infty$	3.29×10^{-1}
$\|e_3\|_\infty$	3.29×10^{-1}
$20 \leq t \leq 35$	Error
$\|e_1\|_\infty$	3.43×10^{-2}
$\|e_2\|_\infty$	4.67×10^{-3}
$\|e_3\|_\infty$	1.18×10^{-3}
$35 \leq t \leq 50$	Error
$\|e_1\|_\infty$	1.06×10^{-2}
$\|e_2\|_\infty$	1.53×10^{-3}
$\|e_3\|_\infty$	2.22×10^{-4}

For $\beta = 0.1$

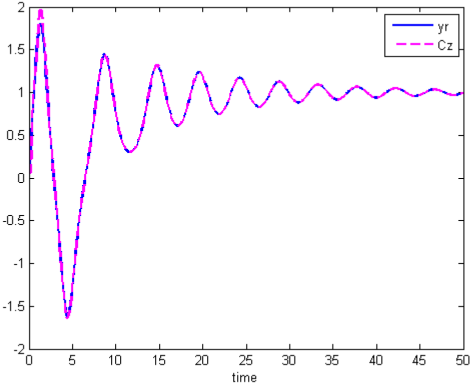


Figure 6 : y_r and $C(z)$

Example 3: Thermal regulation of a Navier-Stokes Flow

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\left[\mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\right] + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0,$$

$$\frac{\partial T}{\partial t} = \alpha\Delta T - \mathbf{v}\cdot\nabla T + B_{\text{in}}u + B_{\text{d}}d, \quad y(t) = CT(t),$$

$$y_r(t) = a + b\sin(\omega_1 t), \quad d(t) = c + d\sin(\omega_2 t)$$

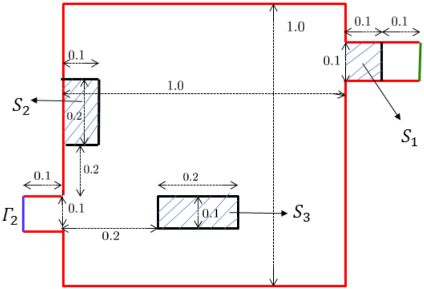
$$\mathbf{v}(x,0) = 0, p(x,0) = 0, T(x,0) = 0$$

$$\mathbf{v} = 0 \text{ on } \Gamma_w,$$

$$\mathbf{v} = \begin{pmatrix} f(s) \\ 0 \end{pmatrix}, \; T = 0 \text{ on } \Gamma_1,$$

$$-\alpha\nabla T\cdot\mathbf{n} = 0 \text{ on } \Gamma_2\cup\Gamma_w,$$

$$\boldsymbol{\tau} = 0 \text{ on } \Gamma_2.$$



Here,

$$A = \alpha\Delta - \mathbf{v}\cdot\nabla, \quad \mathcal{D}(A) = \left\{\varphi \in H^2(\Omega) : \varphi\big|_{\Gamma_1} = 0, -\alpha\nabla\varphi\cdot\mathbf{n}\big|_{\Gamma_2\cup\Gamma_w} = 0\right\}$$

in the Hilbert state space $\mathcal{Z} = L^2(\Omega)$.

$$CT(t) = \frac{1}{|S_3|}\int_{S_3} T ds,$$

$$B_{\text{in}}u(t) = \frac{1}{|S_1|}\chi_{S_1}u(t), \quad B_{\text{d}}d(t) = \frac{1}{|S_2|}\chi_{S_2}d(t).$$

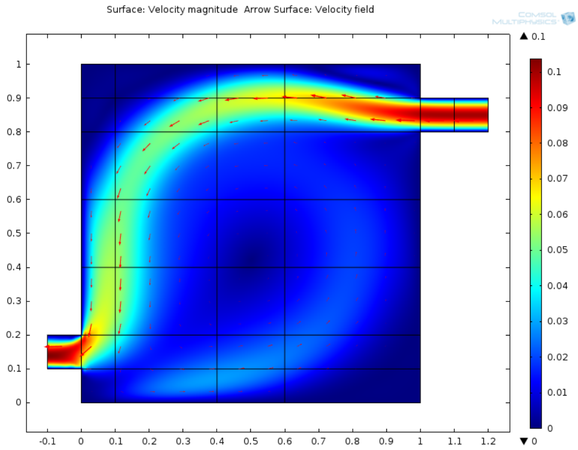


Figure 7 : The Velocity Profile.

For $\beta = 0.1$

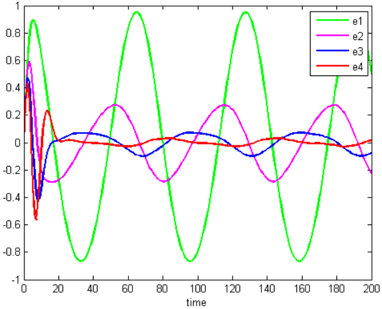


Figure 8 : The first three iteration errors.

$$D \sim 0.3$$

$0 \leq t \leq 100$	Error
$\|e_1\|_\infty$	9.51×10^{-1}
$\|e_2\|_\infty$	5.94×10^{-1}
$\|e_3\|_\infty$	4.72×10^{-1}
$\|e_4\|_\infty$	5.63×10^{-1}
$100 \leq t \leq 200$	Error
$\|e_1\|_\infty$	9.49×10^{-1}
$\|e_2\|_\infty$	2.86×10^{-1}
$\|e_3\|_\infty$	9.86×10^{-2}
$\|e_4\|_\infty$	3.10×10^{-2}

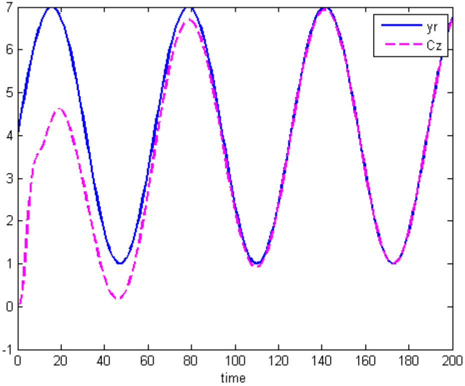


Figure 9 : y_r and the measured output of the closed loop system CT.

Future Work

- Obtain error estimates of the β-iterative method for control systems with
  - unbounded B
  - nonlinear state
- Build c++ PDE toolbox for solving Regulator Problem using the β-iterative method

References

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Thank You