LOW-TYPE SUBMANIFOLDS OF COMPLEX SPACE FORMS

IVKO DIMITRIĆ

1. Introduction

A submanifold $x : M^n \to E^N_{(K)}$ of (pseudo) Euclidean space is said to be of finite type in $E^N_{(K)}$ if the position vector x can be decomposed into a finite sum of vector eigenfunctions of the Laplacian Δ_M on M, viz.

$$x = x_0 + x_1 + \cdots + x_k, \tag{1}$$

where $x_0 = \text{const}, \quad x_i \neq \text{const}, \quad \text{and} \quad \Delta x_i = \lambda_i x_i, \ \ i = 1, ..., k$. For a compact submanifold, x_0 is the center of mass. If λ_i are all different the submanifold is of Chen-type k, or simply of $k-$type. Note that 1-type submanifolds of a Euclidean E^N space are precisely those that are minimal in some hypersphere of the ambient space or minimal in E^N. This notion can be extended to submanifolds $x : M^n \to \bar{M}$ of a more general manifold $\bar{M}$ as long as there is a reasonably "nice" embedding $\Phi : \bar{M} \to E^N_{(K)}$ of the ambient manifold $\bar{M}$ into a suitable (pseudo) Euclidean space, in which case M is said to be of Chen-type k (via Φ) if the composite immersion $\Phi \circ x$ is of Chen-type k.

The complex projective space $\mathbb{C}P^m(4)$ and the complex hyperbolic space $\mathbb{C}H^m(-4)$ (jointly denoted by $\mathbb{C}Q^m(4c), \ c = \pm 1,$) can be equivarianty embedded into a certain (pseudo) Euclidean space $E^N_{(K)}$ of suitable Hermitian matrices by the projection operators. In the case of $\mathbb{C}P^m$, this is the so-called first standard embedding. We will use the symbol $\Phi : \mathbb{C}Q^m(4c) \to E^N_{(K)}$ for the embedding that associates to every complex line in $\mathbb{C}^{m+1}$ the operator (i.e. its matrix) of the orthogonal projection onto it.

Consider the standard Hermitian form Ψ_c on $\mathbb{C}^{m+1}$ given by $\Psi_c(z, w) = c\bar{z}_0 w_0 + \sum_{j=1}^m \bar{z}_j w_j, \ \ z, w \in \mathbb{C}^{m+1}$ and the quadric hypersurface $N^{2m+1} := \{z \in \mathbb{C}^{m+1} \,|\, \Psi_c(z, z) = c\}$. When $c = 1, \ N^{2m+1}$ is the ordinary hypersphere S^{2m+1} of $\mathbb{C}^{m+1} = \mathbb{R}^{2m+2}$ and when $c = -1, \ N^{2m+1}$ is the anti - de Sitter space H^{2m+1}_1 in $\mathbb{C}^{m+1}_1$. The orbit space under the natural action of the circle group S^1 on N^{2m+1} defines $\mathbb{C}Q^m(4c)$. The standard embedding Φ into the set of $\Psi-$Hermitian matrices $H^{(1)}(m+1)$ is achieved by identifying a point, that is a complex line (or a time-like complex line in the hyperbolic case) with the projection operator onto it. Then one gets the following matrix representation of Φ at a point $p = [z]$, where $z = (z_j) \in N^{2m+1} \subset \mathbb{C}^{m+1}_{(1)}$

$$\Phi([z]) = \begin{pmatrix} |z_0|^2 & cz_0\bar{z}_1 & \cdots & cz_0\bar{z}_m \\ z_1\bar{z}_0 & c|z_1|^2 & \cdots & cz_1\bar{z}_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m\bar{z}_0 & cz_m\bar{z}_1 & \cdots & c|z_m|^2 \end{pmatrix}. \tag{2}$$

The second fundamental form of this embedding is parallel and the image

$$\Phi(\mathbb{C}Q^m) = \{P \in H^{(1)}(m+1) | P^2 = P, \operatorname{tr} P = 1\} \tag{3}$$

of the space form is contained in the intersection of the hyperplane $\{\operatorname{tr} A = 1\}$ with the hyperquadric of $H^{(1)}(m+1)$ centered at $I/(m+1)$ and defined by the equation $\langle P - I/(m+1),\, P - I/(m+1)\rangle = \frac{cm}{2(m+1)}$, where I denotes the $(m+1)\times(m+1)$ identity matrix. A submanifold contained in this hyperquadric is said to be mass-symmetric if its center of mass is $\tilde{x}_0 = \frac{I}{m+1}$.

If now $x : M^n \to \mathbb{C}Q^m(4c)$ is an isometric immersion of a Riemannian $n-$manifold as a submanifold of a complex space form then we have the associated composite immersion $\tilde{x} = \Phi \circ x$, which realizes M as a submanifold of the (pseudo) Euclidean space $E^N_{(K)} := H^{(1)}(m+1)$, equipped with the usual trace metric $\langle A, B\rangle = \frac{c}{2}\operatorname{tr}(AB)$.

The study of finite-type submanifolds $x : M^n \to \mathbb{C}Q^m(4c)$ is then the study of the spectral behavior of the associated immersion $\tilde{x} = \Phi \circ x$ of M^n into $E^N_{(K)}$, i.e. of the possibility of decomposing $\tilde{x}$ into finitely many eigenfunctions of Δ_M.

A $k-$type immersion x satisfies a polynomial equation in the Laplacian, $P(\Delta)(x - x_0) = 0$. The most promissing study is that of submanifolds of low type: 1, 2, or 3.

2. **Some Classification Results**

The study of 1-type submanifolds of $\mathbb{C}P^m$ was begun in works of A. Ros (1983-4) and parallel investigation for hypersurfaces of $\mathbb{C}H^m$ was carried out by O. J. Garay and A. Romero (1990). The complete classification of 1-type submanifolds of a non-flat complex space form $\mathbb{C}Q^m(4c)$ was achieved by Dimitric (1991, 1997). These submanifolds are of three kinds:

Theorem 1. *(Dimitric, 1997) A smooth connected submanifold M^n of $\mathbb{C}Q^m(4c)$ is of 1-type if and only if it is one of the following*

(i) A canonically embedded complex space form of a lower dimension .
(ii) A totally real minimal submanifold M^n of a canonically embedded $\mathbb{C}Q^n \subset \mathbb{C}Q^m$,
(iii) ($\mathbb{C}P^m$ only) A geodesic sphere of radius $\arctan\sqrt{n+2}$ of a canonical $\mathbb{C}P^n \subset \mathbb{C}P^m$.

The next step was a study of 2-type submanifolds of non-flat complex space forms. An example of 2-type Kaehler submanifold of $\mathbb{C}P^m$ is a complex quadric hypersurface defined in homogeneous coordinates $(z_0, z_1, \cdots, z_m)$ by $z_0^2 + z_1^2 + \cdots + z_m^2 = 0$. Ros characterized 2-type Kaehler submanifols of $\mathbb{C}P^m$ concluding that they are Einstein and parallel submanifolds. Udagawa refined the study to produce the following classification

Theorem 2. *(Udagawa, 1986) Let $x : M^n \to \mathbb{C}P^m(4)$ be a full isometric holomorphic immersion of a compact Kähler manifold, which is not totally geodesic. Then $\tilde{x} = \Phi \circ x$ is of 2-type if and only if M is an Einstein Kähler parallel submanifold of degree 2, i.e. one of the following:*

(i) $\mathbb{C}P^n(1/2)$ with complex codimension $n(n+1)/2$.
(ii) A complex quadric Q^n with complex codimension 1.
(iii) $\mathbb{C}P^n \times \mathbb{C}P^n$ with complex codimension n^2.
(iv) $U(s+2)/U(2) \times U(s),\ s \geq 3,$ with cx codimension $s(s-1)/2$.
(v) $SO(10)/U(5)$ with cx dimension 10 and cx codimension 5.
(vi) $E_6/Spin(10) \times T$ with cx dimension 16 and cx codimension 10.

In contrast to this situation, in $\mathbb{C}H^m(-4)$ we have the following

Theorem 3. *(Dimitric and Djoric) There are no holomorphic immersions of Kaehler manifolds into $\mathbb{C}H^m$ which are of 2-type in $H^1_{\mathbb{C}}(m+1)$.*

Minimal surfaces of type 2 in complex projective space were classified by Shen (1995) and the study of real hypersurfaces of type-2 in complex space forms yielded some of the more interesting classifications. Martinez and Ros (1984) and Udagawa (1987) considered minimal resp. CMC real hypersurfaces of $\mathbb{C}P^m$ which are of 2- type. Using a weaker assumption that a hypersurface is a Hopf hypersurface, which means that the structure vector field (the Reeb vector field) $U := -J\xi$ is principal for the shape operator, we produced a complete local classification of 2-type Hopf hypersurfaces both in $\mathbb{C}P^m$ and $\mathbb{C}H^m$.

Theorem 4. *(Dimitric, 2011) Let M^{2m-1} be a Hopf hypersurface of $\mathbb{C}P^m(4)$. Then M^{2m-1} is of 2-type in $H(m+1)$ via Φ if and only if it is an open portion of one of the following*

- *(i) A geodesic hypersphere of any radius r except for $r = \cot^{-1}\sqrt{\frac{1}{2m+1}}$;*
- *(ii) The tube of radius $r = \cot^{-1}\sqrt{\frac{k+1}{m-k}}$ about a canonically embedded totally geodesic $\mathbb{C}P^k(4) \subset \mathbb{C}P^m(4)$, for any $k = 1, ..., m-2$;*
- *(iii) The tube of radius $r = \cot^{-1}\sqrt{\frac{2k+1}{2(m-k)+1}}$ about a canonically embedded $\mathbb{C}P^k(4) \subset \mathbb{C}P^m(4)$, for any $k = 1, ..., m-2$.*
- *(iv) The tube of radius $r = \cot^{-1}(\sqrt{m} + \sqrt{m+1})$ about the complex quadric $Q^{m-1} \subset \mathbb{C}P^m(4)$.*
- *(v) The tube of radius $r = \cot^{-1}\sqrt{\sqrt{2m^2-1} + \sqrt{2m^2-2}}$ about the complex quadric $Q^{m-1} \subset \mathbb{C}P^m(4)$.*

The same classification holds when M is assumed to have constant mean curvature (CMC) instead of being Hopf.

Theorem 5. *Let M^{2m-1} be a real hypersurface of $\mathbb{C}H^m(-4)$, $(m \geq 2)$ for which we assume that it is a Hopf hypersurface or has constant mean curvature. Then M^{2m-1} is of 2-type in $H^1(m+1)$ via Φ if and only if it is (an open portion of) either a geodesic hypersphere of arbitrary radius $r > 0$ or a tube of arbitrary radius $r > 0$ about a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{m-1}(-4)$.*

Regarding mass-symmetric hypersurfaces, from the analysis above we have

Corollary 1. *A complete Hopf (or CMC) hypersurface of $\mathbb{C}P^m(4)$ is of 2-type and mass-symmetric in the hypersphere of $H(m+1)$ containing $\Phi(\mathbb{C}P^m)$ if and only if it is one of the hypersurfaces (tubes) in (ii), (iv) and (v) or the geodesic hypersphere of radius $\cot^{-1}(1/\sqrt{m})$. There exists no 2-type mass-symmetric (in particular, no null 2-type) hypersurface of $\mathbb{C}H(-4)$.*

Rulled hypersurfaces of $\mathbb{C}Q^n(4c)$ form another interesting class of real hypersurfaces but there are no such that are of 2-type. A hypersurface $M \subset \mathbb{C}Q^m(4c)$ is said to be ruled if there is a foliation of M by complex hyperplanes $\mathbb{C}Q^{m-1}(4c)$. More precisely given a regular curve $\gamma : I \to \mathbb{C}Q^n(4c)$. For each t let $M_t^{n-1}(4c)$ be a totally geodesic complex hypersurface which is orthogonal to the holomorphic plane $Span\{\dot\gamma, J\dot\gamma\}$. Then a ruled hupersurface $\cup_{t\in I} M_t^{n-1}(4c)$ is generated. A ruled hypersurface is non-Hopf. In fact its shape operator satisfies

(4) $\quad AU = \alpha U + \nu W \ (\nu \neq 0), \quad AW = \nu U, \quad AX = 0, \ \text{for} \ X \perp U, W.$

The vector W is a unit vector of the projection of AU onto the holonmorphic subspace $U^{\perp}$ of TM.

Theorem 6. *(Dimitric and Djoric) There exist no ruled hypersurface of $\mathbb{C}Q^n(4c)$ which is of 2-type in the (pseudo) Euclidean space $H^{(1)}(n+1)$ via the embedding by the projectors.*

Real hypersurfaces, Kähler submanifolds, and totally real submanifolds are all examples of the so-called CR-submanifolds, extend the study of 2-type submanifolds of $\mathbb{C}Q^m$ to a general CR-submanifold of arbitrary dimension and codimension. We recall that a CR-submanifold is a submanifold M of an almost Hermitian manifold $(\bar{M}, g, J)$ whose tangent space splits at each point into a direct sum of two complementary orthogonal distributions (subspaces) $\mathcal{D}$ and $\mathcal{D}^{\perp}$ of constant dimensions so that at each point $p \in M$,

$$T_pM = \mathcal{D}_p \oplus \mathcal{D}_p^{\perp}, \quad \text{with} \quad J\mathcal{D}_p \subset \mathcal{D}_p \quad \text{and} \quad J\mathcal{D}_p^{\perp} \subset T_p^{\perp}M.$$

When dim $\mathcal{D}_p^{\perp} = 0$, a submanifold is said to be holomorphic, and when dim $\mathcal{D}_p = 0$ it is said to be totally real submanifold. Further, if dim $\mathcal{D}_p^{\perp} = 1$ at every point a CR-submanifold is said to be of maximal CR-dimension.

3. Some characterizations of 2-type CR submanifolds

Notation In the following we use the notation:

A_ξ is the shape operator in the direction of a normal vector ξ

D is the connection in the normal bundle $T^{\perp}M$.

h is the second fundamental form of a submanifold

$H = \frac{1}{n}\mathrm{tr}\, h$ is the mean curvature vector field

Q is the $(1,1)$ Ricci tensor on a submanifold

$L\xi = (J\xi)_T$ is the tangential component of $J\xi$ for $\xi \in T_p^{\perp}M$

$K\xi = (J\xi)_N$ is the normal component of $J\xi$ for $\xi \in T_p^{\perp}M$

The ancillary shape operator $\hat{\mathfrak{a}}: \ \xi \to \sum_r \mathrm{tr}\,(A_\xi A_r) e_r$ of a submanifold M is a symmetric endomorphism of $T^{\perp}M$. It is an endomorphism version of the tensor $T(\xi, \eta) = \mathrm{tr}\,(A_\xi A_\eta)$. It plays a kind of a dual role to the Casorati operator $\sum_r A_r^2$ acting on the tangent space - both have trace equal to $\|h\|^2$.

Theorem 7. *Let $x : M^n \to \mathbb{C}Q^m(4c)$ be a totally real isometric immersion of a Riemannian $n-$manifold into a non-flat complex space form of constant holomorphic sectional curvature $4c$, $c = \pm 1$ and let $T^{\perp}M = \mathcal{V} \oplus \mathcal{W}$ be the orthogonal decomposition of the normal bundle into the totally real and holomorphic subbundles. Then if $\tilde{x} = \Phi \circ x : M^n \to H^{(1)}(m+1)$ is 2-type mass-symmetric immersion satisfying $\Delta^2 \tilde{x} - p\Delta\tilde{x} + q(\tilde{x} - I/(m+1)) = 0$, we have*

(i) The mean curvature α is constant.
(ii) $tr\, A_{DH} = 0$;

(iii) $\Delta^{\perp} H + \hat{\mathfrak{a}}(H) + [c(3n+4) - p]\, H + 3cJ(LH) = 0$;
(iv) $Q(X) - L(\hat{\mathfrak{a}}(JX)) - 2nA_H X - kX + \frac{n^2}{2}\langle LH, X\rangle LH = 0$, *for every* $X \in \Gamma(TM)$, *where* $k = \frac{c}{4}[q + 2cf^2 + 4n(n+3) - 2c(n+1)p]$ *is a constant;*
(v) $\langle JD_X H, Y, \rangle = \langle JD_Y H, X\rangle$, *for every* $X, Y \in \Gamma(TM)$;
(vi) $\frac{n^2}{2}\langle LH, X\rangle KH - nK^2(D_X H) - K(\hat{\mathfrak{a}}(JX)) = 0$;
(vii) $\frac{n^2}{2}\langle KH, \xi\rangle LH + n\sum_i \langle D_{e_i} H, \xi\rangle e_i - L(\hat{\mathfrak{a}}(J\xi)) = 0$, *for* $\xi \in \mathcal{W}$;
(viii) $\frac{n^2}{2}\langle H, \xi\rangle H + \frac{n^2}{2}\langle KH, \xi\rangle KH + k'\xi - n\sum_i \langle D_{e_i} H, K\xi\rangle Je_i + \hat{\mathfrak{a}}(\xi) - K(\hat{\mathfrak{a}}(K\xi)) = 0$, *for any* $\xi \in \mathcal{W}$ *where* $k' = \frac{c}{4}[2cnp - 4n(n+1) - 2cf^2 - q]$.

Conversely, if $(i) - (viii)$ *hold then the immersion is mass-symmetric and of type* ≤ 2, *provided that the polynomial* $t^2 - pt + q = 0$ *has simple real roots or* M *is compact.*

For a Lagrangian immersion these conditions reduce to

Corollary 2. If $x : M^n \to \mathbb{C}Q^n(4c)$ is a Lagrangian immersion for which $\tilde{x}$ is mass-symmetric and of 2-type then
(i) $f := n\alpha = const$;
(ii) $\operatorname{tr} A_{DH} = 0$
(iii) $\Delta^{\perp} H + \hat{\mathfrak{a}}(H) + [c(3n+1) - p]\, H = 0$;
(iv) $Q(X) - J(\hat{\mathfrak{a}}(JX)) - 2nA_H X - kX + \frac{n^2}{2}\langle JH, X\rangle JH = 0$, for every $X \in TM$
(v) $\langle JD_X H, Y\rangle = \langle JD_Y H, X\rangle$, for every $X, Y \in \Gamma(TM)$.

Corollary 3. Lagrangian submanifold M^n of $\mathbb{C}Q^n(4c)$ which is mass-symmetric and of 2-type via $\tilde{x}$ has the following properties
(i) $D_{JX} H = 0, \quad \nabla_{JH}(JH) = 0, \quad$ and $\quad A_H(JH) = Jh(JH, JH)$. In particular, the integral curves of JH are geodesics.
(ii) If, in addition, $DH = 0$ then M is an $\mathfrak{a}-$ submanifold, i.e. $\mathfrak{a}(H) = 0$.
(iii) If M is a Maslov submanifold with parallel mean curvature vector then $Q(JH) = qJH$, for a suitable constant q, i.e. JH is a principal vector of the Ricci endomorphism Q, and moreover $\mathfrak{a}(H) = 0$.

Theorem 8. *Let* M^n *be a 3-type holomorphic submanifold fully immered in* $\mathbb{C}Q^m(4c)$ *satisfying a 3-type equation* $\Delta^3\tilde{x} + p\Delta^2\tilde{x} + q\Delta\tilde{x} + r(\tilde{x} - \tilde{x}_0) = 0$. *If* M *has constant scalar curvature* τ *then it is mass-symmetric in* $H^{(1)}(m+1)$ *and the following conditions hold:*
(i) $\tau = \frac{1}{8}\{4cn(n+2)(n+4) + 2pn(n+2) + cnq + \frac{rm}{2(m+1)}\}$;
(ii) $\sum_i (D_{e_i}\hat{\mathfrak{a}})(h(X, e_i)) = \sum_i h(e_i, (\nabla_{e_i} Q)X)$;
(iii) $\sum_{r,s} tr\,(A_r A_s) A_r A_s = \frac{1}{2}\Delta Q - Q^2 + [\frac{p}{2} + c(3n+8)]Q - aI$;
(iv) $(\Delta^{\perp}\hat{\mathfrak{a}})\xi + 2\hat{\mathfrak{a}}^2(\xi) + [p + 4c(n+2)]\, \hat{\mathfrak{a}}(\xi) - 2\sum_i h(e_i, QA_\xi e_i) + b\xi = 0$,

for every $X \in \Gamma(TM)$ *and* $\xi \in \Gamma(T^{\perp}M)$, *where* $a = \frac{c}{16}[\frac{r}{m+1} + 4cq + 16p(n+2) + 16c(n+2)(3n+8)]$ *and* $b = n(n+2)(n+4) - \frac{cr}{8(m+1)}$ *are constant. Conversely, conditions* $(i) - (iv)$ *imply type* ≤ 3 *if the polynomial* $t^3 + pt^2 + qt + r$ *has simple real roots or* M *is compact.*

Corollary 4. There exists no compact Kähler hypersurface of $\mathbb{C}Q^m$ which is of Chen 3-type and mass-symmetric in the hypersphere of $H^{(1)}(m+1)$ containing the image of $\mathbb{C}Q^m$ via Φ.