Vector Parameter Forms of SU(1,1), SL(2, $\mathbb{R}$) and Their Connection with SO(2,1)

VELIKO D. DONCHEV, CLEMENTINA D. MLADENOVA,
IVAÏLO M. MLADENOV

Geometry, Integrability and Quantization,
Varna, June 2015
06.06.2015

The presentation

Summary of the results

The *Cayley* maps for the Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$ converting them into the corresponding Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SO}(2,1)$ along their natural vector parameterizations are examined. Additionally the explicit form of the covering map $\mathrm{SU}(1,1) \to \mathrm{SO}(2,1)$ and its sections are presented. Finally, the vector-parameter forms of the Lie groups $\mathrm{SU}(2)$ and $\mathrm{SU}(1,1)$ are compared and some of their applications are addressed.

- This research is made within a bigger project which is about parameterizing Lie groups with small dimension and its application in physics.

- In [4] the *Cayley* maps for the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ and the corresponding Lie groups SU(2) and $\mathrm{SO}(3, \mathbb{R})$ are examined.

- Parameterizations are used to describe Lie groups in an easier and more intuitive way. Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$. A vector parameterization of G is a map $\mathfrak{g} \to G$, which is diffeomorphic onto its image. Besides the exponential map, there are other alternatives to achieve parameterization. We make use of the *Cayley* map

$$\mathrm{Cay}(\mathrm{X}) = (\mathcal{I} + \mathrm{X})(\mathcal{I} - \mathrm{X})^{-1}. \tag{1.1}$$

Consider the Lie algebra $\mathfrak{so}(2,1)$ with basis

$$\mathrm{P}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \; \mathrm{P}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; \mathrm{P}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.1)$$

The commutation relations of these matrices are as follows:

$$[\mathrm{P}_1, \mathrm{P}_2] = -\mathrm{P}_3, \qquad [\mathrm{P}_2, \mathrm{P}_3] = \mathrm{P}_1, \qquad [\mathrm{P}_3, \mathrm{P}_1] = \mathrm{P}_2. \qquad (2.2)$$

Any $\mathrm{C} \in \mathfrak{so}(2,1)$ has a unique representation

$$\mathbf{c} \mapsto \mathrm{C} = \mathbf{c}.\mathrm{P} = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & -c_1 \\ -c_2 & -c_1 & 0 \end{pmatrix}, \mathbf{c} = (c_1, c_2, c_3). \qquad (2.3)$$

The *Hamilton–Cayley* theorem applied to C from (2.3) reads as

$$\mathrm{C}^3 = (1 - \mathbf{c}.(\eta\mathbf{c}))\mathrm{C} \qquad (2.4)$$

where $\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The *Cayley* map applied for $\mathfrak{so}(2,1)$ is

$$\mathcal{H}(\mathbf{c}) = \mathrm{Cay}_{\mathfrak{so}(2,1)}(\mathrm{C}) = (\mathcal{I} + \mathrm{C})(\mathcal{I} - \mathrm{C})^{-1}. \tag{2.5}$$

One checks immediately that $\mathcal{I} - \mathrm{C}$ is invertable if only if $\mathbf{c}.(\eta\mathbf{c}) \neq 1$ and in this case we can explicitly calculate

$$\begin{aligned} \mathcal{H}(\mathbf{c}) &= \mathrm{Cay}_{\mathfrak{so}(2,1)}(\mathrm{C}) = (\mathcal{I} + \mathrm{C})(\mathcal{I} + \frac{1}{1 - \mathbf{c}.(\eta\mathbf{c})}\mathrm{C} + \frac{1}{1 - \mathbf{c}.(\eta\mathbf{c})}\mathrm{C}^2) \\ &= \mathcal{I} + \frac{2}{1 - \mathbf{c}.(\eta\mathbf{c})}\mathrm{C} + \frac{2}{1 - \mathbf{c}.(\eta\mathbf{c})}\mathrm{C}^2 \qquad (2.6) \\ &= \frac{2}{1 - \mathbf{c}.(\eta\mathbf{c})} \begin{pmatrix} 1 - c_1^2 & c_1c_2 + c_3 & -c_1c_3 - c_2 \\ c_1c_2 - c_3 & 1 - c_2^2 & c_2c_3 - c_1 \\ c_1c_3 - c_2 & -c_2c_3 - c_1 & 1 + c_3^2 \end{pmatrix} - \mathcal{I}. \end{aligned}$$

If $\mathcal{H}(\mathbf{c}_1), \mathcal{H}(\mathbf{c}_1)$ are two SO(2,1) elements represented by the vector parameters and $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_1.(\eta\mathbf{c}_1) \neq 1, \mathbf{c}_2.(\eta\mathbf{c}_2) \neq 1$ and $1 + \mathbf{c}_2.(\eta\mathbf{c}_1) \neq 0$. Then

$$\mathcal{H}(\mathbf{c}_3) = \mathcal{H}(\mathbf{c}_2)\mathcal{H}(\mathbf{c}_1), \quad \mathbf{c}_3 = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle_{\mathrm{SO}(2,1)} = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2.(\eta\mathbf{c}_1)} \tag{2.7}$$

where $\mathbf{c}_2 \curlywedge \mathbf{c}_1 := \eta(\mathbf{c}_2 \times \mathbf{c}_1)$. Equation (2.7) is the vector-parameter form of SO(2,1) obtained by the parameterization given by the *Cayley* map. The same result was obtained independently by usage of pseudo-quaternions [3]. Note that in the elliptic case, i.e., $1 > \mathbf{c}.(\eta\mathbf{c}) > 0$ there exists $\boldsymbol{n} \in \mathbb{R}^{2,1}$ such that $\boldsymbol{n}.(\eta\boldsymbol{n}) = 1$ and

$$\mathbf{c} = \tanh\frac{\theta}{2}\boldsymbol{n}. \tag{2.8}$$

In the hyperbolic case, i.e., $\mathbf{c}.(\eta\mathbf{c}) < 0$ there exist $\boldsymbol{n} \in \mathbb{R}^{2,1}$ such that $\boldsymbol{n}.(\eta\boldsymbol{n}) = -1$ and $\mathbf{c} = \tan\dfrac{\theta}{2}\boldsymbol{n}$.

Structure of $\mathfrak{su}(1,1)$

Let us consider the Lie algebra $\mathfrak{su}(1,1)$ with $\mathbb{R}-$basis

$$\mathrm{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{e}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \mathrm{e}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.1}$$

The matrices E_1, E_2 and E_3 defined by

$$E_1 = \frac{1}{2}\mathrm{e}_1, \qquad E_2 = \frac{1}{2}\mathrm{e}_2, \qquad E_3 = \frac{1}{2}\mathrm{e}_3 \tag{3.2}$$

also form a $\mathbb{R}-$basis of $\mathfrak{su}(1,1)$. Also

$$[E_1, E_2] = -E_3, \qquad [E_2, E_3] = E_1, \qquad [E_3, E_1] = E_2. \tag{3.3}$$

Obviously, the map

$$\mathrm{m}_1 E_1 + \mathrm{m}_2 E_2 + \mathrm{m}_3 E_3 \longrightarrow \mathrm{m}_1 P_1 + \mathrm{m}_2 P_2 + \mathrm{m}_3 P_3 \tag{3.4}$$

is a linear isomorphism between the Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{so}(2,1)$.

Let $\mathrm{M} = m.E = \begin{pmatrix} i\frac{\mathrm{m}_3}{2} & \frac{\mathrm{m}_1}{2} + i\frac{\mathrm{m}_2}{2} \\ \frac{\mathrm{m}_1}{2} - i\frac{\mathrm{m}_2}{2} & -i\frac{\mathrm{m}_3}{2} \end{pmatrix} \in \mathfrak{su}(1,1)$. The *Hamilton–Cayley* theorem applied to M reads as

$$\mathrm{M}^2 = \frac{m}{2} \cdot (\eta \frac{m}{2}) \mathcal{I} \tag{3.5}$$

The *Cayley* map applied for $\mathfrak{su}(1,1)$ is

$$\mathcal{L}(m) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathrm{M}) = (\mathcal{I} + \mathrm{M})(\mathcal{I} - \mathrm{M})^{-1}. \tag{3.6}$$

Let us define

$$\Delta_m = 1 - \frac{\mathrm{m}_1^2 + \mathrm{m}_2^2 - \mathrm{m}_3^2}{4} = 1 - \frac{m}{2} \cdot (\eta \frac{m}{2}) \tag{3.7}$$

which is exactly $\det(\mathcal{I} - \mathrm{M})$. Thus, $\mathrm{Cay}_{\mathfrak{su}(1,1)}$ is well-defined when $\Delta_m \neq 0$.

In this case we have $(\mathcal{J} - \mathrm{M})^{-1} = \dfrac{1}{\Delta_m}(\mathcal{J} + \mathrm{M})$ and thus we can explicitly calculate

$$\mathcal{L}(m) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathrm{M}) = \frac{1}{\Delta_m}(\mathcal{J} + \mathrm{M})^2 = \frac{1}{\Delta_m}(\mathcal{J} + 2\mathrm{M} + \mathrm{M}^2) = \frac{2 - \Delta_m}{\Delta_m}\mathcal{J} \tag{3.8}$$

$$= \frac{2 - \Delta_m}{\Delta_m}\mathcal{J} + \frac{2}{\Delta_m} m.E = \frac{1}{\Delta_m}\begin{pmatrix} 2 - \Delta_m + i\mathrm{m}_3 & \mathrm{m}_1 + i\mathrm{m}_2 \\ \mathrm{m}_1 - i\mathrm{m}_2 & 2 - \Delta_m - i\mathrm{m}_3 \end{pmatrix}. \tag{3.9}$$

Now direct calculation shows that $\det(\mathcal{L}(m)) = 1$ and also $\mathcal{L}(m)^\dagger \eta \mathcal{L}(m) = \eta$, $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $\mathcal{L}(m) \in \mathrm{SU}(1,1)$.

To invert the *Cayley* map $\mathrm{Cay}_{\mathfrak{su}(1,1)} : \mathfrak{su}(1,1) \longrightarrow \mathrm{SU}(1,1)$ let us consider an arbitrary $\mathrm{SU}(1,1)$ matrix

$$\mathcal{L}(\alpha_1,\alpha_2,\beta_1,\beta_2) = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ \beta_1 - i\beta_2 & \alpha_1 - i\alpha_2 \end{pmatrix}, \quad \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1.$$

$\mathcal{L} \in \Im\mathrm{Cay}_{\mathfrak{su}(1,1)}$ if and only if there exist $\mathrm{m} \in \mathbb{R}^3 : \Delta_{\mathrm{m}} \neq 0$ such that $\mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathrm{A}(m)) = \mathcal{L}$. This is only possible if $\alpha_1 \neq -1$ and in this case the inversion is

$$(\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3) = \frac{2}{1+\alpha_1}(\beta_1,\beta_2,\alpha_2). \tag{3.10}$$

Finally, the *Cayley* acts

$$\begin{aligned}\mathrm{Cay}_{\mathfrak{su}(1,1)} &: \{m.E \in \mathfrak{su}(1,1);\ \Delta_m \neq 0\} \\ &\longrightarrow \{\mathcal{L}(\alpha_1,\alpha_2,\beta_1,\beta_2) \in \mathrm{SU}(1,1);\ \alpha_1 \neq -1\}.\end{aligned}$$

Theorem

Let $\mathrm{M}, \mathrm{A} \in \mathfrak{su}(1,1)$

$$\mathrm{M} = m.E, \quad m = (\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3), \qquad \mathrm{A} = a.E, \quad a = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3)$$

be such that $\Delta_m \neq 0, \Delta_a \neq 0$ *and*

$$(a.(\eta a))(m.(\eta m)) + 8a.(\eta m) + 16 \neq 0. \tag{3.11}$$

Let $\mathcal{L}(m) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathrm{M}), \mathcal{W}(a) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\mathrm{A})$*. Then, if* $\widetilde{\mathcal{L}} = \mathcal{W}.\mathcal{L}$ *is the composition of the images in* $\mathrm{SU}(1,1)$ *then* $\widetilde{\mathcal{L}} = \mathrm{Cay}_{\mathfrak{su}(1,1)}(\widetilde{\mathrm{A}})$ *where* $\widetilde{\mathrm{A}} = \widetilde{m}.E$ *and*

$$\widetilde{m} = \frac{(1 + \frac{m}{2} \cdot (\eta \frac{m}{2}))a + (1 + \frac{a}{2} \cdot (\eta \frac{a}{2}))m + a \curlywedge m}{1 + 2\frac{a}{2} \cdot (\eta \frac{m}{2}) + (\frac{a}{2} \cdot (\eta \frac{a}{2}))(\frac{m}{2} \cdot (\eta \frac{m}{2}))}. \tag{3.12}$$

Idea for proof 1

Let us calculate

$$\mathcal{W}(a).\mathcal{L}(m) = \left(\frac{2-\Delta_a}{\Delta_a}\mathcal{I} + \frac{2}{\Delta_a}a.E\right)\left(\frac{2-\Delta_m}{\Delta_m}\mathcal{I} + \frac{2}{\Delta_m}m.E\right)$$
$$\stackrel{(??)}{=} \frac{(2-\Delta_a)(2-\Delta_m)+a\cdot m}{\Delta_a\Delta_m}\mathcal{I} + \frac{(2-\Delta_m)a+(2-\Delta_a)m+a\curlywedge m}{\Delta_a\Delta_m}\cdot E.$$

Now, the condition for existence of $\widetilde{m}$ is

$$\frac{(2-\Delta_a)(2-\Delta_m)+a\cdot m}{\Delta_a\Delta_m} \neq -1 \Leftrightarrow 1+(1-\Delta_a)(1-\Delta_m)+2\frac{a}{2}\cdot(\eta\frac{m}{2})$$

and after simplification we obtain that it is equivalent to (3.11).

Idea for proof 2

Thus, $\widetilde{m}$ exists and is such that

$$\frac{2-\Delta_{\widetilde{m}}}{\Delta_{\widetilde{m}}} = \frac{(2-\Delta_a)(2-\Delta_m)+a.(\eta m)}{\Delta_a\Delta_m} \tag{3.14}$$

$$\frac{2}{\Delta_{\widetilde{m}}}\widetilde{m} = \frac{(2-\Delta_m)a+(2-\Delta_a)m+a\curlywedge m}{\Delta_a\Delta_m}. \tag{3.15}$$

From (3.14) we immediately find

$$\Delta_{\widetilde{m}} = \frac{\Delta_a\Delta_m}{1+2\frac{a}{2}\cdot(\eta\frac{m}{2})+(\frac{a}{2}\cdot(\eta\frac{a}{2}))(\frac{m}{2}\cdot(\eta\frac{m}{2}))} \neq 0 \tag{3.16}$$

and thus after some algebraic simplifications the composition law in vector-parameter form follows.

It is well know fact that $\mathrm{SU}(1,1)$ is isomorphic as a group to the real special linear group of order 2, i.e., $\mathrm{SL}(2,\mathbb{R})$ by the map

$$\varphi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SU}(1,1),$$
$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{3.17}$$

Now, to automatically obtain the vector-parameter form we need to invert φ and calculate $\varphi^{-1}(\mathcal{L}(m))$ where $m \in \mathbb{R}^3$, $\Delta_m \neq 0$. A straight forward calculation shows that

$$\psi(m) := \varphi^{-1}(\mathcal{L}(m)) = M(m) = \frac{1}{\Delta_m}\begin{pmatrix} 2-\Delta_m + m_2 & m_1 + m_3 \\ m_1 - m_3 & 2 - \Delta_m - m_2 \end{pmatrix}.$$

We have

$$\operatorname{tr}\mathcal{L}(m) = \operatorname{tr} M(m) = \frac{4-2\Delta_m}{\Delta_m}, \qquad \Delta_m \neq 0. \tag{3.18}$$

In a straight forward manner we obtain that if $\Delta_m \neq 0$ then

$$M(m) \text{ is } \begin{cases} \text{hyperbolic} & \text{if} \quad \Delta_m < 1 \quad \Leftrightarrow \quad m.(\eta m) > 0 \\ \text{elliptic} & \text{if} \quad \Delta_m > 1 \quad \Leftrightarrow \quad m.(\eta m) < 0 \\ \text{parabolic} & \text{if} \quad \Delta_m = 1 \quad \Leftrightarrow \quad m.(\eta m) = 0. \end{cases} \tag{3.19}$$

Corollary

In the terms of Theorem 1, *for the composition vector* $\widetilde{m}$ *we have*

$$M(\widetilde{m}) \text{ is } \begin{cases} \text{hyperbolic} & \text{if} \quad \xi\zeta > 0 \\ \text{elliptic} & \text{if} \quad \xi\zeta < 0 \\ \text{parabolic} & \text{if} \quad \xi = 0 \end{cases}$$

$$\xi = \xi(a, m) = (a + m).(\eta(a + m)),$$

$$\zeta = \zeta(a, m) = (4 + (a.m))^2 - (a \curlywedge m).(\eta(a \curlywedge m)).$$

Consider the homomorphism map [1] $\varphi : \mathrm{SU}(1,1) \longrightarrow \mathrm{SO}(2,1)$ which sends the SU(1,1) matrix $\mathcal{L} = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ where $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$ and $\alpha\overline{\alpha} - \beta\overline{\beta} = 1$ into

$$\varphi(\mathcal{L}) = \begin{pmatrix} -\frac{1}{2}(\beta^2 + \overline{\beta}^2 - \alpha^2 - \overline{\alpha}^2) & \frac{i}{2}(\overline{\alpha}^2 + \overline{\beta}^2 - \alpha^2 - \beta^2) & i(\overline{\alpha}\overline{\beta} - \alpha\beta) \\ -\frac{i}{2}(\beta^2 - \overline{\beta}^2 - \alpha^2 + \overline{\alpha}^2) & \frac{1}{2}(\overline{\alpha}^2 + \overline{\beta}^2 + \alpha^2 + \beta^2) & \alpha\beta + \overline{\alpha}\overline{\beta} \\ i(\overline{\alpha}\beta - \alpha\overline{\beta}) & \overline{\alpha}\beta + \alpha\overline{\beta} & \alpha\overline{\alpha} + \beta\overline{\beta} \end{pmatrix}.$$

The homomorphism map φ is a double cover with $\ker\varphi = \{\pm\mathcal{I}\}$ and $\mathrm{SU}(1,1)/\mathbb{Z}_2 \cong \mathrm{SO}(2,1)$. We are interested in the vector-parameter form of φ.

Consider the SU(1,1) matrix $\mathcal{L}(m) = \mathrm{Cay}_{\mathfrak{su}(1,1)}(m)$. We have

$$\alpha = \frac{2-\Delta_m}{\Delta_m} + i\frac{\mathrm{m}_3}{\Delta_m}, \qquad \beta = \frac{\mathrm{m}_1}{\Delta_m} + i\frac{\mathrm{m}_2}{\Delta_m}. \tag{4.1}$$

Substitution of (4.1) in (4.1) leads to the matrix $\varphi(m) = \mathcal{H}(m)$ i.e.,

$$\frac{2}{\Delta_m^2}\begin{pmatrix} \mathrm{m}_2^2-\mathrm{m}_3^2 & \mathrm{m}_1\mathrm{m}_2+\mathrm{m}_3(2-\Delta_m) & \mathrm{m}_1\mathrm{m}_3+\mathrm{m}_2(2-\Delta_m) \\ \mathrm{m}_1\mathrm{m}_2-\mathrm{m}_3(2-\Delta_m) & \mathrm{m}_1^2-\mathrm{m}_3^2 & -\mathrm{m}_2\mathrm{m}_3+\mathrm{m}_1(2-\Delta_m) \\ -\mathrm{m}_1\mathrm{m}_3+\mathrm{m}_2(2-\Delta_m) & \mathrm{m}_2\mathrm{m}_3+\mathrm{m}_1(2-\Delta_m) & \mathrm{m}_1^2+\mathrm{m}_2^2 \end{pmatrix} + \mathcal{I}.$$

This vector parameter form will allow us to obtain the connection between the vector parameter in SO(2,1) and its cover SU(1,1).

Theorem

Let $\mathcal{L}(m)$ is an $\mathrm{SU}(1,1)$ element, represented by the vector-parameter m such that $\frac{m}{2}\cdot(\eta\frac{m}{2}) \neq -1$. Then in $\mathrm{SO}(2,1)$ this element is represented by $\mathcal{H}(\mathbf{c})$ where

$$\mathbf{c} = -\frac{\eta m}{1+\frac{m}{2}\cdot(\eta\frac{m}{2})}. \tag{4.2}$$

On the other hand, if $\mathbf{c}$ represents the $\mathrm{SO}(2,1)$ element $\mathcal{H}(\mathbf{c})$, then the in $\mathrm{SU}(1,1)$ the elements represented by the vector-parameters

$$m_+(\mathbf{c}) = \frac{-2-2\sqrt{1-\mathbf{c}.(\eta\mathbf{c})}}{\mathbf{c}.(\eta\mathbf{c})}\eta\mathbf{c}, \; m_-(\mathbf{c}) = \frac{-2+2\sqrt{1-\mathbf{c}.(\eta\mathbf{c})}}{\mathbf{c}.(\eta\mathbf{c})}\eta\mathbf{c}$$

correspond to $\mathcal{H}(\mathbf{c})$.

Moreover, the following relations hold

$$m_+.(\eta m_-) = 4, \qquad (m_+.(\eta m_+))(m_-.(\eta m_-)) = 16 \tag{4.3}$$

$$m_+ = -\frac{4}{m_-.(\eta m_-)} m_-, \qquad m_- = -\frac{4}{m_+.(\eta m_+)} m_+. \tag{4.4}$$

Idea for proof 1

Let us equate the following expressions from $\mathcal{H}(\mathbf{c})$ (cf. (**??**)) and $\mathcal{H}(m)$ (cf. (4.2))

$$\operatorname{tr}\mathcal{H}(\mathbf{c}) = \operatorname{tr}\mathcal{H}(m) \tag{4.5}$$

$$\begin{aligned} \mathcal{H}(\mathbf{c})_{1,2} - \mathcal{H}(\mathbf{c})_{2,1} &= \mathcal{H}(m)_{1,2} - \mathcal{H}(m)_{2,1} \\ \mathcal{H}(\mathbf{c})_{1,3} + \mathcal{H}(\mathbf{c})_{3,1} &= \mathcal{H}(m)_{1,3} + \mathcal{H}(m)_{3,1}, \\ \mathcal{H}(\mathbf{c})_{2,3} + \mathcal{H}(\mathbf{c})_{3,2} &= \mathcal{H}(m)_{2,3} + \mathcal{H}(m)_{3,2}. \end{aligned} \tag{4.6}$$

Equation (4.5) is equivalent to

$$\frac{2}{1-\mathbf{c}.(\eta\mathbf{c})}(3-\mathbf{c}.(\eta\mathbf{c})) - 3 = \frac{4}{\Delta_m^2} m.(\eta m) + 3 \tag{4.7}$$

from where after some algebraic manipulation

$$\frac{2}{1-\mathbf{c}.(\eta\mathbf{c})} = 2\frac{(2-\Delta_m)^2}{\Delta_m^2}. \tag{4.8}$$

Idea for proof 2

Equation (4.6) reads as

$$\frac{4}{1-\mathbf{c}.(\eta\mathbf{c})}c_3 = 4\frac{2-\Delta_m}{\Delta_m^2}d_3 \tag{4.9}$$

where as (**??**) reads as

$$\frac{-4}{1-\mathbf{c}.(\eta\mathbf{c})}c_2 = 4\frac{2-\Delta_m}{\Delta_m^2}d_2, \qquad \frac{-4}{1-\mathbf{c}.(\eta\mathbf{c})}c_1 = 4\frac{2-\Delta_m}{\Delta_m^2}d_1. \tag{4.10}$$

It is obvious that from (4.9) and (4.10)

$$\frac{1}{1-\mathbf{c}.(\eta\mathbf{c})}\mathbf{c} = -\frac{2-\Delta_m}{\Delta_m^2}\eta m \tag{4.11}$$

and thus substituting (4.8) into (4.11) we obtain the result (4.2).

Idea for proof 3

Let us find the vector-parameter form of the sections of the homomorphism i.e., invert (4.2). From it we have obtain

$$\mathbf{c}.(\eta\mathbf{c}) = \frac{-\eta m}{1+\frac{m}{2}\cdot(\eta\frac{m}{2})}\cdot\eta\frac{-\eta m}{1+\frac{m}{2}\cdot(\eta\frac{m}{2})} = \frac{m.(\eta m)}{\left(1+\frac{m.(\eta m)}{4}\right)^2} \quad (4.12)$$

This leads to the quadratic equation for $x = m.(\eta m)$

$$\mathbf{c}.(\eta\mathbf{c})x^2 + 8(\mathbf{c}.(\eta\mathbf{c}) - 2)x + 16\mathbf{c}.(\eta\mathbf{c}) = 0. \quad (4.13)$$

Note that the expression $1-\mathbf{c}.(\eta\mathbf{c}) = 1 - \dfrac{x}{(1+\frac{x}{4})^2}$ and straight forward calculus shows that $1-\mathbf{c}.(\eta\mathbf{c}) \geq 0$ with $1-\mathbf{c}.(\eta\mathbf{c}) = 0$ if only if $x = 4$. Thus, equation (4.13) has two real roots

$$m_+.(\eta m_+) = 4\frac{2-\mathbf{c}.(\eta\mathbf{c}) \pm 2\sqrt{1-\mathbf{c}.(\eta\mathbf{c})}}{\mathbf{c}.(\eta\mathbf{c})}. \quad (4.14)$$

Idea for proof 4

Direct calculation shows that

$$1+\frac{m_\pm}{2}\cdot(\eta\frac{m_\pm}{2})=\frac{-2\mp 2\sqrt{1-\mathbf{c}.(\eta\mathbf{c})}}{\mathbf{c}.(\eta\mathbf{c})} \tag{4.15}$$

and thus the two sections of (4.2) are

$$m_+(\mathbf{c})=\frac{-2-2\sqrt{1-\mathbf{c}.(\eta\mathbf{c})}}{\mathbf{c}.(\eta\mathbf{c})}\eta\mathbf{c}, \qquad m_-(\mathbf{c})=\frac{-2+2\sqrt{1-\mathbf{c}.(\eta\mathbf{c})}}{\mathbf{c}.(\eta\mathbf{c})}\eta\mathbf{c}. \quad (4.1$$

Note that the properties (4.3) follow directly from (4.14) and (4.16).

Remark

Note that if we assume the axis-rapidity representation of SO(2,1) elliptic vector-parameter

$$\mathbf{c} = \tanh\frac{\theta}{2}\boldsymbol{n}, \qquad \boldsymbol{n}.(\eta\boldsymbol{n}) = 1, \quad \theta \in \mathbb{R} \tag{5.1}$$

then for $m_+(\mathbf{c})$ and $m_-(\mathbf{c})$ we obtain

$$m_-(\mathbf{c}) = -2\tanh\frac{\theta}{4}\eta\boldsymbol{n}, \qquad m_+(\mathbf{c}) = -2\frac{1}{\tanh\frac{\theta}{4}}\eta\boldsymbol{n} = -2\coth\frac{\theta}{4}\eta\boldsymbol{n} \tag{5.2}$$

If we assume the representation of SO(2,1) hyperbolic vector-parameter

$$\mathbf{c} = \tan\frac{\theta}{2}\boldsymbol{n}, \qquad \boldsymbol{n}.(\eta\boldsymbol{n}) = -1, \quad \theta \in [0,\pi) \tag{5.3}$$

then for $m_+(\mathbf{c})$ and $m_-(\mathbf{c})$ we obtain

1. Basu D., *Introduction to Classical and Modern Analysis and Their Application to Group Representation Theory*, World Scientific, Singapore 2009.
2. Brezov D., Mladenova C. and Mladenov I., *Vector Decompositions of Rotations*, J. Geom. Symmetry Phys. **28** (2012) 67-103.
3. Brezov D., Mladenova C. and Mladenov I., *Vector Parameters in Classical Hyperbolic Geometry*, J. Geom. Symmetry Phys. **30** (2013) 19-48.
4. Donchev, V., Mladenova K. and Mladenov I. *Vector Parameter Form of the* SU2 $\to$ SO$(3,\mathbb{R})$ *Map*, accepted for publication in Ann. Univ. Sofia (2015).
5. Fedorov F., *The Lorentz Group* (in Russian), Nauka, Moscow 1979.
6. Serre J., *Lie Algebras and Lie Groups*, 2nd Edn, Springer, Berlin 2006.

Thank you for your attention!