CONFORMALITY IN SEMI-RIEMANNIAN CONTEXT

Cornelia-Livia BEJAN & Șemsi EKEN

"Gheorghe Asachi" Technical University & Karadeniz Technical University

05-10 June, 2015

Conformality in semi-Riemannian Context

1. Picture,

Conformality in semi-Riemannian Context

1. Picture,
2. History,

Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,

Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,
4. Introduction,

Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,
4. Introduction,
5. Preliminaries,

Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,
4. Introduction,
5. Preliminaries,
6. Semi-Riemannian map in semi-Riemannian context,

Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,
4. Introduction,
5. Preliminaries,
6. Semi-Riemannian map in semi-Riemannian context,
7. Conformal semi-Riemannian map in semi-Riemannian manifolds,

1. Picture,
2. History,
3. Motivation,
4. Introduction,
5. Preliminaries,
6. Semi-Riemannian map in semi-Riemannian context,
7. Conformal semi-Riemannian map in semi-Riemannian manifolds,
8. Generalized Eikonal Equation,

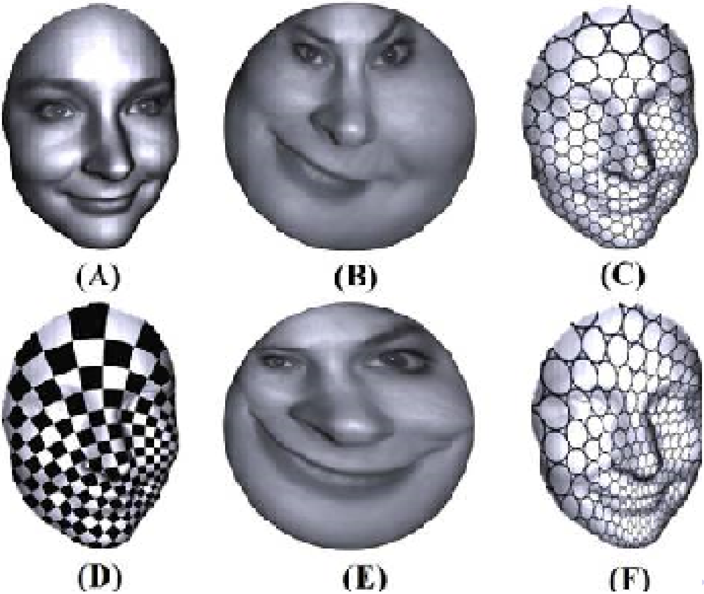
Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,
4. Introduction,
5. Preliminaries,
6. Semi-Riemannian map in semi-Riemannian context,
7. Conformal semi-Riemannian map in semi-Riemannian manifolds,
8. Generalized Eikonal Equation,
9. Relation with Harmonicity,

Conformality in semi-Riemannian Context

1. Picture,
2. History,
3. Motivation,
4. Introduction,
5. Preliminaries,
6. Semi-Riemannian map in semi-Riemannian context,
7. Conformal semi-Riemannian map in semi-Riemannian manifolds,
8. Generalized Eikonal Equation,
9. Relation with Harmonicity,
10. References.

1. Picture



1. Riemannian Context:

- O'Neill: Riemannian submersions,

2. History

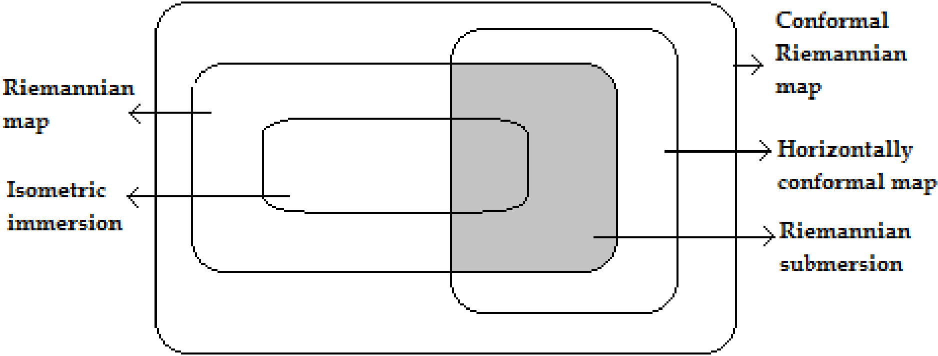
1. Riemannian Context:

- O'Neill: Riemannian submersions,
- Fischer: Riemannian map,

2. History

1.Riemannian Context:

- O'Neill: Riemannian submersions,
- Fischer: Riemannian map,
- Şahin: Conformal Riemannian map.


Riemannian map
Isometric immersion
Conformal Riemannian map
Horizontally conformal map
Riemannian submersion

Definition (O'Neill)

Let M and N be Riemannian manifolds. A Riemannian submersion $F : (M^m, g) \longrightarrow (N^n, h)$ is a mapping of M onto N satisfying the following axioms, $S1$ and $S2$:

$S1$. F has maximal rank;

$S2$. F_* preserves lengths of horizontal vectors.

Definition (Fischer)

A smooth map $F : (M^m, g) \to (N^n, h)$ is called Riemannian map at $p \in M$ if the horizontal restriction $F_{*p} : H_p \to \mathrm{Im}\, F_{*p}$ is a linear isometry between inner product spaces $(H_p, g_p \mid_{H_p})$ and $(\mathrm{Im}\, F_{*p}, h_{F(p)} \mid_{\mathrm{Im}\, F_{*p}})$.

Definition (Şahin)

Let (M^m, g) and (N^n, h) be Riemannian manifolds and $F : (M^m, g) \longrightarrow (N^n, h)$ a smooth map between them. Then we say that F is a conformal Riemannian map at $p \in M$ if $0 < rankF_{*p} \leq \min\{m, n\}$ and F_{*p} maps the horizontal space $H_p = (\ker(F_{*p}))^{\perp}$ conformally onto $\operatorname{Im} F_{*p}$, i.e., there exists a number $\lambda^2(p) \neq 0$ such that

$$h(F_{*p}X, F_{*p}Y) = \lambda^2(p)g(X, Y)$$

for $X, Y \in H_p$. Also, F is called conformal Riemannian if F is conformal Riemannian at each $p \in M$.

2. Semi-Riemannian Context:

- O'Neill: Semi-Riemannian submersion,

2.. Semi-Riemannian Context:

- O'Neill: Semi-Riemannian submersion,
- Garcia-Rio & Kupeli: Semi-Riemannian map,

2..Semi-Riemannian Context:

- O'Neill: Semi-Riemannian submersion,
- Garcia-Rio & Kupeli: Semi-Riemannian map,
- Here: Conformal semi-Riemannian map.

Definition (O'Neill)

A semi-Riemannian submersion $F : (M^m, g) \longrightarrow (N^n, h)$ is a submersion of semi-Riemannian manifolds such that:

i) The fibres $F^{-1}(y),\ y \in N$, are semi-Riemannian submanifolds of M.

ii) F_* preserves scalar products of vectors normal to fibres.

Definition (Garcia-Rio & Kupeli)

Let $f : (M, g) \longrightarrow (N, h)$ be a map between semi-Riemannian manifolds. Then f is called semi-Riemannian at $p \in M$ if

$$\overline{f}_{*p} : (\overline{H}(p), g_{/\overline{H}(p)}) \longrightarrow (\overline{A}_2(p), h_{/\overline{A}_2(p)})$$

is an (into) isometry, where $(\overline{H}(p), g_{/\overline{H}(p)})$ and $(\overline{A}_2(p), h_{/\overline{A}_2(p)})$ are the quotient inner product spaces is given by:

$$\begin{aligned}\overline{H}(p) &= H_p / Rad(V), \\ \overline{A}_2(p) &= Imf_{*p} / Rad(Imf_{*p})\end{aligned}$$

and $\overline{f}_{*p}$ is the quotient of f_{*p}. Moreover, f is called semi-Riemannian if f is semi-Riemannian at each $p \in M$.

Definition (B & E)

Let $F : (M^m, g) \to (N^n, h)$ be a smooth map between semi-Riemannian manifolds.

i) We say that F is conformal semi-Riemannian at $p \in M$ if $0 < rankF \leq min\{m, n\}$ and the screen tangent map F^S_{*p} is conformal, that is, there exists a non-zero real number $\Lambda(p)$ (called square dilation) such that:

$$F^S_{*p} = F_{*p/S(H_p)} : (S(H_p), g_{/S(H_p)}) \to (S(ImF_{*p}), h_{/S(ImF_{*p})})$$

satisfies:

$$h_{F(p)}(F_{*p}X, F_{*p}Y) = \Lambda(p) g_p(X, Y), \quad \forall X, Y \in S(H_p).$$

ii) Moreover, we call F a conformal semi-Riemannian map if F is conformal semi-Riemannian at each $p \in M$.

3. Motivation

- We aim to unify and generalize both conformal Riemannian maps (see [18]) and semi-Riemannian maps (see [9]).

3. Motivation

- We aim to unify and generalize both conformal Riemannian maps (see [18]) and semi-Riemannian maps (see [9]).
- We extend a result of Fischer and a result of Şahin obtained in the Riemannian case, to the semi-Riemannian context.

3. Motivation

- We aim to unify and generalize both conformal Riemannian maps (see [18]) and semi-Riemannian maps (see [9]).
- We extend a result of Fischer and a result of Şahin obtained in the Riemannian case, to the semi-Riemannian context.
- We generalize a result concerning the eikonal equation.

4. Introduction

In the Riemannian context, two fundamental notions, namely the Riemannian maps introduced by Fischer [6] on one side and horizontally conformal maps given by Fuglede [8] and Ishihara [12] on the other side, were both generalized by conformal Riemannian maps defined in [18]. Șahin motivated there the importance of this new class of maps between Riemannian manifolds by several geometric properties and practical applications in computer vision, computer graphics and medical imaging fields (see [11], [15], [19]). Conformal maps between cortical surfaces were computed in [20].

Next, Fuglede extended the notion of horizontally conformal map from the Riemannian context (see [8]) to the semi-Riemannian one (see [7]) with the purpose to characterize harmonic morphisms between semi-Riemannian manifolds (see [2]). Some theoretical applications to gravity of these horizontally conformal maps between semi-Riemannian manifolds were provided by Mustafa in [16]. Moreover, these maps were described in terms of jets in [13]. The class of horizontally conformal maps contains in particular semi-Riemannian submersions, for which we refer to [17] and [5]. Semi-Riemannian submersions are generalized by the semi-Riemannian maps between semi-Riemannian manifolds. The importance of this subject in semi-Riemannian geometry was exposed by García-Río and Kupeli in their monograph [9] devoted to study of the semi-Riemannian maps between semi-Riemannian manifolds.

Our goal is to introduce in this paper a new class of maps between semi-Riemannian manifolds with the purpose to unify and generalize the above two concepts, namely the one treated in [18] (i.e. conformal Riemannian maps between Riemannian manifolds) and the other one studied in [9] (i.e. semi-Riemannian maps between semi-Riemannian manifolds). This class of maps, which we call conformal semi-Riemannian maps between semi-Riemannian manifolds contains semi-Riemannian submersions (see [5]) and isometric immersions between semi-Riemannian manifolds as particular cases. Different from the approach of [9] by using quotient spaces, in our approach we use the screen distributions introduced by [4], which we present in Section 5.

Next, we characterize the semi-Riemannian maps between semi-Riemannian manifolds and we show some properties of them in Section 6. The main notion of our paper, namely conformal semi-Riemannian map between semi-Riemannian manifolds, is given by Section 7 which provides several classes of examples. Section 8 is devoted to the generalized eikonal equation. As it was mentioned in ([9], page 92), Fischer's result for Riemannian map (and similar for Şahin's result for conformal Riemannian map) is not valid in the semi-Riemannian case. By using conformal semi-Riemannian maps defined here, we adapt both these results in order to remain valid in the semi-Riemannian context. The last section relates this new notion of conformality with that of harmonicity used in many branches of mathematics.

We assume throughout this paper the manifolds and maps to be smooth.

Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds. At any point $p \in M$ one has the following linear spaces:

$$V_p = \{X \in T_pM \mid F_{*p}X = 0\} = KerF_{*p},$$
$$H_p = \{Y \in T_pM \mid g(Y, X) = 0\} = V_p^{\perp},$$
$$Rad(V_p) = V_p \cap H_p.$$

which denote respectively the vertical, the horizontal and the radical space.

In the Riemannian case, we don't need the following assumption, but in the semi-Riemannian case, we make the assumption that we obtain a distribution which we call vertical (resp. horizontal) if we assign to each $p \in M \to V_p$ the vertical (resp. $p \in M \to H_p$ the horizontal) space:

$$V = \bigcup_{p \in M} V_p = kerF_*,$$
$$H = \bigcup_{p \in M} H_p = V^{\perp}.$$

Suppose that the mapping $p \in M \to Rad(V_p)$ which assigns to each $p \in M$ the radical subspace $Rad(V_p)$ of V_p with respect to g_p defines a smooth distribution $Rad(V)$ of rank $r \in \mathbb{N}$ on M. Obviously $Rad(V)$ is a totally degenerate distribution on M since g restricted to $Rad(V)$ is identically zero.
We note that the leaves of the vertical distribution are lightlike (resp. semi-Riemannian) submanifolds of M provided $r > 0$ (resp. $r = 0$).

Consider a complementary distribution $S(V)$ to $Rad(V)$ in V. The fibres of $S(V)$ are $S(V_p)$ defined such that

$$V_p = Rad(V_p) \oplus S(V_p),$$

where $p \in M$. As these fibres of $S(V)$ are screen subspaces of V_p, $p \in M$ (see [4]), we call $S(V)$ the vertical screen distribution on M.
Similarly, let $S(H)$ be a complementary distribution to $Rad(V)$ in H. The fibres of $S(H)$ are $S(H_p)$ defined such that

$$H_p = Rad(V_p) \oplus S(H_p),$$

where $p \in M$. Analogous, we call $S(H)$ the horizontal screen distribution on M.
Let $\pi_H : H \to S(H) \to$ denote the projection of $H = RadV \oplus S(H)$ on $S(H)$.
Claim: From now on, we assume that all screen distributions related to F are arbitrary fixed.

Lemma

The following properties hold good:

(i) *The distribution $Rad(V)$ is degenerate, while $S(V)$ and $S(H)$ are nondegenerate;*
(ii) *We have $S(H) \perp V$ and $S(V) \perp H$, since $S(H_p) \perp V_p$ and $S(V_p) \perp H_p$, $\forall p \in M$;*
(iii) $\dim V + \dim H = \dim M$;
(iv) $(V^{\perp})^{\perp} = H^{\perp} = V$;
(v) *The following equivalences hold:*
$(V, g_{/V})$ *is a nondegenerate distribution* $\Leftrightarrow Rad(V) = \{0\} \Leftrightarrow TM = V \oplus H$;
(vi) *Any leaf of the vertical distribution V is either a lightlike submanifold of M (provided $(V, g_{/V})$ is degenerate) or a semi-Riemannian manifold (provided $(V, g_{/V})$ is nondegenerate).*

Proposition

If the vertical distribution $(V, g_{/V})$ is lightlike of type (r, ν', η'), then H is a lightlike distribution on (M, g) in TM, of type $(r, \nu - r - \nu', m - \nu - r - \eta')$ where $m = dimM$ and ν is the index of M. Moreover,

$$[Rad(V)]^{\perp}_{g/V} = V + H$$

is a lightlike distirbution on (M, g) in TM of type $(r, \nu - r, m - \nu - r)$.

Corollary

In particular, when the vertical leaves are degenerate hypersurfaces of (M, g), then $Rad(V) = H$. Hence the horizontal distribution is of dimension 1 and (V, g_V) is of type $(1, \nu - 1, m - \nu - 1)$.

Suppose that the mapping

$$p \in M \longrightarrow Rad(ImF_{*p}) = ImF_{*p} \cap (ImF_{*p})^{\perp}$$

which assigns to each $p \in M$ the radical subspace $Rad(ImF_{*p})$ of ImF_{*p} (with respect to h) is a vector bundle on M. Consider a complementary vector subbundle $S(ImF_*)$ to $Rad(ImF_*)$ (with respect to h) in

$$ImF_* = \bigcup_{p \in M} ImF_{*p}.$$

The fibres of $S(ImF_*)$ are $S(ImF_{*p})$ defined such that

$$ImF_{*p} = S(ImF_{*p}) \oplus Rad(ImF_{*p})$$

for any $p \in M$. We call $S(ImF_*)$ the screen vector subbundle of the image of F_* and let

$$\pi_{ImF_*} : ImF_* \longrightarrow S(ImF_*)$$

denote the projection of $ImF_* = S(ImF_*) \oplus Rad(ImF_*)$ to the first component of the direct sum.

Definition

Under the above notations, for any $p \in M$, we define the restriction of F_{*p} as the following linear transformation:

$$F^S_{*p} = F_{*p/S(H_p)} : (S(H_p), g_{/S(H)}) \to (S(ImF_{*p}), h_{/S(ImF_{*p})}),$$

given by

$$F^S_{*p}(\overline{X}) = \pi_{ImF_{*p}}(F_{*p}X),$$

where $X \in H$ and $\pi_H(X) = \overline{X}$. The rank of the F^S_{*p} is called the nondegenerate rank of F_{*p}.

Remark

i) We note that in $p \in M$, the screen tangent map F^S_{*p} may be neither injective nor surjective.
ii) For any $p \in M$, the linear transformation F^S_{*p}depends on the screen distribution, while the rank of F^S_{*p} is independent on it. Therefore, the nondegenerate rank of F_{*p} is well defined.

Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds. Then the square norm of F is defined in any point $p \in M$ by

$$\|F_{*p}\|^2 =< F_{*p}, F_{*p} >= trace_g(F_{*p}\cdot, F_{*p}\cdot) = \sum_{i=1}^{m} \varepsilon_i h(F_{*p}u_i, F_{*p}u_i)$$

where $\{u_1, u_2, ..., u_m\}$ is an orthonormal basis in T_pM and $\varepsilon_i = g(u_i, u_i) \in \{-1, 1\}, \ i = 1, 2, ..., m.$

Lemma

Let $F : (M^m, g) \rightarrow (N^n, h)$ *be a map between semi-Riemannian manifolds and* $p \in M$*. Then*

$$\|F_{*p}\|^2 = \|F^S_{*p}\|^2.$$

Proof.

Let $\{f_1, ..., f_s\}$ be a local orthonormal frame of the screen vertical distribution $S(V)$ and $\{e_1, ..., e_t\}$ be a local orthonormal frame of the screen horizontal distribution $S(H)$. Note that $span\{f_1, ..., f_s, e_1, ..., e_t\}$ is a nondegenerate subspace in T_pM and denote by $(span\{f_1, ..., f_s, e_1, ..., e_t\})^{\perp}$ its orthogonal complamantary space in T_pM. Since g is a nondegenerate metric on M it follows that $(span\{f_1, ..., f_s, e_1, ..., e_t\})^{\perp}$ is also a nondegenerate subspace and we may take $\{z_1, w_1, ..., z_k, w_k\}$ to be an orthonormal basis of it, such that

$$g(z_i, z_j) = \delta_{ij} = -g(w_i, w_j)$$

and

$$g(z_i, w_j) = 0, \quad \forall i, j \in \{1, 2, ..., k\}.$$

So, $z_i + w_i \in Rad(V)$ for $i = 1, 2, ..., k$. Then $\{f_1, ..., f_s, e_1, ..., e_t, z_1, w_1, ..., z_k, w_k\}$ is a local orthonormal frame on (M, g). □

Proof.

[Continuation of proof] Hence

$$\begin{aligned}
\|F_{*p}\|^2 &= \sum_{i=1}^{s} g(f_i, f_i) g(({}^*F_{*p} \circ F_{*p}) f_i, f_i) \\
&\quad + \sum_{i=1}^{t} g(e_i, e_i) g(({}^*F_{*p} \circ F_{*p}) e_i, e_i) \\
&\quad + \sum_{i=1}^{k} g(z_i, z_i) g(({}^*F_{*p} \circ F_{*p}) z_i, z_i) \\
&\quad + \sum_{i=1}^{k} g(w_i, w_i) g(({}^*F_{*p} \circ F_{*p}) w_i, w_i) \\
&= \sum_{i=1}^{t} g(e_i, e_i) h(F_{*p} e_i, F_{*p} e_i) + \sum_{i=1}^{k} g(z_i, z_i) h(F_{*p} z_i, F_{*p} z_i) \\
&\quad + \sum_{i=1}^{k} g(w_i, w_i) h(F_{*p} w_i, F_{*p} w_i).
\end{aligned}$$

Proof.

[Continuation of proof] But since for $i = 1, 2, ..., k$ we have $z_i + w_i \in Rad(V) \subseteq V$, then

$$0 = F_{*p}z_i + F_{*p}w_i$$

and

$$0 = g(z_i, z_i) + g(w_i, w_i).$$

Thus $F_{*p}z_i = -F_{*p}w_i$ and $g(z_i, z_i) = -g(w_i, w_i),\ i = 1, 2, ..., k$. Hence

$$\begin{aligned} \|F_{*p}\|^2 &= \sum_{i=1}^{t} g(e_i,\ e_i)h(F_{*p}e_i,\ F_{*p}e_i) \\ &= \sum_{i=1}^{t} g_{S(H)}(\pi_H(e_i),\ \pi_H(e_i))h_{S(ImF_{*p})}(F^S_{*p}\pi_H(e_i),\ F^S_{*p}\pi_H(e_i)) \\ &= \|F^S_{*p}\|^2, \end{aligned}$$

which proves the required equality. □

Definition (Garcia-Rio & Kupeli)

Let $f : (M, g) \to (N, h)$ be a map between semi-Riemannian manifolds. For any $p \in M$, we introduce the screen tangent map F^S_{*p} defined as the restriction of F_{*p}:

$$\overline{f}_{*p} : (\overline{H}(p), g_{/\overline{H}(p)}) \to (\overline{A}_2(p), h_{/\overline{A}_2(p)})$$

is an (into) isometry, where $(\overline{H}(p), g_{/\overline{H}(p)})$ and $(\overline{A}_2(p), h_{/\overline{A}_2(p)})$ are the quotient inner product spaces is given by:

$$\begin{aligned} \overline{H}(p) &= H_p / Rad(V), \\ \overline{A}_2(p) &= Imf_{*p} / Rad(Imf_{*p}) \end{aligned}$$

and $\overline{f}_{*p}$ is the quotient of f_{*p}. Moreover, f is called semi-Riemannian if f is semi-Riemannian at each $p \in M$.

Proposition

A map $F : (M, g) \longrightarrow (N, h)$ between semi-Riemannian manifolds is semi-Riemannian at $p \in M$ if and only if F_{*p} preserves inner products on the screen horizontal vectors, that is the screen tangent map F^S_{*p} is an (into) isometry map. Moreover, F is a semi-Riemannian map if and only if F is semi-Riemannian at each $p \in M$.

Remark

An (into) isometry map F^S_* remains an (into) isometry when one changes the screen distribution, since this fact can easily be justified by using basis. Hence, it follows that the above Proposition is independent on the screen distribution chosen and therefore this notion is well defined.

Lemma

Let F be a semi-Riemannian map. Then:

1. $rankF_*^S = rankF - \dim Rad(V) = \dim S(H)$;
2. $\|F_*\|^2 = rankF_*^S$;
3. *In the particular case, when* (M, g) *and* (N, h) *are Riemannian manifolds, then the semi-Riemannian map F becomes a Riemannian map, defined by Fischer, in [6].*

Definition (Baird & Wood)

Let us recall that a C^1 map $F : (M^m, g) \longrightarrow (N^n, h)$ between semi-Riemannian manifolds is called horizontally weakly conformal at $p \in M$ with square dilation $\Lambda(p)$ if

$$g({}^*F_{*p}U, {}^*F_{*p}U) = \Lambda(p)h(U, V) \quad (U, V \in T_{F(p)}N)$$

for some $\Lambda(p) \in \mathbb{R}$; it is said to be horizontally weakly conformal (on M) if it is horizontally weakly conformal at every point $p \in M$.

Note that under the condition $\Lambda(p) \in \mathbb{R} \backslash \{0\}$, we obtain that F is horizontally conformal at $p \in M$. Moreover, we say that F is horizontally homothetic if F is horizontally conformal on M and the square dilation $\Lambda : M \longrightarrow \mathbb{R} \backslash \{0\}$ is constant.

7. Conformal semi-Riemannian maps in semi-Riemannian manifolds

Definition

Let $F : (M^m, g) \to (N^n, h)$ be a smooth map between semi-Riemannian manifolds.

i) We say that F is conformal semi-Riemannian at $p \in M$ if $0 < rankF \leq min\{m, n\}$ and the screen tangent map F^S_{*p} is conformal, that is, there exists a non-zero real number $\Lambda(p)$ (called square dilation) such that:

$$F^S_{*p} = F_{*p/S(H_p)} : (S(H_p), g_{/S(H_p)}) \to (S(ImF_{*p}), h_{/S(ImF_{*p})})$$

satisfies:

$$h_{F(p)}(F_{*p}X, F_{*p}Y) = \Lambda(p) g_p(X, Y), \quad \forall X, Y \in S(H_p). \qquad (1)$$

ii) Moreover, we call F a conformal semi-Riemannian map if F is conformal semi-Riemannian at each $p \in M$;

Remark

If F^S_{*p} is a conformal map with $\Lambda(p) \neq 0$, then F^S_{*p} is injective. Indeed, if we suppose that there exist $X, Y \in S(H_p)$ such that:

$$F^S_{*p}(X) = F^S{*p}(Y), \qquad (2)$$

then the following equivalences hold good: (2) $\Leftrightarrow F_{*p}X = F_{*p}Y \quad \Leftrightarrow$

$$\begin{aligned} h_{F(p)}(F_{*p}X, Z) &= h_{F(p)}(F_{*p}Y, Z) && \Leftrightarrow \\ & \qquad \forall Z \in ImF_{*p} = \{F_{*p}U, \forall U \in S(H_p)\} \\ h_{F(p)}(F_{*p}X, F_{*p}U) &= h_{F(p)}(F_{*p}Y, F_{*p}U), \ \ \forall U \in S(H_p) && \Leftrightarrow \\ \Lambda(p)g(X, U) &= \Lambda(p)g(Y, U), \ \ \forall U \in S(H_p) \ \Leftrightarrow \ X = Y, \end{aligned}$$

since $\Lambda(p) \neq 0$ and $S(H_p)$ is nondegenerate.

Example

Let $\mathbb{R}^n_q$ denote the semi-Euclidean space of dimension n and index q, endowed with the inner product:

$$g(x, y) = -x_1y_1 - ... - x_qy_q + x_{q+1}y_{q+1} + ... + x_ny_n,$$

$\forall x = (x_1, ..., x_n),\ y = (y_1, ..., y_n) \in \mathbb{R}^n_q$.

We construct here a map

$$F : \mathbb{R}^3_1 \longrightarrow \mathbb{R}^4_2$$

between the semi-Euclidean spaces $(\mathbb{R}^3_1, g)$ and $(\mathbb{R}^4_2, h)$ defined for any $a, b \in \mathbb{R}$, by

$$F(x_1, x_2, x_3) = (e^{x_2} sin x_3, e^{x_2} cos x_3, a, b).$$

Example (Continuing)

Then we have

$$V = kerF_* = span\{\partial x_1\}$$

and

$$H = (kerF_*)^{\perp} = span\{\partial x_2, \partial x_3\}.$$

It follows that $rankF = 2$ and we get

$$\begin{aligned} F_*\partial x_2 &= e^{x_2} sinx_3 \partial y_1 + e^{x_2} cosx_3 \partial y_2 \\ F_*\partial x_3 &= e^{x_2} cosx_3 \partial y_1 - e^{x_2} sinx_3 \partial y_2. \end{aligned}$$

Therefore we have:

$$h(F_*\partial x_k, F_*\partial x_k) = -e^{2x_2} g(\partial x_k, \partial x_k), \quad k = 2, 3.$$

We conclude that F is a conformal semi-Riemannian map with the square dilation, $\Lambda : \mathbb{R}^3 \to \mathbb{R}$, defined by $\Lambda(x_1, ..., x_3) = -e^{2x_2}$.

Proposition

Let $F : (M, g) \to (N, h)$ *be a map between semi-Riemannian manifolds. Then* F *is horizontally weakly conformal with non-zero square dilation if and only if* F *is conformal semi-Riemannian with* $ImF_* = TN$ *and* $Rad(V) = \{0\}$.

Proposition

Let $F : (M, g) \to (N, h)$ *be a semi-Riemannian map between semi-Riemannian manifolds. Then* F *is a conformal semi-Riemannian map with the square dilation* $\Lambda = 1$.

Proof.

It is easy to see that we can identify by linear isometry the quotient spaces constructed in Garcia-Rio & Kupeli's book at each point $p \in M$, as follows:

$$\begin{aligned} L_1 &\cong Rad(V); \\ A_1 &\cong V + H = [Rad(V)]^{\perp}; \\ A_2 &\cong ImF_*; \\ L_2 &\cong ImF_* \cap (ImF_*)^{\perp} = Rad(ImF_*); \\ \overline{H} &= H/L_1 \cong S(H); \\ \overline{A}_2 &= A_2/L_2 \cong S(ImF_*). \end{aligned}$$

Then we complete the proof in a straightforward way. □

Corollary

Let $F : (M, g) \to (N, h)$ *be a map between semi-Riemannian manifolds.*

a) *ImF is an isometric immersed submanifold (see O'Neill's book) in N if and only if F is a conformal semi-Riemannian map with* $KerF_* = \{0\}$ *and* $\Lambda = 1$;

b) *F is a semi-Riemannian submersion (see Falcitelli, Ianus & Pastore's book) if and only if F is a conformal semi-Riemannian map with* $ImF_* = TN$, $Rad(V) = \{0\}$ *and* $\Lambda = 1$.

c) *F is a horizontally weakly conformal map of square dilation* $\Lambda = 1$ *if and only if it is a conformal semi-Riemannian map with* $ImF_* = TN$ *and* $\Lambda = 1$. *We note that a semi-Riemannian submersion defined in Falcitelli, Ianus & Pastore's book is a horizontally weakly conformal map (see O'neill's book) with the square dilation* $\Lambda = 1$.

Example (in Riemannian context)

Let $F : M \to N$ be a conformal Riemannian map between Riemannian manifolds, with dilation λ, defined by Şahin in his paper. Then F provides an example of a conformal semi-Riemannian map with positive square dilation $\Lambda = \lambda^2 : M \to \mathbb{R} \setminus \{0\}$.

8. Generalized Eikonal Equation

Eikonal equations are an interesting topic for both PDE and differential geometry (see Kupeli, Garcia-Rio & Kupeli and Şahin). We provide here a generalized eikonal equation which states a relation between the square norm of the tangent map and the nondegenerate rank of a conformal semi-Riemannian map.

Proposition

Let $F : (M^m, g) \longrightarrow (N^n, h)$ *be a map between semi-Riemannian manifolds which is conformal semi-Riemannian map in* $p \in M$ *with* $\Lambda(p) \neq 0$. *Then:*

$$\|F_{*p}\|^2 = \Lambda(p) rank F^S_{*p}. \quad (3)$$

Proof.

We have that the map

$$F_{*p}^{S} : (S(H_p), g_{/S(H)}) \longrightarrow (S(ImF_{*p}), h_{/S(ImF_{*p})})$$

is conformal and let

$$^{*}F_{*p}^{S} : (S(ImF_{*p}), h_{/S(ImF_{*p})}) \longrightarrow (S(H_p), g_{/S(H)})$$

be the adjoint of $\overline{F}_{*p}$. Then:

$$\begin{aligned} g_{/S(H)}(^{*}F_{*p}^{S} \circ F_{*p}^{S} u, v) &= h_{/S(ImF_{*p})}(F_{*p}^{S} u, F_{*p}^{S} v) \\ &= \Lambda(p) g_{/S(H)}(u, v), \quad \forall u, v \in S(H_p). \end{aligned} \tag{4}$$

Then, we have:

$$rank F_{*p}^{S} = \dim S(H_p) = \dim H_p - \dim Rad(V_p). \tag{5}$$

□

Proof.

[Continuation of proof] Hence, by applying consequently the relations (4) and (5), one has:

$$\begin{aligned} \|F_*\|^2 = \|F^S_{*p}\|^2 &= trace_g\, {}^*F^S_{*p} \circ F^S_{*p} = \sum_{i=1}^{t} \varepsilon_i \; g_{S(H)}({}^*F^S_{*p} \circ F^S_{*p}\; e_i, e_i) \\ &= \Lambda(p) \sum_{i=1}^{t} \varepsilon_i \; g_{/S(H)}(e_i, e_i) = \Lambda(p)\; \mathrm{dim} S(H_p) \\ &= \mathrm{rank}\; F^S_{*p}, \end{aligned}$$

where $\{e_1, ..., e_t\}$ is an orthonormal basis (with respect to g) of the nondegenerate screen horizontal distribution $S(H)$ and $\varepsilon_i = g(e_i, e_i) \in \{-1, 1\}$, $i = 1, ..., t$, which complete the proof. □

Remark

i) *The statement of above Lemma is independent of the screen horizontal distribution which was chosen in the proof.*

ii) *When F is a homothetic semi-Riemannian map, then the right hand side of the relation (1) is constant on each connected component of M, since the map $\|F_*\|^2 : M \longrightarrow \mathbb{R}$, defined by $\|F_*\|^2(p) = \|F_{*p}\|^2$ is a continuous function.*

iii) *From the above remark, it follows that F is a solution of the generalized eikonal equation, provided that F is a homothetic semi-Riemannian map.*

Let $F : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between semi-Riemannian manifolds. For any $p \in M$, let define the linear transformation

$$Q_p : S(ImF_{*p}) \longrightarrow S(ImF_{*p}), \; by$$
$$Q_p = F^S_{*p} \circ^* F^S_{*p}$$

to obtain the following characterization of conformal semi-Riemannian map.

Theorem

A smooth map $F : (M^m, g) \to (N^n, h)$ *is conformal semi-Riemannian if and only if for any* $p \in M$, *there exists a smooth function*

$$\Lambda : M \to \mathbb{R}$$

such that

$$Q_p^2 = \Lambda(p) Q_p. \tag{6}$$

Proof.

We have the following sequence of equivalences:
(6)⇔(7), where

$$Q_p^2 W = \Lambda(p) Q_p W, \quad \forall W \in S(ImF_{*p}). \tag{7}$$

Since $S(ImF_{*p})$ is nondegenerate, then (7)⇔

$$\begin{aligned}
h_{F(p)}(U, Q_p^2 W) &= \Lambda(p) h_{F(p)}(U, Q_p W), \quad \Leftrightarrow \\
&\quad \forall U, W \in S(ImF_{*p}) \\
h_{F(p)}(U, F^S_{*p} \circ^* F^S_{*p} \circ F^S_{*p} \circ^* F^S_{*p} W) &= \Lambda(p) h_{F(p)}(U, F^S_{*p} \circ^* F^S_{*p} W), \Leftrightarrow \\
&\quad \forall U, W \in S(ImF_{*p}) \\
g_p({}^* F^S_{*p} U, {}^* F^S_{*p} \circ F^S_{*p} \circ^* F^S_{*p} W) &= \Lambda(p) g_p({}^* F^S_{*p} U, {}^* F^S_{*p} W), \quad \Leftrightarrow \\
&\quad \forall U, W \in S(ImF_{*p}) \\
h_{F(p)}(F^S_{*p} \circ^* F^S_{*p} U, F^S_{*p} \circ^* F^S_{*p} W) &= \Lambda(p) g_p({}^* F^S_{*p} U, {}^* F^S_{*p} W), \qquad (8) \\
&\quad \forall U, W \in S(ImF_{*p}).
\end{aligned}$$

Proof.

[Continuation of proof] The direct statement follows immediately, since if F is a conformal semi-Rieman-nian map at p, then the last equality (8) is satisfied.

Conversely, if we suppose that the relation (5) is true, then by the above equivalence, the relation (8) is satisfied. To prove that F is a conformal semi-Riemannian map, we note first that the map ${}^*F^S_{*p} : S(ImF_{*p}) \rightarrow S(H_p)$ is onto. Indeed, the image of ${}^*F^S_{*p} : S(ImF_{*p}) \rightarrow S(H_p)$ is $S(H_p)$, since if we suppose, otherwise, then there exists a non-zero vector field $\xi \in S(H_p)$, such that $g_p({}^*F^S_{*p}Z, \xi) = 0$, $\forall Z \in S(ImF_{*p})$. □

Proof.

[Continuation of proof] Therefore we have:

$$0 = g_p({}^*F^S_{*p}Z, \xi) = h_{F(p)}(Z, F_{*p}\xi), \quad \forall Z \in S(ImF_{*p}).$$

Now, as $S(ImF_{*p})$ is nondegenerate with respect to h, then $F_{*p}\xi = 0$, that is $\xi \in Ker(F_{*p}) = V_p$ which is orthogonal to $S(H_p)$. As $\xi \in S(H_p)$, ξ is orthogonal to $S(H_p)$ and $S(H_p)$ is nondegenerate with respect to g, then $\xi = 0$ which is a contradiction. Therefore, the relation (8) is equivalent to (1), since for any $X, Y \in S(H_p)$, there exist $U, W \in S(ImF_{*p})$ such that $X = {}^*F^S_{*p}U$ and $Y = {}^*F^S_{*p}W$, which shows that F is conformal semi-Riemannian in any point $p \in M$, and complete the proof. □

Remark

1) *If* (M, g) *and* (N, h) *are Riemannian manifolds we reobtain Fischer's result that is,* F *is a Riemannian map if and only if* $Q_p = F_{*p} \circ^* F_{*p}$ *is a projection of* T_pM*, i.e.* $Q_p^2 = Q_p$.

2) *If* (M, g) *and* (N, h) *are semi-Riemannian manifolds, then we reobtain Şahin's result, that is* F *is a conformal semi-Riemannian map if and only if the operator* Q_p *defined on* T_pM *by* $Q_p = F_{*p} \circ^* F_{*p}$ *satisfies the relation* (5).

3) *As it is noticed in* $[Garica - RioKupeli, page92]$*, Fischer's theorem is not valid when* M *and* N *are semi-Riemannian manifolds and when* F *is a semi-Riemannian map if we take* Q_p *as an operator of* T_pM*. To generalize Fischer's result we state the last theorem by taking* Q_p *defined on a screen distribution* $S(ImF_{*p})$.

9. Relation with Harmonicity

Definition

Let $F : (M^m, g) \to (N^n, h)$ be a smooth map between semi-Riemannian manifolds and let ∇^M and $\nabla^{F^{-1}TN}$ denote respectively the Levi-Civita connection on M and the pull-back connection. Then F is harmonic if its tension field $\tau(F)$ vanishes identically, that is

$$\tau(F) = trace_g(\nabla_{\cdot} F_{*\cdot}) = \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i) = 0,$$

where $\{e_i\}$ is an orthonormal frame on M and the second fundamental form ∇F_* of F is given by

$$(\nabla F_*)(X, Y) = \nabla_X^{F^{-1}(TN)} F_* Y - F_*(\nabla_X^M Y), \quad \forall X, Y \in \Gamma(TM).$$

We recall the following geometric notion:

Definition (Baird & Wood)

Let $F : (M, g) \to (N, h)$ be a C^2 map between semi-Riemannian manifolds. Then F is a harmonic morphism if, for any C^2 harmonic function f defined on an open subset $\overline{N}$ of N with $F^{-1}(\overline{N})$ non-empty, the composition $f \circ F$, is harmonic on $F^{-1}(\overline{N})$.

The above notion was characterized by the following:

Theorem

A C^2 map between semi-Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal.

If D is a nondegenerate differentiable distribution of rank k on a semi-Riemannian manifold (M, g) with the Levi-Civita connection ∇, then TM splits into the direct sum $TM = D \oplus D^{\perp}$, where $D^{\perp}$ is the orthogonal distribution of D with respect to g. Moreover, D is called minimal if at each $p \in M$, the mean curvature field $\mu(D) \in \Gamma(F^{-1}TN)$ of D vanishes, i.e.,

$$\mu(D) = \frac{1}{k} trace_g(\nabla_{\cdot}\cdot)^{\perp} = \frac{1}{k}\sum_{i=1}^{k} g(e_i, e_i)(\nabla_{e_i} e_i)^{\perp} = 0,$$

where $(\nabla_{e_i} e_i)^{\perp}$ denotes the component of $\nabla_{e_i} e_i$ in the orthonormal complementary distribution $D^{\perp}$ on M and $\{e_i\}_{i=1,\ldots,k}$ is an orthonormal basis of D.

When the distribution D is integrable, then D is minimal if and only if any leaf of D is a minimal submanifold of M. (For degenerate distributions we refer the reader to Bejan & Duggal).

Then the calculation in the semi-Riemannian context follows the same steps as in the Riemannian case (see Șahin) and consequently, Theorem 4.1 from Șahin is now valid in the semi-Riemannian case, as follows:

Theorem

Let $F : (M^m, g) \to (N^n, h)$ be a non-constant proper conformal semi-Riemannian map between semi-Riemannian manifolds, such that the vertical distribution is nondegenerate and of codimension greater than 2. Then any three conditions imply fourth one:

i) *F is harmonic;*
ii) *F horizontally homothetic;*
iii) *The vertical distribution is minimal;*
iv) *The distribution ImF_* is minimal.*

In view of the first definition of Section 7, we note that in the semi-Riemannian context, both minimal immersions and harmonic morphisms are particular classes of harmonic maps which are conformal semi-Riemannian and hence both these classes can be studied in a unitary manner.

REFERENCES

[1] Atçeken, M., Kılıç, E., *Semi-Invariant Lightlike submanifolds of semi-Riemannian Product Manifold*, Kodai Math. J., 30(2007), 361-378.

[2] Baird, P., Wood, J. C., *Harmonic Morphisms between Riemannian Manifolds*. Clarendon Press, Oxford (2003).

[3] Bejan, C. L., Duggal, K. L., *Global Lightlike Manifolds and Harmonicity*, Kodai Math. J., 28(2005), 131-145.

[4] Duggal, K. L., Bejancu, A., *Lightlike Submanifolds of Semi-Riemannian Manifolds and Their Applicaitons*, Kluwer Academic, Dordrecht (1996).

[5] Falcitelli, M., Ianus, S. and Pastore, A. M., *Riemannian Submersions and Related Topics*, World Scientific, Singapore (2004).

[6] Fischer, A. E., *Riemannian maps between Riemannian manifolds*, Contemp. Math., 132(1992), 331-336 .

[7] Fuglede, B., *Harmonic Morphisms between Semi-Riemannian Manifolds*, Ann. Ac. Sci.Fen. Math., 21(1996), 31-50.

[8] Fuglede, B. *Harmonic morphisms between Riemannian manifolds*. Ann. Inst. Fourier (Grenoble) 28(1978), 107-144 .

[9] García-Río, E., Kupeli, D. N., *Semi-Riemannian Maps and Their Applications*, Kluwer Academic, Dordrecht (1999).

[10] Gray, A., *Pseudo-Riemannian almost product manifolds and submersions*. J. Math. Mech. 16(1967), 715-737.

[11] Gu, X., Wang, Y., Yau, S.T., *Computing conformal invariants:periodic matrices*. Commun. Inf. Syst. 3(3)(2003), 153-169.

[12] Ishihara, T., *A mapping of Riemannian manifolds which preserves harmonic functions*. J. Math. Kyoto Univ. 19(1979), 215-229.

[13] Kock, A., *A geometric theory of harmonic and semi-conformal maps*. Cent. Eur. J. Math. 2(5)(2004), 708-724.

[14] Kupeli, D. N., *The eikonal equation of an indefinite metric*, Acta Appl. Math., 40(1995), 245-253.

[15] Miao, J., Wang, Y., Gu, X., Yau, S.T., *Optimal global conformal surface parametrization for visualization*. Commun. Inf. Syst. 4(2)(2005), 117-134.

[16] Mustafa, M.T., *Applications of harmonic morphisms to gravity*. J. Math. Phys. 41(10)(2000), 6918-6929.

[17] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York (1983).

[18] Şahin, B., *Conformal Riemannian maps between Riemannian Manifolds, Their Harmonicty and Decomposition Theorems*, Acta Appl. Math., 109(2010), 829-847.

[19] Wang, Y., Gu, X., Chan, T.F., Thompson, P.M., Yau, S.T., *Brain surface conformal parametrization with the Ricci flow.* In: IEEE International Symposium on Biomedical Imaginging From Nano to Macro (ISBI), pp. 1312-1315, Washington D.C. (2007)

[**20**] Wang, Y., Yin, X., Zhang, J., Gu, X., Chan, T.F., Thompson, P.M., Yau, S.T., In: *Brain Mapping with the Ricci Flow Conformal Parameterization and Multivariate Statistics on Deformation Tensors*, 2nd MICCAI Workshop on Mathematical Foundations of Computational Anatomy, New York, pp. (2008), 36-47.

Thank you for attention !...