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Systems of MKdV equations related to the
affine Lie algebras

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PLAN

- The inverse scattering method
- Hierarchies of integrable nonlinear evolution equations (NLEE)
- Reductions of polynomial bundles
- mKdV equations related to simple Lie algebras
- The ISM as a GFT
- Conclusions and open questions
Based on:


- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with
\( \mathbb{Z}_N \) and \( \mathbb{D}_N \)-Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).


- V. S. Gerdjikov, D. M. Mladenov, A. A. Stefanov, S. K. Varbev. Integrable equations and recursion operators related to the affine Lie algebras \( A_r^{(1)} \). *ArXiv: 1411.0273v1 [nlin-SI]* Submitted to JMP.

The inverse scattering method


Lax representation:

\[
[L, M] \equiv 0, \\
L\psi \equiv i\frac{\partial \psi}{\partial x} + (U_1(x, t) - \lambda J)\psi(x, t, \lambda) = 0, \\
M\psi \equiv i\frac{\partial \psi}{\partial t} + (V_1(x, t) - \lambda K)\psi(x, t, \lambda) = 0,
\]

where $J, K$ – constant diagonal matrices.

\[
\lambda^2 \quad \text{a)} \quad [J, K] = 0, \\
\lambda \quad \text{b)} \quad [U_1, K] + [J, V_1] = 0, \\
\lambda^0 \quad \text{c)} \quad iV_{1,x} - iU_{1,t} + [U_1, V_1] = 0.
\]

Eq. a) is satisfied identically.
Eq. b) is satisfied identically if:

\[ U_1(x, t) = [J, Q_1(x, t)], \quad V_1(x, t) = [K, Q_1(x, t)], \]

Then eq. c) becomes the N-wave equation:

\[ i \left[ J, \frac{\partial Q_1}{\partial t} \right] - i \left[ K, \frac{\partial Q_1}{\partial x} \right] + [[K, Q_1], [J, Q_1]] = 0. \]

Simplest non-trivial case:

\[ N = 3, \quad \mathfrak{g} \simeq sl(3), \quad Q_1(x, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ u_1^* & 0 & u_2 \\ u_3^* & u_2^* & 0 \end{pmatrix}. \]

Then the 3-wave equations take the form:

\[ \begin{align*}
\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2} \frac{\partial u_1}{\partial x} + \kappa \epsilon_1 \epsilon_2 u_2^* u_3 &= 0, \\
\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3} \frac{\partial u_2}{\partial x} + \kappa \epsilon_1 u_1^* u_3 &= 0, \\
\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3} \frac{\partial u_3}{\partial x} + \kappa \epsilon_2 u_1^* u_2^* &= 0,
\end{align*} \]
where
\[ \kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2). \]

Solving Nonlinear Cauchy problems by the Inverse scattering method

Find solution to the \( N \)-wave eqs. such that
\[ Q_1(x, t = 0) = q_0(x). \]

\[
q_0 \rightarrow L_0 \quad L|_{t>0} \rightarrow q(x, t)
\]

\[
\begin{array}{c}
T(0, \lambda) \rightarrow II T(t, \lambda)
\end{array}
\]

**Step I:** Given \( Q_1(x, t = 0) = q_0(x) \) construct the scattering matrix \( T(\lambda, 0) \).
Jost solutions:

\[ L\phi(x, \lambda) = 0, \quad \lim_{x \to -\infty} \phi(x, \lambda)e^{i\lambda Jx} = 1, \]
\[ L\psi(x, \lambda) = 0, \quad \lim_{x \to \infty} \psi(x, \lambda)e^{i\lambda Jx} = 1, \]

\[ T(\lambda, 0) = \psi^{-1}(x, \lambda)\phi(x, \lambda). \]

**Step II:** From the Lax representation there follows:

\[ i \frac{\partial T}{\partial t} - \lambda[K, T(\lambda, t)] = 0, \]

i.e.

\[ T(\lambda, t) = e^{-i\lambda Kt}T(\lambda, 0)e^{i\lambda Kt}. \]

**Step III:** Given \( T(\lambda, t) \) construct the potential \( Q_1(x, t) \) for \( t > 0 \).

For \( \mathfrak{g} \simeq sl(2) \) – GLM eq. – Volterra type integral equations

For higher rank simple Lie algebras – GLM eq. become very complicated.

But it can be reduced to Riemann-Hilbert problem.

**Important:** Thus the nonlinear Cauchy problem reduces to a sequence of three linear Cauchy problems; each has unique solution!
Hierarchies of integrable nonlinear evolution equations

We can choose more complicated $M$-operators:

for the NLS type eqs:

$$V(x, t, \lambda) = V_2(x, t) + \lambda V_1(x, t) - \lambda^2 K.$$  

Then

$$i \frac{\partial T}{\partial t} - \lambda^2 [K, T(\lambda, t)] = 0,$$

for the MKdV type eqs:

$$V(x, t, \lambda) = V_3(x, t) + \lambda V_2(x, t) + \lambda^2 V_1(x, t) - \lambda^3 K.$$  

$$i \frac{\partial T}{\partial t} - \lambda^3 [K, T(\lambda, t)] = 0,$$

With each Lax operator $L$ one can relate a hierarchy of integrable NLEEE.
Reductions of Lax pairs

a) \( AU^\dagger(x, t, \epsilon \lambda^*) \hat{A} = -U(x, t, \lambda) \), \( AV^\dagger(x, t, \epsilon \lambda^*) \hat{A} = -V(x, t, \lambda) \),
b) \( BU^\dagger(x, t, \epsilon \lambda^*) \hat{B} = U(x, t, \lambda) \), \( BV^\dagger(x, t, \epsilon \lambda^*) \hat{B} = V(x, t, \lambda) \),
c) \( CUT(x, t, -\lambda) \hat{C} = -U(x, t, \lambda) \), \( CV^\dagger(x, t, -\lambda) \hat{C} = -V(x, t, \lambda) \),

where \( \epsilon^2 = 1 \) and \( A, B \) and \( C \) are elements of the group \( \mathfrak{G} \) such that \( A^2 = B^2 = C^2 = 1 \). As for the fundamental analytic solutions we have

a) \( A\xi^+, \dagger(x, t, \epsilon \lambda^*) \hat{A} = \hat{\xi}^-(x, t, \lambda) \),
b) \( B\xi^+, \dagger(x, t, \epsilon \lambda^*) \hat{B} = \xi^-(x, t, \lambda) \),
c) \( C\xi^+, T(x, t, -\lambda) \hat{C} = \hat{\xi}^-(x, t, \lambda) \),

For the \( \mathbb{Z}_N \)-reductions we may have:

\( D\xi^\pm(x, t, \omega \lambda) \hat{D} = \xi^\pm(x, t, \lambda) \),

\( DU(x, t, \omega \lambda) \hat{D} = U(x, t, \lambda) \), \( DV(x, t, \omega \lambda) \hat{D} = V(x, t, \lambda) \),

where \( \omega^N = 1 \) and \( D^N = 1 \).
NLS and MKdV eqs with $sl(n)$-series

DNLS type equations

Special examples of DNLS systems of equations can be found in VSG - 1988. We will give some particular examples when $M$ operator is from second and third degree in $\lambda$.

Those equations admit the following Hamiltonian formulation

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\delta H}{\delta q_{r+1-i}} \right).$$

The first interesting nontrivial case is when $M$ is quadratic polynomial in $\lambda$ and $g \simeq A_2^{(1)}$ algebra. The potential of $L$ is given by

$$U(x,t,\lambda) = \begin{pmatrix} 0 & q_1 & q_2 \\ q_2 & 0 & q_1 \\ q_1 & q_2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$. This gives us the system of integrable nonlinear partial
differential equations

\[ i \frac{\partial q_1}{\partial t} + i \gamma \frac{\partial}{\partial x} (q_2^2) + \gamma \frac{\sqrt{3}}{3} \frac{\partial^2 q_1}{\partial x^2} = 0, \]

\[ i \frac{\partial q_2}{\partial t} + i \gamma \frac{\partial}{\partial x} (q_1^2) - \gamma \frac{\sqrt{3}}{3} \frac{\partial^2 q_2}{\partial x^2} = 0. \]

The corresponding Hamiltonian is

\[ H = \frac{i \gamma \sqrt{3}}{6} \left( q_2 \frac{\partial q_1}{\partial x} - q_1 \frac{\partial q_2}{\partial x} \right) - \frac{\gamma}{3} (q_1^3 + q_2^3). \]

In the case of $A_3^{(1)}$ algebra using the potential

\[ U(x, t, \lambda) = \begin{pmatrix}
0 & q_1 & q_2 & q_3 \\
q_3 & 0 & q_1 & q_2 \\
q_2 & q_3 & 0 & q_1 \\
q_1 & q_2 & q_3 & 0
\end{pmatrix} - \lambda \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}, \]

we obtain the system of integrable nonlinear partial differential equations
\[ i \frac{\partial q_1}{\partial t} + 2i \gamma \frac{\partial}{\partial x} (q_2 q_3) + \gamma \frac{\partial^2 q_1}{\partial x^2} = 0, \]
\[ i \frac{\partial q_2}{\partial t} + i \gamma \frac{\partial}{\partial x} (q_1^2) + i \gamma \frac{\partial}{\partial x} (q_3^2) = 0, \]
\[ i \frac{\partial q_3}{\partial t} + 2i \gamma \frac{\partial}{\partial x} (q_1 q_2) - \gamma \frac{\partial^2 q_3}{\partial x^2} = 0. \]

The corresponding Hamiltonian is
\[ H = \frac{i \gamma}{2} \left( q_3 \frac{\partial q_1}{\partial x} - q_1 \frac{\partial q_3}{\partial x} + \frac{1}{2} \frac{\partial}{\partial x} (q_2^2) \right) - \gamma q_2 (q_1^2 + q_3^2). \]

**Systems of equations of mKdV type**

These are equations with cubic dispersion laws, therefore the $M$-operators are also cubic polynomials in $\lambda$.

In the case of $A_1^{(1)}$ algebra, with the following potential
\[ U(x, t, \lambda) = \begin{pmatrix} 0 & q_1 \\ q_1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
we obtain the well-known focusing mKdV equation

\[ \alpha \frac{\partial q_1}{\partial t} = -\frac{1}{4} \frac{\partial^3 q_1}{\partial x^3} - \frac{1}{2} \frac{\partial}{\partial x} (q_1^3), \]

where \( \alpha = \frac{a^3}{b} \). In this case the Hamiltonian is

\[ H = \frac{1}{8\alpha} \left( \left( \frac{\partial q_1}{\partial x} \right)^2 - q_1^4 \right). \]

In the case of \( A_2^{(1)} \) algebra we obtain a trivial system of equations \( \partial_t q_1 = 0 \) and \( \partial_t q_2 = 0 \) and the corresponding Hamiltonian is bilinear with respect to \( q_1 \) and \( q_2 \).

In the case of \( A_3^{(1)} \) algebra the potential of the Lax operator is parameterized by

\[ U(x, t, \lambda) = \begin{pmatrix} 0 & q_1 & q_2 & q_3 \\ q_3 & 0 & q_1 & q_2 \\ q_2 & q_3 & 0 & q_1 \\ q_1 & q_2 & q_3 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \]
which is related to the following system of mKdV type equations

\[
\begin{align*}
\alpha \frac{\partial q_1}{\partial t} &= \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial^2 q_1}{\partial x^2} + 3 \frac{\partial q_2}{\partial x} q_3 + 3q_1q_2^2 + q_3^3 \right), \\
\alpha \frac{\partial q_2}{\partial t} &= \frac{1}{4} \frac{\partial}{\partial x} \left( -\frac{\partial^2 q_2}{\partial x^2} + 3 \frac{\partial}{\partial x} (q_1^2 - q_3^2) + 12q_1q_2q_3 - 2q_2^3 \right), \\
\alpha \frac{\partial q_3}{\partial t} &= \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial^2 q_3}{\partial x^2} - 3 \frac{\partial q_2}{\partial x} q_1 + 3q_3q_2^2 + q_1^3 \right).
\end{align*}
\]

The corresponding Hamiltonian is

\[
H = \frac{1}{\alpha} \int_{-\infty}^{\infty} dx \left( \frac{1}{4} q_1^4 - \frac{1}{8} q_2^4 + \frac{1}{4} q_3^4 + \frac{3}{2} q_1q_2q_3 + \frac{1}{2} q_1q_2 \frac{\partial q_1}{\partial x} - \frac{1}{2} q_1^2 \frac{\partial q_2}{\partial x} \\
+ \frac{1}{2} q_2 \frac{\partial q_2}{\partial x} - \frac{1}{6} \left( \frac{\partial q_1}{\partial x} \right) \left( \frac{\partial q_3}{\partial x} \right) + \frac{1}{24} \left( \frac{\partial q_2}{\partial x} \right)^2 - \frac{1}{2} q_2q_3 \frac{\partial q_3}{\partial x} \\
+ \frac{1}{6} q_3 \frac{\partial^2 q_1}{\partial x^2} - \frac{1}{12} q_2 \frac{\partial^2 q_2}{\partial x^2} + \frac{1}{6} q_1 \frac{\partial^2 q_3}{\partial x^2} \right).
\]
The next example is related to $A_4^{(1)}$. The potential of the Lax operator now is

$$U(x, t, \lambda) = \begin{pmatrix}
0 & q_1 & q_2 & q_3 & q_4 \\
q_4 & 0 & q_1 & q_2 & q_3 \\
q_3 & q_4 & 0 & q_1 & q_2 \\
q_2 & q_3 & q_4 & 0 & q_1 \\
q_1 & q_2 & q_3 & q_4 & 0
\end{pmatrix} - \lambda \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 & 0 \\
0 & 0 & \omega^2 & 0 & 0 \\
0 & 0 & 0 & \omega^3 & 0 \\
0 & 0 & 0 & 0 & \omega^4
\end{pmatrix}, \quad \omega = e^{2\pi i/5}$$

The set of equations is

$$\alpha \frac{\partial q_1}{\partial t} = \frac{\partial}{\partial x} \left( \frac{c_1}{2s_1^2} \frac{\partial^2 q_1}{\partial x^2} + \frac{3}{2s_1} q_4 \frac{\partial q_2}{\partial x} + \frac{3}{2s_2} q_3 \frac{\partial q_3}{\partial x} + 3q_1q_2q_3 + q_2^3 + 3q_3q_4^2 \right),$$

$$\alpha \frac{\partial q_2}{\partial t} = \frac{\partial}{\partial x} \left( -\frac{c_2}{2s_2^2} \frac{\partial^2 q_2}{\partial x^2} - \frac{3}{2s_2} q_3 \frac{\partial q_4}{\partial x} + \frac{3}{2s_1} q_1 \frac{\partial q_1}{\partial x} + 3q_1q_2q_4 + q_4^3 + 3q_1q_3^2 \right),$$

$$\alpha \frac{\partial q_3}{\partial t} = \frac{\partial}{\partial x} \left( -\frac{c_2}{2s_2^2} \frac{\partial^2 q_3}{\partial x^2} + \frac{3}{2s_2} q_2 \frac{\partial q_1}{\partial x} - \frac{3}{2s_1} q_4 \frac{\partial q_4}{\partial x} + 3q_1q_3q_4 + q_1^3 + 3q_4q_2^2 \right),$$

$$\alpha \frac{\partial q_4}{\partial t} = \frac{\partial}{\partial x} \left( \frac{c_1}{2s_1^2} \frac{\partial^2 q_4}{\partial x^2} - \frac{3}{2s_1} q_1 \frac{\partial q_3}{\partial x} - \frac{3}{2s_2} q_2 \frac{\partial q_2}{\partial x} + 3q_2q_3q_4 + q_3^3 + 3q_2q_1^2 \right),$$
where

\[ s_k = \sin \left( \frac{k\pi}{5} \right), \quad c_k = \cos \left( \frac{k\pi}{5} \right), \quad s_1 = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}, \]

\[ c_1 = \frac{1}{4}(1 + \sqrt{5}), \quad s_2 = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}, \quad c_2 = \frac{1}{4}(\sqrt{5} - 1). \]

The Hamiltonian is

\[
H = \frac{2b}{3a^3} \int_{-\infty}^{\infty} dx \left( -\frac{c_1}{2s_1^2} \frac{\partial q_1}{\partial x} \frac{\partial q_4}{\partial x} + \frac{c_2}{2s_2^2} \frac{\partial q_2}{\partial x} \frac{\partial q_3}{\partial x} + q_1 q_3 + q_2 q_4 + q_3 q_4 \right.
\]

\[
+ \frac{3}{8s_1} \left( q_4^2 \frac{\partial q_2}{\partial x} - 2q_2 q_4 \frac{\partial q_4}{\partial x} + 2q_1 q_3 \frac{\partial q_1}{\partial x} - q_1^2 \frac{\partial q_3}{\partial x} \right) + 3q_1 q_2 q_3 q_4 + q_1^3 q_2 \]

\[
+ \frac{3}{8s_2} \left( q_2^2 \frac{\partial q_1}{\partial x} - 2q_1 q_2 \frac{\partial q_2}{\partial x} + 2q_3 q_4 \frac{\partial q_3}{\partial x} - q_3^2 \frac{\partial q_4}{\partial x} \right) \bigg). \]
Additional Involutions. Real Hamiltonian forms

Along with the $\mathbb{Z}_{r+1}$-reduction we can introduce one of the following involutions ($\mathbb{Z}_2$-reductions) on the Lax pair:

a) $K_0\kappa_1^{-1}U^\dagger(x, t, \kappa_1(\lambda))K_0 = U(x, t, \lambda), \quad \kappa_1(\lambda) = \omega^{-1}\lambda^*;$

b) $K_0\kappa_1^{-1}U^*(x, t, \kappa_1(\lambda))K_0 = -U(x, t, \lambda), \quad \kappa_1(\lambda) = -\omega^{-1}\lambda^*;$

c) $U^T(x, t, -\lambda) = -U(x, t, \lambda),$

where $K_0^2 = 1$. If we choose

$$K_0 = \sum_{k=1}^{r+1} E_{k,r-k+2}$$

then the action of $K_0$ on the basis is as follows

$$K_0 \left( J_s^{(k)} \right)^\dagger K_0 = \omega^k J_s^{(k)}, \quad K_0 \left( J_s^{(k)} \right)^* K_0 = \omega^{-k} J_{-s}^{(k)}.$$
An immediate consequences are the constraints on the potentials

a) \( K_0^{-1} Q^\dagger(x, t) K_0 = Q(x, t), \quad K_0^{-1} (J_0^{(1)})^\dagger K_0 = \omega^{-1} J_0^{(1)}, \)

b) \( K_0^{-1} Q^*(x, t) K_0 = -Q(x, t), \quad K_0^{-1} (J_0^{(1)})^* K_0 = \omega^{-1} J_0^{(1)}, \)

c) \( Q^T(x, t) = -Q(x, t), \quad (J_0^{(1)})^T = J_0^{(1)}. \)

Thus we obtain the algebraic relations below

a) \( q_j^*(x, t) = q_j(x, t), \quad \alpha = \alpha^*; \)

b) \( q_j^*(x, t) = -q_{r-j+1}(x, t), \quad \alpha = \alpha^*; \)

c) \( q_j(x, t) = -q_{r-j+1}(x, t), \)

where \( j = 1, \ldots, r, \) are compatible with the evolution of the mKdV equations.

If we apply case a) we get the same set of mKdV equations with \( q_1, q_2 \) and \( q_3 \) being purely real functions. In the case b) we put \( q_1 = -q_3^* = u \)
and \( q_2 = -q_2^* = iv \) and we get

\[
\alpha \frac{\partial v}{\partial t} = -\frac{1}{4} \frac{\partial^3 v}{\partial x^3} + \frac{3}{2} \frac{\partial^2}{\partial x^2} \left( u^2 - (u^*)^2 \right) - 3 \frac{\partial}{\partial x}(|u|^2 v) + \frac{1}{2} \frac{\partial}{\partial x} v^3,
\]

\[
\alpha \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^3 u}{\partial x^3} - \frac{3}{2} \frac{\partial}{\partial x} \left( u^* \frac{\partial v}{\partial x} \right) - \frac{3}{2} \frac{\partial}{\partial x} (uv^2) - \frac{\partial}{\partial x} (u^*)^3,
\]

where \( u \) is a complex function but \( v \) is a purely real function. The corresponding Hamiltonian is

\[
H = \frac{1}{\alpha} \left( \frac{1}{4} u^4 - \frac{1}{8} v^4 + \frac{1}{4} (u^*)^4 + \frac{3}{2} |u|^2 v^2 + \frac{i}{2} uv \frac{\partial u}{\partial x} - \frac{i}{2} u^2 \frac{\partial v}{\partial x} + \frac{i}{2} (u^*)^2 \frac{\partial v}{\partial x} + \frac{1}{6} \frac{\partial u}{\partial x} \right) - \frac{1}{24} \left( \frac{\partial v}{\partial x} \right)^2 - \frac{i}{2} u^* v \frac{\partial u^*}{\partial x} - \frac{1}{6} u^* \frac{\partial^2 u}{\partial x^2} + \frac{1}{12} v \frac{\partial^2 v}{\partial x^2} - \frac{1}{6} u \frac{\partial^2 u^*}{\partial x^2} \right).
\]

The case c) leads to the well known defocusing mKdV equation

\[
\alpha \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x} (u^3),
\]
where $u$ is a complex function. The corresponding Hamiltonian is

$$H = -\frac{1}{4\alpha} \left( \left( \frac{\partial u}{\partial x} \right)^2 + u^4 \right).$$

And finally, considering $A_5^{(1)}$ algebra with $\mathbb{D}_6$-reduction, case c) we find

$$\alpha \frac{\partial u}{\partial t} = 2 \frac{\partial^3 u}{\partial x^3} - 2\sqrt{3} \frac{\partial}{\partial x} \left( u \frac{\partial v}{\partial x} \right) - 6 \frac{\partial}{\partial x} (uv^2),$$

$$\alpha \frac{\partial v}{\partial t} = \sqrt{3} \frac{\partial^2}{\partial x^2} (u^2) - 6 \frac{\partial}{\partial x} (u^2 v),$$

where $u$ and $v$ are complex functions. The Hamiltonian is given by

$$H = -\frac{1}{\alpha} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \sqrt{3}u^2 \left( \frac{\partial v}{\partial x} \right) + 3u^2 v^2 \right).$$
Figure 1: The contour for the RHP of $L$ with $\mathbb{Z}_6$-symmetry.
ISP and RHP

Fundamental analytic solutions of $L \chi_n u(x, t, \lambda)$ and solutions to the RHP:

$$m_\nu(x, t, \lambda) = \chi(x, t, \lambda)e^{iJ\lambda x}.$$  

The rays $l_\nu$ are defined by:

$$\text{Im } \lambda \alpha(J) = 0, \quad \Leftrightarrow \quad \alpha \in \delta_\nu \quad \Leftrightarrow \quad g_\nu \subset g.$$  

The RHP is:

$$m_\nu^+(x, \lambda) = m_\nu^-(x, \lambda)e^{-iJ\lambda x}g_\nu(\lambda)e^{iJ\lambda x}$$

$$g_\nu(\lambda) = \hat{S}_\nu^-(\lambda)S_\nu^+(\lambda) = \hat{D}_\nu^-(\lambda)\hat{T}_\nu^+(\lambda)T_\nu^-(\lambda)D_\nu^+(\lambda).$$  

Here $S_\nu^\pm(\lambda), T_\nu^\pm(\lambda), D_\nu^\pm(\lambda)$ are defined by the asymptotic of $m_\nu^\pm(x, \lambda)$ when $x \to \pm \infty$:

$$S_\nu^\pm(\lambda) = \lim_{x \to -\infty} (e^{iJ\lambda x}m_\nu^\pm(x, \lambda)e^{-iJ\lambda x}) = \lim_{x \to -\infty} e^{iJ\lambda x}\chi_\nu^\pm(x, \lambda)$$

$$T_\nu^\pm(\lambda)D_\nu^\pm(\lambda) = \lim_{x \to \infty} (e^{iJ\lambda x}m_\nu^\pm(x, \lambda)e^{-iJ\lambda x}) = \lim_{x \to +\infty} e^{iJ\lambda x}\chi_\nu^\pm(x, \lambda).$$  

(2)
One could write $S_{\nu}^{\pm}, T_{\nu}^{\pm}, D_{\nu}^{\pm}$ also into the form

$$S_{\nu}^{\pm}(\lambda) = \exp \sum_{\alpha \in \delta_{\nu}^+} s_{\nu,\alpha}^{\pm}(\lambda) E_{\pm \alpha}, \quad T_{\nu}^{\pm}(\lambda) = \exp \sum_{\alpha \in \delta_{\nu}^+} t_{\nu,\alpha}^{\pm}(\lambda) E_{\pm \alpha} \quad (3)$$

$$D_{\nu,\alpha}^{\pm}(\lambda) = \exp(\pm \sum_{\alpha \in \pi_{\nu}} d_{\nu,\alpha}^{\pm}(\lambda) H_{\alpha}). \quad (4)$$

In other words $S_{\nu}^{\pm}, T_{\nu}^{\pm}, D_{\nu}^{\pm}$ belong to the subgroup $G_{\nu}$ with Lie algebra $g_{\nu}$. The fact that the factors $S_{\nu}^{\pm}, T_{\nu}^{\pm}, D_{\nu}^{\pm}$ have the above form is a consequence of the following relations that hold for $\lambda \in l_{\nu}$

$$\lim_{x \to \pm \infty} \langle E_{-\alpha}, m_{\nu}^{\pm} E_{\beta} \hat{m}_{\nu}^{\pm} \rangle = 0, \quad \alpha, \beta \in \Delta, \quad \text{Im} \ (\lambda(\alpha - \beta)(J)) \neq 0$$

$$\lim_{x \to \pm \infty} \langle H, m_{\nu}^{\pm} E_{\beta} \hat{m}_{\nu}^{\pm} \rangle = 0, \quad \beta \in \Delta, \quad H \in \mathfrak{h}, \quad \text{Im} \ (\lambda \beta(J)) \neq 0$$

$$\lim_{x \to \pm \infty} \langle E_{\beta}, m_{\nu}^{\pm} H \hat{m}_{\nu}^{\pm} \rangle = 0, \quad \beta \in \Delta, \quad H \in \mathfrak{h}, \quad \text{Im} \ (\lambda \beta(J)) \neq 0. \quad (5)$$

The minimal sets of scattering data that determine uniquely $T(\lambda)$ and
\( Q(x, t) \) are

\[
\mathcal{T}_S = \bigcup_{\nu=0}^{2} \{ s_{\nu, \alpha}^\pm (\lambda) : \alpha \in \delta^+_{\nu}, \lambda \in l_{\nu} \} \tag{6}
\]

\[
\mathcal{T}_T = \bigcup_{\nu=0}^{2} \{ t_{\nu, \alpha}^\pm (\lambda) : \alpha \in \delta^+_{\nu}, \lambda \in l_{\nu} \}. \tag{7}
\]

Completeness of ‘squared solutions’ and generalized Fourier transforms.

**Theorem** The sets of ‘squared solutions’ \( e_{\nu, il}(x, \lambda) \) form complete sets of functions in \( \mathcal{M}_J \). The completeness relation has the form:

\[
\delta(x - y) \Pi_0 = \\
= \frac{1}{\pi} \sum_{\nu=0}^{2h-1} (-1)^\nu \int_{l_{\nu}} d\lambda (G_{\nu+1}(x, y, \lambda) - G_\nu(x, y, \lambda)) - 2i \sum_{j=1}^{N} \text{Res}_{\lambda = \lambda_j} G_\nu(x, y, \lambda)
\]
$$
\Pi_0 = \sum_{\alpha > 0} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha)
$$

$$
G_{\nu+1}(x, y, \lambda) = \sum_{\alpha \in \Delta^+_\nu} e_{\nu+1,\alpha}(x, \lambda) \otimes e_{\nu+1,-\alpha}(y, \lambda),
$$

$$
G_{\nu}(x, y, \lambda) = \sum_{\alpha \in \Delta^-_\nu} e_{\nu,-\alpha}(x, \lambda) \otimes e_{\nu,\alpha}(y, \lambda) + \sum_{s=1}^{2} h_{\nu,s}(x, \lambda) \otimes h_{\nu,s}(y, \lambda).
$$
Figure 2: The contours $\gamma_\nu = l_\nu \cup \gamma_{\nu,\infty} \cup l_{\nu+1}$. 
Expansions over the ‘squared solutions’:

\[
Q(x, t) = \frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^{\nu} \alpha_\nu(J) \int_{l_\nu} d\lambda \left( s^+_{\alpha_\nu, \nu}(\lambda)e_{\nu+1; \alpha}(x, \lambda) - s^-_{\alpha_\nu, \nu}(\lambda)e_{\nu; -\alpha}(x, \lambda) \right) \\
+ \sum_{\text{DS}} \cdots
\]

(8)

\[
\text{ad } \frac{1}{J} \delta Q(x, t) = \\
\frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^{\nu} \int_{l_\nu} d\lambda \left( \delta s^+_{\alpha_\nu, \nu}(\lambda)e_{\nu+1; \alpha}(x, \lambda) + \delta - s^-_{\alpha_\nu, \nu}(\lambda)e_{\nu; -\alpha}(x, \lambda) \right) + \sum_{\text{DS}} \cdots
\]

(9)

e_{\alpha_\nu; \nu}(x, \lambda) are generalizations of \( e^{-i\lambda x} \). We need the analogs of \( id/dx \) for which \( i(d/dx)e^{-i\lambda x} = \lambda e^{-i\lambda x} \)

\[
(\Lambda_+ - \lambda)e_{\alpha_\nu; \nu}(x, \lambda) = 0, \quad (\Lambda_- - \lambda)e_{-\alpha_\nu; \nu}(x, \lambda) = 0,
\]

\[
\Lambda_{\pm}X(x) \equiv \text{ad } \frac{1}{J} \left( \frac{dX}{dx} + i \left[ [J, Q(x)], \int_{\pm\infty} dy \left[ [J, Q(y)], X(y) \right] \right] \right).
\]
In order to treat NLEE consider variations of the form:

$$\delta Q \simeq \frac{\partial Q}{\partial t} \delta t + \mathcal{O}((\delta t)^2), \quad (10)$$

and keep only first order of $\delta t$. Then we have the expansion:

$$i \text{ad}_J^{-1} \frac{\partial Q(x, t)}{\partial t} =$$

$$\frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^\nu \int_{l_\nu} d\lambda \left( i \frac{\partial s^+_{\alpha,\nu}}{\partial t} e_{\nu+1;\alpha}(x, \lambda) + i \frac{\partial s^-_{\alpha,\nu}}{\partial t} e_{\nu;\alpha}(x, \lambda) \right) + \sum_{DS} \cdots$$

$$\Lambda \text{ad}_J^{-1} [J^2, Q(x, t)] =$$

$$= \frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^\nu \alpha_{\nu}(J) \int_{l_\nu} d\lambda \lambda^3 \left( s^+_{\alpha,\nu}(\lambda) e_{\nu+1;\alpha}(x, \lambda) - s^-_{\alpha,\nu}(\lambda) e_{\nu;\alpha}(x, \lambda) \right) + \cdots$$
\[ i \text{ad}_{J}^{-1} \frac{\partial Q(x, t)}{\partial t} + \Lambda \text{ad}_{J}^{-1}[J^{2}, Q(x, t)] \equiv \text{mKdV} = \]

\[
\frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^{\nu} \int_{\lambda_{\nu}} d\lambda \left( \left( i \frac{\partial s_{\alpha_{\nu}, \nu}^{+}}{\partial t} + \lambda^{3} s_{\alpha_{\nu}, \nu}^{+} \right) e_{\nu+1; \alpha}(x, \lambda) + \right.
\]

\[
+ \left( i \frac{\partial s_{-\alpha_{\nu}, \nu}^{-}}{\partial t} - \lambda^{3} s_{\alpha_{\nu}, \nu}^{-}(\lambda) \right) e_{\nu; -\alpha}(x, \lambda) \right) + \sum_{\text{DS}} \cdots = 0
\tag{11} \]

i.e. these mKdV equations are equivalent to the following linear equations

\[
i \frac{\partial s_{\alpha_{\nu}, \nu}^{+}}{\partial t} + \lambda^{3} s_{\alpha_{\nu}, \nu}^{+} = 0,
\]

\[
i \frac{\partial s_{-\alpha_{\nu}, \nu}^{-}}{\partial t} - \lambda^{3} s_{\alpha_{\nu}, \nu}^{-} = 0.
\tag{12} \]

These GFT linearize the NLEE of mKdV type!

**Solving the RHP and soliton solutions**

Assume we have a regular solution of the RHP
\[ \xi_{\nu+1}^0(x, t, \lambda) = \xi_{\nu}^0(x, t, \lambda)G_{\nu}^0(x, t, \lambda) \]

Regular: \( \det m_{\nu}^0 \neq 0 \) for \( \lambda \in \Omega_{\nu} \)

Construct a new, singular solution of the RHP
\[ \xi_{\nu}^1(x, t, \lambda) = u(x, t, \lambda)\xi_{\nu+1}^0(x, t, \lambda), \]

\( u(x, t, \lambda) \) is the dressing factor, which may have poles and zeroes in \( \lambda \). The regular solution corresponds to potential \( Q_0 \) of \( L \); we may even choose \( Q_0 = 0 \).

The new singular solution of RHP corresponds to new potential \( Q \) which will depend on additional parameters.

One soliton solution of first type:
\[ u(x, t, \lambda) = 1 + \frac{1}{3} \left( \frac{A_1}{\lambda - \lambda_1} + \frac{J^{-1}A_1J}{\lambda \omega^2 - \lambda_1} + \frac{J^{-2}A_1J^2}{\lambda \omega - \lambda_1} \right) \] (13)

where \( A_1(\xi, \eta) \) is a \( 3 \times 3 \) degenerate matrix of the form
\[ A_1(x, t) = |n(x, t)\rangle\langle m^T(x, t)| \quad (A_1)_{ij}(x, t) = n_i(x, t)m_j(x, t). \] (14)
By construction $u(x, t, \lambda)$ satisfies the $\mathbb{Z}_3$-symmetry. The $\mathbb{Z}_2$-symmetry on $u(x, t, \lambda)$ can be put in the form

$$u(\xi, \eta, \lambda)A_0^{-1}u^\dagger(\xi, \eta, \lambda^*)A_0 = 1.$$  

(15)

and leads to algebraic equations which allow us to express the components of $n_j(x, t)$ in terms of $m_k(x, t)$:

$$n_1 = \frac{2\lambda_1^3 m_3^*}{\lambda_1^2 m_3^* m_1 + |\lambda_1|^2 |m_2|^2 + \lambda_1^2, m_1^* m_3} = \frac{2i\rho_1 m_3}{2m_1 m_3 - m_2^2}$$

$$n_2 = \frac{2\lambda_1^3 m_2^*}{\lambda_1^2, m_3^* m_1 + \lambda_1^2 |m_2|^2 + |\lambda_1|^2 m_1^* m_3} = \frac{2i\rho_1}{m_2}$$

$$n_3 = \frac{2\lambda_1^3 m_1^*}{|\lambda_1|^2 m_3^* m_1 + \lambda_1^2, |m_2|^2 + \lambda_1^2 m_1^* m_3} = \frac{2i\rho_1 m_1}{m_2^2}.$$  

(16)

After putting $\lambda_1 = i\rho_1$ we obtain the 1-soliton solution of the first
type for Tzitzeica eq.:

\[
\phi_{1s}(x,t) = \frac{1}{2} \ln \left| \frac{|\mu_{01}|^2 e^{-3x_1} \left( 4 \cos^2(\tilde{\Omega}_1) - 6 \right) - 8|\mu_{01}|\mu_{02} \cos(\tilde{\Omega}_1) + \mu_{02}^2 e^{3x_1}}{4|\mu_{01}|^2 e^{-3x_1} \cos^2(\tilde{\Omega}_1) + 4|\mu_{01}|\mu_{02} \cos(\tilde{\Omega}_1) + \mu_{02}^2 e^{3x_1}} \right|.
\]

(17)

where

\[
x_1 = \frac{1}{2} \left( \rho_1 x - \frac{t}{\rho_1} \right), \quad \Omega_1 = \frac{\sqrt{3}}{2} \left( \rho_1 x + \frac{t}{\rho_1} \right).
\]

(18)

Note: it is not traveling wave solution; it may have singularities! In the limit \( \mu_{02} \to 0 \) we obtain a traveling wave solution of the form

\[
\phi(x,t) = \frac{1}{2} \ln \left[ \frac{3}{2} \tanh^2 \left( \frac{\sqrt{3}}{2} (\rho_1 \xi + \rho_1^{-1} \eta) - \alpha_{01} \right) + \frac{1}{2} \right].
\]

(19)

One Soliton Solutions of Second Type
Now the anzatz for the dressing factor is

\[
  u(x, t, \lambda) = 1 + \frac{1}{3} \left( \frac{A_1}{\lambda - \lambda_1} + \frac{J^{-1} A_1 J}{\lambda \omega^2 - \lambda_1} + \frac{J^{-2} A_1 J^2}{\lambda \omega - \lambda_1} \right) \\
  - \frac{1}{3} \left( \frac{A_1^*}{\lambda + \lambda_1^*} + \frac{J^{-1} A_1^* J}{\lambda \omega^2 + \lambda_1^*} + \frac{J^{-2} A_1^* J^2}{\lambda \omega + \lambda_1^*} \right)
\]

which obviously satisfies the \( \mathbb{Z}_3 \)-reduction and the first \( \mathbb{Z}_2 \)-reduction.

Again we obtain an algebraic relations between \( n_j(x, t) \) in terms of \( m_k(x, t) \) which are more complicated:

\[
  |\mu\rangle = \begin{pmatrix} m_3 \\ m_2 \\ m_1 \\ m_3^* \\ m_2^* \\ m_1^* \end{pmatrix}, \\
  |\nu\rangle = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_1^* \\ n_2^* \\ n_3^* \end{pmatrix}, \\
  |\mu\rangle = \mathcal{M} |\nu\rangle
\]
where

\[
\mathcal{M} = \begin{pmatrix}
c_1 P_1 & 0 & 0 & \zeta_1 K_1 & 0 & 0 \\
0 & c_1 P_2 & 0 & 0 & \zeta_1 K_2 & 0 \\
0 & 0 & c_1 P_3 & 0 & 0 & \zeta_1 K_3 \\
\zeta_1 K_1^* & 0 & 0 & c_1 P_1^* & 0 & 0 \\
0 & \zeta_1 K_2^* & 0 & 0 & c_1 P_2^* & 0 \\
0 & 0 & \zeta_1 K_3^* & 0 & 0 & c_1 P_3^*
\end{pmatrix}
\]  \tag{22}

The result is

\[
|\nu\rangle = \mathcal{M}^{-1} |\nu\rangle
\]

\[
\mathcal{M}^{-1} = \begin{pmatrix}
-c_1^* \tilde{P}_1^* & 0 & 0 & \zeta_1 \tilde{K}_1 & 0 & 0 \\
0 & -c_1^* \tilde{P}_2^* & 0 & 0 & \zeta_1 \tilde{K}_2 & 0 \\
0 & 0 & -c_1^* \tilde{P}_3^* & 0 & 0 & \zeta_1 \tilde{K}_3 \\
\zeta_1 \tilde{K}_1^* & 0 & 0 & -c_1 \tilde{P}_1 & 0 & 0 \\
0 & \zeta_1 \tilde{K}_2^* & 0 & 0 & -c_1 \tilde{P}_2 & 0 \\
0 & 0 & \zeta_1 \tilde{K}_3^* & 0 & 0 & -c_1 \tilde{P}_3
\end{pmatrix}
\]  \tag{23}
where

\[ \tilde{P}_s^* = \frac{P_s^*}{d_s}, \quad \tilde{P}_s = \frac{P_s}{d_s}, \quad \tilde{K}_s = \frac{K_s}{d_s}, \quad \tilde{K}_s^* = \frac{K_s^*}{d_1} \]

\[ d_1 = \zeta_1 \zeta_1^* K_1 K_1^* - c_1 c_1^* P_1 P_1^* \quad d_2 = \zeta_1 \zeta_1^* K_2 K_2^* - c_1 c_1^* P_2 P_2^* \]

\[ d_3 = \zeta_1 \zeta_1^* K_3 K_3^* - c_1 c_1^* P_3 P_3^*. \]

From the above equations we obtain \(|n\rangle\) in terms of \(\langle m^T|\)

\[ n_1 = \frac{1}{d_1} (-c_1^* P_1^* m_3 + \zeta_1 K_1 m_3^*) \quad n_2 = \frac{1}{d_2} (-c_1^* P_2^* m_2 + \zeta_1 K_2 m_2^*) \]

\[ n_3 = \frac{1}{d_3} (-c_1^* P_3^* m_1 + \zeta_1 K_3 m_1^*). \]

The explicit \(x, t\) dependence of \(m_j(x, t)\) is

\[ m_1 = \omega^2 \mu_{01} e^{ix_1 - y_1} + \mu_{02} e^{ix_2 - y_2} + \omega \mu_{03} e^{ix_3 - y_3} \]

\[ m_2 = \mu_{01} e^{ix_1 - y_1} + \mu_{02} e^{ix_2 - y_2} + \mu_{03} e^{ix_3 - y_3} \]

\[ m_3 = \omega \mu_{01} e^{ix_1 - y_1} + \mu_{02} e^{ix_2 - y_2} + \omega^2 \mu_{03} e^{ix_3 - y_3} \]
where

\[ x_1 = - \left( x \rho_1 + \frac{t}{\rho_1} \right) \cos \left( \beta_1 - \frac{2\pi}{3} \right), \quad y_1 = - \left( x \rho_1 - \frac{t}{\rho_1} \right) \sin \left( \beta_1 - \frac{2\pi}{3} \right) \]

\[ x_2 = - \left( x \rho_1 + \frac{t}{\rho_1} \right) \cos (\beta_1), \quad y_2 = - \left( x \rho_1 - \frac{t}{\rho_1} \right) \sin (\beta_1) \]

\[ x_3 = - \left( x \rho_1 + \frac{t}{\rho_1} \right) \cos \left( \beta_1 + \frac{2\pi}{3} \right), \quad y_3 = - \left( x \rho_1 - \frac{t}{\rho_1} \right) \sin \left( \beta_1 + \frac{2\pi}{3} \right) \]

(27)

We determine the 1-soliton solution for the second kind of solitons using exactly the same technique

\[ \Phi = - \frac{1}{2} \ln \left| 1 - \frac{1}{\lambda_1} n_1 m_1 - \frac{1}{\lambda_1^*} n_1^* m_1^* \right|. \]

(28)

Multisoliton solutions \( N = N_1 + N_2 \) with \( N_1 \) solitons of first type and \( N_2 \) solitons of second type can also be derived: They would correspond to \( 6N_1 + 12N_2 \) singularities of the RHP.
Reconstructing the potential $Q(x,t)$ from $u(x,t,\lambda)$

After constructing the dressing factor we use the fact that it satisfies the equation:

\[
    i \frac{\partial u}{\partial x} + (Q(x,t) - \lambda J)u(x,t,\lambda) - u(x,t,\lambda)(Q_0(x,t) - \lambda J) = 0, \quad (29)
\]

Take the limit $\lambda \to \infty$ and use that

\[
    \lim_{\lambda \to \infty} u(x,t,\lambda) = 1,
\]

and choose also $Q_0(x,t) = 0$. Then

\[
    Q(x,t) = \lim_{\lambda \to \infty} \lambda(J - u(x,t,\lambda)J\hat{u}(x,t,\lambda)) \quad (30)
\]

which allows you to express $Q(x,t)$ in terms of the residue $A_1(x,t) = |\vec{n}_1\rangle\langle m_1|$. 

Figure 3: The discrete eigenvalues of $L$ with $\mathbb{Z}_3$-symmetry and $\mathbb{Z}_2$-symmetries. Two types of discrete eigenvalues, two types of soliton solutions.
Conclusions and some open questions

- The mKdV eqs. are Hamiltonian. View the jets $U(x, t, \lambda)$ and $V(x, t, \lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.

- Each of these eqs. has **two types** of soliton solutions. Find constraints on the soliton parameters that render them regular.

- One can derive their soliton interactions by evaluating the limits of the dressing factors for $x \to \pm \infty$.

Thank you for your attention!