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Systems of MKdV equations related to the affine Lie algebras

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PLAN

- The inverse scattering method
- Hierarchies of integrable nonlinear evolution equations (NLEE)
- Reductions of polynomial bundles
- mKdV equations related to simple Lie algebras
- The ISM as a GFT
- Conclusions and open questions

Based on:

- Mon1 V. S. Gerdjikov, G. Vilasi, A. B. Yanovski. *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods* Lecture Notes in Physics **748**, Springer Verlag, Berlin, Heidelberg, New York (2008). ISBN: 978-3-540-77054-1.

- V S Gerdjikov, A B Yanovski. CBC systems with Mikhailov reductions by Coxeter Automorphism. I. Spectral Theory of the Recursion Operators. Studies in Applied Mathematics **134** (2), 145–180 (2015). DOI: 10.1111/sapm.12065.

- V. S. Gerdjikov. On new types of integrable 4-wave interactions. AIP Conf. proc. **1487** pp. 272-279; (2012).

- V. S. Gerdjikov. Riemann-Hilbert Problems with canonical normalization and families of commuting operators. Pliska Stud. Math. Bulgar. **21**, 201–216 (2012).

- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with

$\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).

- V. S. Gerdjikov, A. B. Yanovski *On soliton equations with $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions: conservation laws and generating operators.* J. Geom. Symmetry Phys. **31**, 57–92 (2013).

- V. S. Gerdjikov, A B Yanovski. Riemann-Hilbert Problems, families of commuting operators and soliton equations Journal of Physics: Conference Series **482** (2014) 012017 doi:10.1088/1742-6596/482/1/012017

- V. S. Gerdjikov, D. M. Mladenov, A. A. Stefanov, S. K. Varbev. Integrable equations and recursion operators related to the affine Lie algebras $A_r^{(1)}$. **ArXiv: 1411.0273v1 [nlin-SI]** Submitted to JMP.

- V.S. Gerdjikov, D.M. Mladenov, A. A. Stefanov, S.K. Varbev. MKdV-type of equations related to $\mathfrak{sl}(N,\mathbb{C})$ algebra In Mathematics in Industry, Ed. A. Slavova. Cambridge Scholar Publishing, pp. 335–344 (2014).

The inverse scattering method

The inverse scattering method for the N-wave equations – Zakharov, Shabat, Manakov (1973).

Lax representation:

$$
\begin{aligned}
&[L, M] \equiv 0, \\
&L\psi \equiv i\frac{\partial \psi}{\partial x} + (U_1(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \\
&M\psi \equiv i\frac{\partial \psi}{\partial t} + (V_1(x,t) - \lambda K)\psi(x,t,\lambda) = 0,
\end{aligned}
$$

where J, K – constant diagonal matrices.

$$
\begin{array}{lll}
\lambda^2 & \text{a)} & [J, K] = 0, \\
\lambda & \text{b)} & [U_1, K] + [J, V_1] = 0, \\
\lambda^0 & \text{c)} & iV_{1,x} - iU_{1,t} + [U_1, V_1] = 0.
\end{array}
$$

Eq. a) is satisfied identically.

Eq. b) is satisfied identically if:

$$U_1(x,t) = [J, Q_1(x,t)], \qquad V_1(x,t) = [K, Q_1(x,t)],$$

Then eq. c) becomes the N-wave equation:

$$i\left[J, \frac{\partial Q_1}{\partial t}\right] - i\left[K, \frac{\partial Q_1}{\partial x}\right] + [[K, Q_1], [J, Q_1]] = 0.$$

Simplest non-trivial case:

$$N = 3, \qquad \mathfrak{g} \simeq sl(3), \qquad Q_1(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ u_1^* & 0 & u_2 \\ u_3^* & u_2^* & 0 \end{pmatrix}.$$

Then the 3-wave equations take the form:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0, \\
\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0, \\
\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0,
\end{aligned}$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

Solving Nonlinear Cauchy problems by the Inverse scattering method

Find solution to the N-wave eqs. such that

$$Q_1(x, t = 0) = q_0(x).$$

$$\begin{array}{ccccccc}
q_0 & \longrightarrow & L_0 & & L|_{t>0} & \longrightarrow & q(x,t) \\
 & & \mathrm{I}\Big\downarrow & & \Big\uparrow\mathrm{III} & & \\
 & & T(0,\lambda) & \xrightarrow{\mathrm{II}} & T(t,\lambda) & &
\end{array}$$

Step I: Given $Q_1(x, t = 0) = q_0(x)$ construct the scattering matrix $T(\lambda, 0)$.

Jost solutions:

$$L\phi(x,\lambda) = 0, \qquad \lim_{x\to-\infty} \phi(x,\lambda) e^{i\lambda J x} = \mathbb{1},$$

$$L\psi(x,\lambda) = 0, \qquad \lim_{x\to\infty} \psi(x,\lambda) e^{i\lambda J x} = \mathbb{1},$$

$$T(\lambda, 0) = \psi^{-1}(x,\lambda)\phi(x,\lambda).$$

Step II: From the Lax representation there follows:

$$i\frac{\partial T}{\partial t} - \lambda[K, T(\lambda,t)] = 0,$$

i.e.

$$T(\lambda,t) = e^{-i\lambda K t} T(\lambda, 0) e^{i\lambda K t}.$$

Step III: Given $T(\lambda,t)$ construct the potential $Q_1(x,t)$ for $t>0$.
For $\mathfrak{g} \simeq sl(2)$ – GLM eq. – Volterra type integral equations
For higher rank simple Lie algebras – GLM eq. become very complicated.
But it can be reduced to Riemann-Hilbert problem.

Important: Thus the nonlinear Cauchy problem reduces to a sequence of three **linear Cauchy problems**; each has unique solution!

Hierarchies of integrable nonlinear evolution equations

We can choose more complicated M-operators:

for the NLS type eqs:

$$V(x,t,\lambda) = V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K.$$

Then

$$i\frac{\partial T}{\partial t} - \lambda^2 [K, T(\lambda,t)] = 0,$$

for the MKdV type eqs:

$$V(x,t,\lambda) = V_3(x,t) + \lambda V_2(x,t) + \lambda^2 V_1(x,t) - \lambda^3 K.$$

$$i\frac{\partial T}{\partial t} - \lambda^3 [K, T(\lambda,t)] = 0,$$

With each Lax operator L one can relate a hierarchy of integrable NLEE.

Reductions of Lax pairs

$$\begin{array}{lll} \text{a)} & AU\dagger(x,t,\epsilon\lambda^*)\hat{A} = -U(x,t,\lambda), & AV^\dagger(x,t,\epsilon\lambda^*)\hat{A} = -V(x,t,\lambda), \\ \text{b)} & BU*(x,t,\epsilon\lambda^*)\hat{B} = U(x,t,\lambda), & BV^*(x,t,\epsilon\lambda^*)\hat{B} = V(x,t,\lambda), \\ \text{c)} & CUT(x,t,-\lambda)\hat{C} = -U(x,t,\lambda), & CV^\dagger(x,t,-\lambda)\hat{C} = -V(x,t,\lambda), \end{array}$$

where $\epsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the fundamental analytic solutions we have

$$\begin{array}{ll} \text{a)} & A\xi^{+,\dagger}(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^-(x,t,\lambda), \\ \text{b)} & B\xi^{+,*}(x,t,\epsilon\lambda^*)\hat{B} = \xi^-(x,t,\lambda), \\ \text{c)} & C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^-(x,t,\lambda), \end{array}$$

For the $\mathbb{Z}_N$-reductions we may have:

$$D\xi^\pm(x,t,\omega\lambda)\hat{D} = \xi^\pm(x,t,\lambda),$$

$$DU(x,t,\omega\lambda)\hat{D} = U(x,t,\lambda), \; DV(x,t,\omega\lambda)\hat{D} \quad = V(x,t,\lambda),$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

NLS and MKdV eqs with $sl(n)$-series

DNLS type equations

Special examples of DNLS systems of equations can be found in VSG - 1988. We will give some particular examples when M operator is from second and third degree in λ.

Those equations admit the following Hamiltonian formulation

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta q_{r+1-i}}\right).$$

The first interesting nontrivial case is when M is quadratic polynomial in λ and $\mathfrak{g} \simeq A_2^{(1)}$ algebra. The potential of L is given by

$$U(x,t,\lambda) = \begin{pmatrix} 0 & q_1 & q_2 \\ q_2 & 0 & q_1 \\ q_1 & q_2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$. This gives us the system of integrable nonlinear partial differential equations

$$i\frac{\partial q_1}{\partial t} + i\gamma\frac{\partial}{\partial x}(q_2^2) + \gamma\frac{\sqrt{3}}{3}\frac{\partial^2 q_1}{\partial x^2} = 0,$$

$$i\frac{\partial q_2}{\partial t} + i\gamma\frac{\partial}{\partial x}(q_1^2) - \gamma\frac{\sqrt{3}}{3}\frac{\partial^2 q_2}{\partial x^2} = 0.$$

The corresponding Hamiltonian is

$$H = \frac{i\gamma\sqrt{3}}{6}\left(q_2\frac{\partial q_1}{\partial x} - q_1\frac{\partial q_2}{\partial x}\right) - \frac{\gamma}{3}(q_1^3 + q_2^3).$$

In the case of $A_3^{(1)}$ algebra using the potential

$$U(x,t,\lambda) = \begin{pmatrix} 0 & q_1 & q_2 & q_3 \\ q_3 & 0 & q_1 & q_2 \\ q_2 & q_3 & 0 & q_1 \\ q_1 & q_2 & q_3 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

we obtain the system of integrable nonlinear partial differential equations

$$
\begin{aligned}
&i\frac{\partial q_1}{\partial t} + 2i\gamma\frac{\partial}{\partial x}(q_2 q_3) + \gamma\frac{\partial^2 q_1}{\partial x^2} = 0, \\
&i\frac{\partial q_2}{\partial t} + i\gamma\frac{\partial}{\partial x}(q_1^2) + i\gamma\frac{\partial}{\partial x}(q_3^2) = 0, \\
&i\frac{\partial q_3}{\partial t} + 2i\gamma\frac{\partial}{\partial x}(q_1 q_2) - \gamma\frac{\partial^2 q_3}{\partial x^2} = 0.
\end{aligned}
$$

The corresponding Hamiltonian is

$$
H = \frac{i\gamma}{2}\left(q_3\frac{\partial q_1}{\partial x} - q_1\frac{\partial q_3}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}(q_2^2)\right) - \gamma q_2(q_1^2 + q_3^2).
$$

Systems of equations of mKdV type

These are equations with cubic dispersion laws, therefore the M-operators are also cubic polynomials in λ.

In the case of $A_1^{(1)}$ algebra, with the following potential

$$
U(x,t,\lambda) = \begin{pmatrix} 0 & q_1 \\ q_1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

we obtain the well-known focusing mKdV equation

$$\alpha \frac{\partial q_1}{\partial t} = -\frac{1}{4}\frac{\partial^3 q_1}{\partial x^3} - \frac{1}{2}\frac{\partial}{\partial x}(q_1^3)\,,$$

where $\alpha = \frac{a^3}{b}$. In this case the Hamiltonian is

$$H = \frac{1}{8\alpha}\left(\left(\frac{\partial q_1}{\partial x}\right)^2 - q_1^4\right).$$

In the case of $A_2^{(1)}$ algebra we obtain a trivial system of equations $\partial_t q_1 = 0$ and $\partial_t q_2 = 0$ and the corresponding Hamiltonian is bilinear with respect to q_1 and q_2.

In the case of $A_3^{(1)}$ algebra the potential of the Lax operator is parameterized by

$$U(x,t,\lambda) = \begin{pmatrix} 0 & q_1 & q_2 & q_3 \\ q_3 & 0 & q_1 & q_2 \\ q_2 & q_3 & 0 & q_1 \\ q_1 & q_2 & q_3 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

which is related to the following system of mKdV type equations

$$
\begin{aligned}
\alpha \frac{\partial q_1}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^2 q_1}{\partial x^2} + 3\frac{\partial q_2}{\partial x} q_3 + 3q_1 q_2^2 + q_3^3\right), \\
\alpha \frac{\partial q_2}{\partial t} &= \frac{1}{4}\frac{\partial}{\partial x}\left(-\frac{\partial^2 q_2}{\partial x^2} + 3\frac{\partial}{\partial x}\left(q_1^2 - q_3^2\right) + 12 q_1 q_2 q_3 - 2q_2^3\right), \\
\alpha \frac{\partial q_3}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^2 q_3}{\partial x^2} - 3\frac{\partial q_2}{\partial x} q_1 + 3q_3 q_2^2 + q_1^3\right).
\end{aligned}
$$

The corresponding Hamiltonian is

$$
\begin{aligned}
H = \frac{1}{\alpha}\int_{-\infty}^{\infty} dx \, \Bigg(& \frac{1}{4}q_1^4 - \frac{1}{8}q_2^4 + \frac{1}{4}q_3^4 + \frac{3}{2}q_1 q_2^2 q_3 + \frac{1}{2}q_1 q_2 \frac{\partial q_1}{\partial x} - \frac{1}{2}q_1^2 \frac{\partial q_2}{\partial x} \\
& + \frac{1}{2}q_3^2 \frac{\partial q_2}{\partial x} - \frac{1}{6}\left(\frac{\partial q_1}{\partial x}\right)\left(\frac{\partial q_3}{\partial x}\right) + \frac{1}{24}\left(\frac{\partial q_2}{\partial x}\right)^2 - \frac{1}{2}q_2 q_3 \frac{\partial q_3}{\partial x} \\
& + \frac{1}{6}q_3 \frac{\partial^2 q_1}{\partial x^2} - \frac{1}{12}q_2 \frac{\partial^2 q_2}{\partial x^2} + \frac{1}{6}q_1 \frac{\partial^2 q_3}{\partial x^2}\Bigg).
\end{aligned}
$$

The next example is related to $A_4^{(1)}$. The potential of the Lax operator now is

$$U(x,t,\lambda) = \begin{pmatrix} 0 & q_1 & q_2 & q_3 & q_4 \\ q_4 & 0 & q_1 & q_2 & q_3 \\ q_3 & q_4 & 0 & q_1 & q_2 \\ q_2 & q_3 & q_4 & 0 & q_1 \\ q_1 & q_2 & q_3 & q_4 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & 0 & \omega^4 \end{pmatrix}, \qquad \omega = e^{2\pi i/5}$$

The set of equations is

$$\alpha \frac{\partial q_1}{\partial t} = \frac{\partial}{\partial x}\left(\frac{c_1}{2s_1^2}\frac{\partial^2 q_1}{\partial x^2} + \frac{3}{2s_1} q_4 \frac{\partial q_2}{\partial x} + \frac{3}{2s_2} q_3 \frac{\partial q_3}{\partial x} + 3q_1q_2q_3 + q_2^3 + 3q_3q_4^2 \right),$$

$$\alpha \frac{\partial q_2}{\partial t} = \frac{\partial}{\partial x}\left(-\frac{c_2}{2s_2^2}\frac{\partial^2 q_2}{\partial x^2} - \frac{3}{2s_2} q_3 \frac{\partial q_4}{\partial x} + \frac{3}{2s_1} q_1 \frac{\partial q_1}{\partial x} + 3q_1q_2q_4 + q_4^3 + 3q_1q_3^2 \right),$$

$$\alpha \frac{\partial q_3}{\partial t} = \frac{\partial}{\partial x}\left(-\frac{c_2}{2s_2^2}\frac{\partial^2 q_3}{\partial x^2} + \frac{3}{2s_2} q_2 \frac{\partial q_1}{\partial x} - \frac{3}{2s_1} q_4 \frac{\partial q_4}{\partial x} + 3q_1q_3q_4 + q_1^3 + 3q_4q_2^2 \right),$$

$$\alpha \frac{\partial q_4}{\partial t} = \frac{\partial}{\partial x}\left(\frac{c_1}{2s_1^2}\frac{\partial^2 q_4}{\partial x^2} - \frac{3}{2s_1} q_1 \frac{\partial q_3}{\partial x} - \frac{3}{2s_2} q_2 \frac{\partial q_2}{\partial x} + 3q_2q_3q_4 + q_3^3 + 3q_2q_1^2 \right),$$

where

$$s_k = \sin\left(\frac{k\pi}{5}\right), \qquad c_k = \cos\left(\frac{k\pi}{5}\right), \qquad s_1 = \frac{1}{4}\sqrt{10-2\sqrt{5}},$$
$$c_1 = \frac{1}{4}(1+\sqrt{5}), \qquad s_2 = \frac{1}{4}\sqrt{10-2\sqrt{5}}, \qquad c_2 = \frac{1}{4}(\sqrt{5}-1).$$

The Hamiltonian is

$$\begin{aligned} H = \frac{2b}{3a^3}\int_{-\infty}^{\infty} dx \Bigg(&- \frac{c_1}{2s_1^2}\frac{\partial q_1}{\partial x}\frac{\partial q_4}{\partial x} + \frac{c_2}{2s_2^2}\frac{\partial q_2}{\partial x}\frac{\partial q_3}{\partial x} + q_1 q_3^3 + q_2^3 q_4 + q_3 q_4^3 \\ &+ \frac{3}{8s_1}\left(q_4^2\frac{\partial q_2}{\partial x} - 2q_2 q_4\frac{\partial q_4}{\partial x} + 2q_1 q_3\frac{\partial q_1}{\partial x} - q_1^2\frac{\partial q_3}{\partial x}\right) + 3q_1 q_2 q_3 q_4 + q_1^3 q_2 \\ &+ \frac{3}{8s_2}\left(q_2^2\frac{\partial q_1}{\partial x} - 2q_1 q_2\frac{\partial q_2}{\partial x} + 2q_3 q_4\frac{\partial q_3}{\partial x} - q_3^2\frac{\partial q_4}{\partial x}\right)\Bigg). \end{aligned}$$

Additional Involutions. Real Hamiltonian forms

Along with the $\mathbb{Z}_{r+1}$-reduction we can introduce one of the following involutions ($\mathbb{Z}_2$-reductions) on the Lax pair:

$$
\begin{array}{lrl}
\text{a)} & K_0^{-1}U^\dagger(x,t,\kappa_1(\lambda))K_0 = U(x,t,\lambda), & \kappa_1(\lambda) = \omega^{-1}\lambda^*; \\
\text{b)} & K_0^{-1}U^*(x,t,\kappa_1(\lambda))K_0 = -U(x,t,\lambda), & \kappa_1(\lambda) = -\omega^{-1}\lambda^*; \\
\text{c)} & U^T(x,t,-\lambda) = -U(x,t,\lambda), &
\end{array}
$$

where $K_0^2 = \mathbb{1}$. If we choose

$$
K_0 = \sum_{k=1}^{r+1} E_{k,r-k+2}
$$

then the action of K_0 on the basis is as follows

$$
K_0 \left(J_s^{(k)}\right)^\dagger K_0 = \omega^{k(s-1)} J_s^{(k)}, \qquad K_0 \left(J_s^{(k)}\right)^* K_0 = \omega^{-k} J_{-s}^{(k)}.
$$

An immediate consequences are the constraints on the potentials

$$
\begin{array}{lll}
\text{a)} & K_0^{-1}Q^\dagger(x,t)K_0 = Q(x,t), & K_0^{-1}(J_0^{(1)})^\dagger K_0 = \omega^{-1}J_0^{(1)}, \\
\text{b)} & K_0^{-1}Q^*(x,t)K_0 = -Q(x,t), & K_0^{-1}(J_0^{(1)})^* K_0 = \omega^{-1}J_0^{(1)}, \\
\text{c)} & Q^T(x,t) = -Q(x,t), & (J_0^{(1)})^T = J_0^{(1)}.
\end{array}
$$

Thus we obtain the algebraic relations below

$$
\begin{array}{lll}
\text{a)} & q_j^*(x,t) = q_j(x,t), & \alpha = \alpha^*; \\
\text{b)} & q_j^*(x,t) = -q_{r-j+1}(x,t), & \alpha = \alpha^*; \\
\text{c)} & q_j(x,t) = -q_{r-j+1}(x,t), &
\end{array}
$$

where $j = 1,\ldots,r$, are compatible with the evolution of the mKdV equations.

If we apply case a) we get the same set of mKdV equations with q_1, q_2 and q_3 being purely real functions. In the case b) we put $q_1 = -q_3^* = u$

and $q_2 = -q_2^* = iv$ and we get

$$\alpha \frac{\partial v}{\partial t} = -\frac{1}{4}\frac{\partial^3 v}{\partial x^3} + \frac{3}{4i}\frac{\partial^2}{\partial x^2}\left(u^2 - (u^*)^2\right) - 3\frac{\partial}{\partial x}(|u|^2 v) + \frac{1}{2}\frac{\partial}{\partial x}v^3,$$
$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - i\frac{3}{2}\frac{\partial}{\partial x}\left(u^*\frac{\partial v}{\partial x}\right) - \frac{3}{2}\frac{\partial}{\partial x}(uv^2) - \frac{\partial}{\partial x}(u^*)^3,$$

where u is a complex function but v is a purely real function. The corresponding Hamiltonian is

$$H = \frac{1}{\alpha}\Bigg(\frac{1}{4}u^4 - \frac{1}{8}v^4 + \frac{1}{4}(u^*)^4 + \frac{3}{2}|u|^2 v^2 + \frac{i}{2}uv\frac{\partial u}{\partial x} - \frac{i}{2}u^2\frac{\partial v}{\partial x} + \frac{i}{2}(u^*)^2\frac{\partial v}{\partial x}$$
$$+ \frac{1}{6}\left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{24}\left(\frac{\partial v}{\partial x}\right)^2 - \frac{i}{2}u^* v\frac{\partial u^*}{\partial x} - \frac{1}{6}u^*\frac{\partial^2 u}{\partial x^2} + \frac{1}{12}v\frac{\partial^2 v}{\partial x^2} - \frac{1}{6}u\frac{\partial^2 u^*}{\partial x^2}\Bigg).$$

The case c) leads to the well known defocusing mKdV equation

$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x}(u^3),$$

where u is a complex function. The corresponding Hamiltonian is

$$H = -\frac{1}{4\alpha}\left(\left(\frac{\partial u}{\partial x}\right)^2 + u^4\right).$$

And finally, considering $A_5^{(1)}$ algebra with $\mathbb{D}_6$-reduction, case c) we find

$$\alpha\frac{\partial u}{\partial t} = 2\frac{\partial^3 u}{\partial x^3} - 2\sqrt{3}\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) - 6\frac{\partial}{\partial x}(uv^2),$$
$$\alpha\frac{\partial v}{\partial t} = \sqrt{3}\frac{\partial^2}{\partial x^2}\left(u^2\right) - 6\frac{\partial}{\partial x}(u^2 v),$$

where u and v are complex functions. The Hamiltonian is given by

$$H = -\frac{1}{\alpha}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \sqrt{3}u^2\left(\frac{\partial v}{\partial x}\right) + 3u^2v^2\right).$$

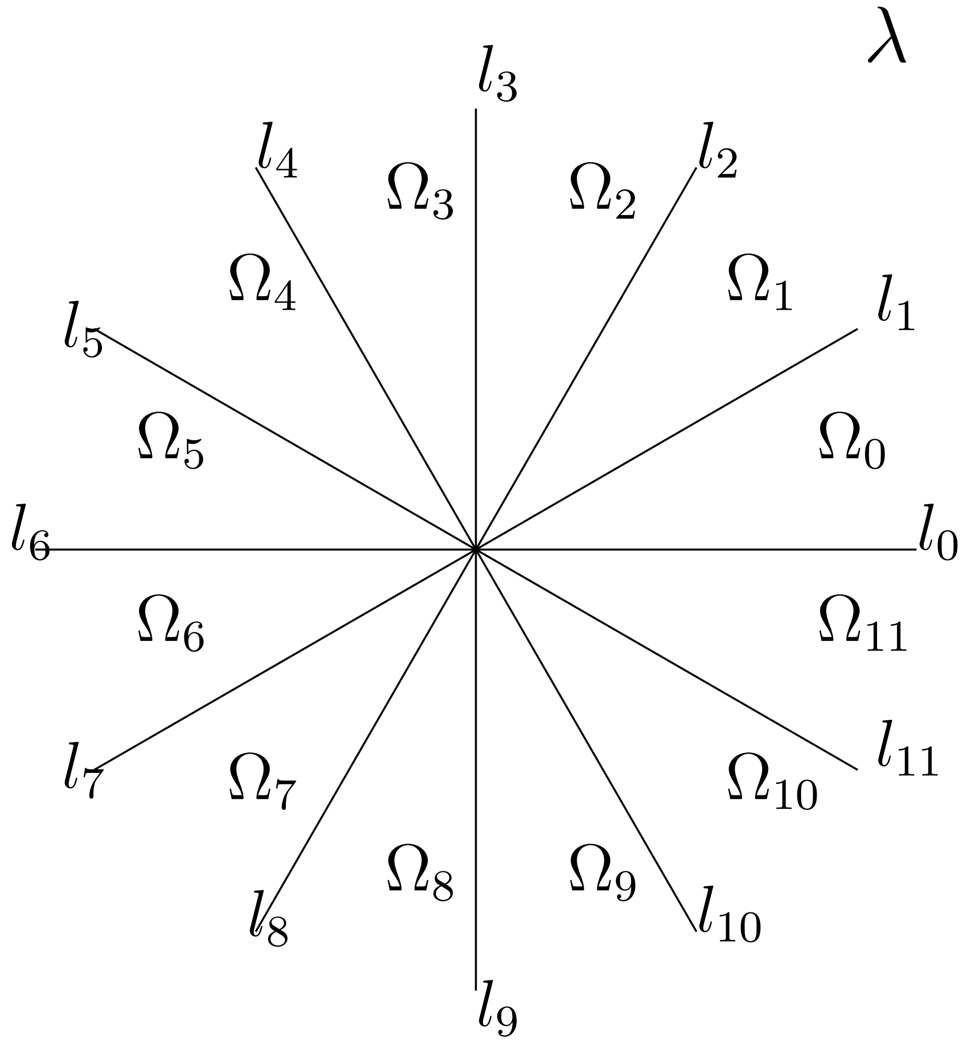

Figure 1: The contour for the RHP of L with $\mathbb{Z}_6$-symmetry.

ISP and RHP

Fundamental analytic solutions of L $\chi_n u(x,t,\lambda)$ and solutions to the RHP:

$$m_\nu(x,t,\lambda) = \chi(x,t,\lambda)e^{iJ\lambda x}.$$

The rays l_ν are defined by:

$$\operatorname{Im}\lambda\alpha(J) = 0, \qquad \Leftrightarrow \qquad \alpha \in \delta_\nu \qquad \Leftrightarrow \qquad \mathfrak{g}_\nu \subset \mathfrak{g}.$$

The RHP is:

$$\begin{aligned} m_\nu^+(x,\lambda) &= m_\nu^-(x,\lambda)e^{-iJ\lambda x} g_\nu(\lambda) e^{iJ\lambda x} \\ g_\nu(\lambda) &= \hat{S}_\nu^-(\lambda) S_\nu^+(\lambda) = \hat{D}_\nu^-(\lambda)\hat{T}_\nu^+(\lambda) T_\nu^-(\lambda) D_\nu^+(\lambda). \end{aligned} \tag{1}$$

Here $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$, $D_\nu^\pm(\lambda)$ are defined by the asymptotic of $m_\nu^\pm(x,\lambda)$ when $x \to \pm\infty$:

$$\begin{aligned} S_\nu^\pm(\lambda) &= \lim_{x\to-\infty} \left(e^{i\lambda Jx} m_\nu^\pm(x,\lambda) e^{-i\lambda Jx} \right) = \lim_{x\to-\infty} e^{iJ\lambda x}\chi_\nu^\pm(x,\lambda) \\ T_\nu^\mp(\lambda) D_\nu^\pm(\lambda) &= \lim_{x\to\infty} \left(e^{i\lambda Jx} m_\nu^\pm(x,\lambda) e^{-i\lambda Jx} \right) = \lim_{x\to+\infty} e^{iJ\lambda x}\chi_\nu^\pm(x,\lambda). \end{aligned} \tag{2}$$

One could write $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ also into the form

$$S_\nu^\pm(\lambda) = \exp \sum_{\alpha\in\delta_\nu^+} s_{\nu,\alpha}^\pm(\lambda) E_{\pm\alpha}, \quad T_\nu^\pm(\lambda) = \exp \sum_{\alpha\in\delta_\nu^+} t_{\nu,\alpha}^\pm(\lambda) E_{\pm\alpha} \quad (3)$$

$$D_{\nu,\alpha}^\pm(\lambda) = \exp(\pm \sum_{\alpha\in\pi_\nu} d_{\nu,\alpha}^\pm(\lambda) H_\alpha). \quad (4)$$

In other words $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ belong to the subgroup G_ν with Lie algebra $\mathfrak{g}_\nu$. The fact that the factors $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ have the above form is a consequence of the following relations that hold for $\lambda \in l_\nu$

$$\begin{aligned}
&\lim_{x\to\pm\infty} \langle E_{-\alpha}, m_\nu^\pm E_\beta \hat{m}_\nu^\pm \rangle = 0, && \alpha, \beta \in \Delta, && \operatorname{Im}(\lambda(\alpha-\beta)(J)) \neq 0 \\
&\lim_{x\to\pm\infty} \langle H, m_\nu^\pm E_\beta \hat{m}_\nu^\pm \rangle = 0, && \beta \in \Delta, \quad H \in \mathfrak{h}, && \operatorname{Im}(\lambda\beta(J)) \neq 0 \\
&\lim_{x\to\pm\infty} \langle E_\beta, m_\nu^\pm H \hat{m}_\nu^\pm \rangle = 0, && \beta \in \Delta, \quad H \in \mathfrak{h}, && \operatorname{Im}(\lambda\beta(J)) \neq 0.
\end{aligned} \quad (5)$$

The minimal sets of scattering data that determine uniquely $T(\lambda)$ and

$Q(x,t)$ are

$$\mathcal{T}_S = \bigcup_{\nu=0}^{2} \{ s^{\pm}_{\nu,\alpha}(\lambda) : \alpha \in \delta_\nu^+, \lambda \in l_\nu \} \tag{6}$$

$$\mathcal{T}_T = \bigcup_{\nu=0}^{2} \{ t^{\pm}_{\nu,\alpha}(\lambda) : \alpha \in \delta_\nu^+, \lambda \in l_\nu \}. \tag{7}$$

Completeness of 'squared solutions' and generalized Fourier transforms.

Theorem The sets of 'squared solutions' $e_{\nu,il}(x,\lambda)$ form complete sets of functions in $\mathcal{M}_J$. The completeness relation has the form:

$$\begin{aligned} &\delta(x-y)\Pi_0 = \\ &= \frac{1}{\pi} \sum_{\nu=0}^{2h-1} (-1)^\nu \int_{l_\nu} d\lambda (G_{\nu+1}(x,y,\lambda) - G_\nu(x,y,\lambda)) - 2i \sum_{j=1}^{N} \mathrm{Res}\,_{\lambda=\lambda_j} G_\nu(x,y,\lambda) \end{aligned}$$

$$
\begin{aligned}
\Pi_0 \; &= \sum_{\alpha>0} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha) \\
& G_{\nu+1}(x,y,\lambda) = \sum_{\alpha\in\Delta_\nu^+} e_{\nu+1,\alpha}(x,\lambda) \otimes e_{\nu+1,-\alpha}(y,\lambda), \\
& G_\nu(x,y,\lambda) = \sum_{\alpha\in\Delta_\nu^-} e_{\nu,-\alpha}(x,\lambda) \otimes e_{\nu,\alpha}(y,\lambda) + \sum_{s=1}^{2} h_{\nu,s}(x,\lambda) \otimes h_{\nu,s}(y,\lambda).
\end{aligned}
$$

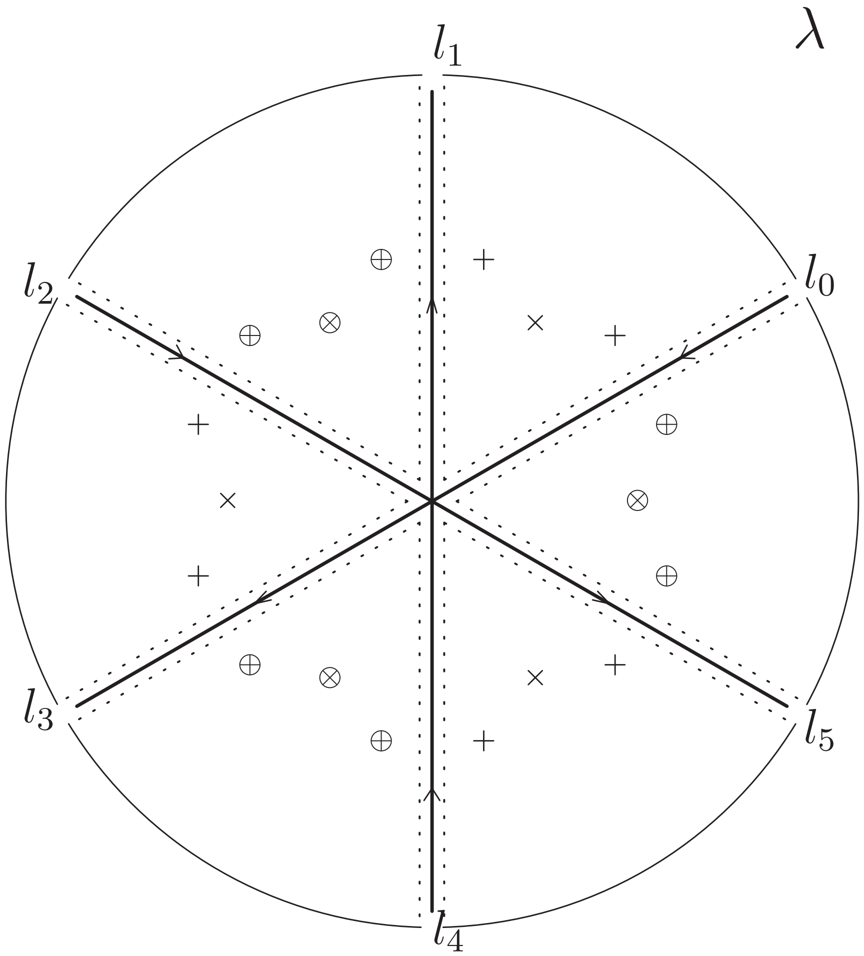

Figure 2: The contours $\gamma_\nu = l_\nu \cup \gamma_{\nu,\infty} \cup l_{\nu+1}$.

Expansions over the 'squared solutions':

$$Q(x,t) = \frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^\nu \alpha_\nu(J) \int_{l_\nu} d\lambda \, \left(s^+_{\alpha_\nu,\nu}(\lambda) e_{\nu+1;\alpha}(x,\lambda) - s^-_{\alpha_\nu,\nu}(\lambda) e_{\nu;-\alpha}(x,\lambda) \right) \\ + \sum_{\mathrm{DS}} \cdots \tag{8}$$

$$\mathrm{ad}\,_J^{-1} \delta Q(x,t) = \\ \frac{i}{2\pi} \sum_{\nu=0}^{5} (-1)^\nu \int_{l_\nu} d\lambda \, \left(\delta s^+_{\alpha_\nu,\nu}(\lambda) e_{\nu+1;\alpha}(x,\lambda) + \delta - s^-_{\alpha_\nu,\nu}(\lambda) e_{\nu;-\alpha}(x,\lambda) \right) + \sum_{\mathrm{DS}} \cdots \tag{9}$$

$e_{\alpha_\nu;\nu}(x,\lambda)$ are generalizations of $e^{-i\lambda x}$. We need the analogs of id/dx for which $i(d/dx)e^{-i\lambda x} = \lambda e^{-i\lambda x}$

$$(\Lambda_+ - \lambda) e_{\alpha_\nu;\nu}(x,\lambda) = 0, \qquad (\Lambda_- - \lambda) e_{-\alpha_\nu;\nu}(x,\lambda) = 0,$$

$$\Lambda_\pm X(x) \equiv \mathrm{ad}\,_J^{-1} \left(i \frac{dX}{dx} + i \left[[J, Q(x)], \int_{\pm\infty}^{x} dy \, [[J, Q(y)], X(y)] \right] \right).$$

In order to treat NLEE consider variations of the form:

$$\delta Q \simeq \frac{\partial Q}{\partial t}\delta t + \mathcal{O}((\delta t)^2), \qquad (10)$$

and keep only first order of δt. Then we have the expansion:

$$i\mathrm{ad}\,_J^{-1}\frac{\partial Q(x,t)}{\partial t} =$$

$$\frac{i}{2\pi}\sum_{\nu=0}^{5}(-1)^\nu\int_{l_\nu} d\lambda\, \left(i\frac{\partial s^+_{\alpha_\nu,\nu}}{\partial t}e_{\nu+1;\alpha}(x,\lambda) + i\frac{\partial s^-_{-\alpha_\nu,\nu}}{\partial t}e_{\nu;-\alpha}(x,\lambda)\right) + \sum_{\mathrm{DS}}\cdots$$

$$\mathbf{\Lambda}\mathrm{ad}\,_J^{-1}[J^2, Q(x,t)] =$$

$$= \frac{i}{2\pi}\sum_{\nu=0}^{5}(-1)^\nu\alpha_\nu(J)\int_{l_\nu} d\lambda\,\lambda^3\left(s^+_{\alpha_\nu,\nu}(\lambda)e_{\nu+1;\alpha}(x,\lambda) - s^-_{\alpha_\nu,\nu}(\lambda)e_{\nu;-\alpha}(x,\lambda)\right) + \cdots$$

$$
\begin{aligned}
& i\mathrm{ad}_J^{-1}\frac{\partial Q(x,t)}{\partial t} + \mathbf{\Lambda}\mathrm{ad}_J^{-1}[J^2, Q(x,t)] \equiv \mathrm{mKdV} = \\
& \frac{i}{2\pi}\sum_{\nu=0}^{5}(-1)^\nu \int_{l_\nu} d\lambda \; \left(\left(i\frac{\partial s^+_{\alpha_\nu,\nu}}{\partial t} + \lambda^3 s^+_{\alpha_\nu,\nu} \right) e_{\nu+1;\alpha}(x,\lambda) + \right. \\
& \left. + \left(i\frac{\partial s^-_{-\alpha_\nu,\nu}}{\partial t} - \lambda^3 s^-_{\alpha_\nu,\nu}(\lambda) \right) e_{\nu;-\alpha}(x,\lambda) \right) + \sum_{\mathrm{DS}} \cdots = 0
\end{aligned}
\tag{11}
$$

i.e. these mKdV equations are equivalent to the following **linear** equations

$$
\begin{aligned}
i\frac{\partial s^+_{\alpha_\nu,\nu}}{\partial t} + \lambda^3 s^+_{\alpha_\nu,\nu} = 0, \\
i\frac{\partial s^-_{-\alpha_\nu,\nu}}{\partial t} - \lambda^3 s^-_{\alpha_\nu,\nu} = 0.
\end{aligned}
\tag{12}
$$

These GFT linearize the NLEE of mKdV type!.

Solving the RHP and soliton solutions

The dressing Zakharov-Shabat method - (1974), Mikhailov - (1981)

Assume we have a regular solution of the RHP

$$\xi^0_{\nu+1}(x,t,\lambda) = \xi^0_\nu(x,t,\lambda) G^0_\nu(x,t,\lambda)$$

Regular: $\det m^0_\nu \neq 0$ for $\lambda \in \Omega_\nu$

Construct a new, singular solution of the RHP

$$\xi^1_\nu(x,t,\lambda) = u(x,t,\lambda)\xi^0_{\nu+1}(x,t,\lambda),$$

$u(x,t,\lambda)$ is the dressing factor, which may have poles and zeroes in λ. The regular solution corresponds to potential Q_0 of L; we may even choose $Q_0 = 0$.

The new singular solution of RHP corresponds to new potential Q which will depend on additional parameters.

One soliton solution of first type:

$$u(x,t,\lambda) = \mathbb{1} + \frac{1}{3}\left(\frac{A_1}{\lambda - \lambda_1} + \frac{J^{-1}A_1 J}{\lambda\omega^2 - \lambda_1} + \frac{J^{-2}A_1 J^2}{\lambda\omega - \lambda_1} \right) \qquad (13)$$

where $A_1(\xi,\eta)$ is a 3×3 degenerate matrix of the form

$$A_1(x,t) = |n(x,t)\rangle\langle m^T(x,t)| \qquad (A_1)_{ij}(x,t) = n_i(x,t)m_j(x,t). \qquad (14)$$

By construction $u(x,t,\lambda)$ satisfies the $\mathbb{Z}_3$-symmetry. The $\mathbb{Z}_2$-symmetry on $u(x,t,\lambda)$ can be put in the form

$$u(\xi,\eta,\lambda)A_0^{-1}u^\dagger(\xi,\eta,\lambda^*)A_0 = \mathbb{1}. \qquad (15)$$

and leads to algebraic equations which allow us to express the components of $n_j(x,t)$ in terms of $m_k(x,t)$:

$$\begin{aligned} n_1 &= \frac{2\lambda_1^3 m_3^*}{\lambda_1^2 m_3^* m_1 + |\lambda_1|^2 |m_2|^2 + \lambda_1^{2,*} m_1^* m_3} = \frac{2i\rho_1 m_3}{2m_1 m_3 - m_2^2} \\ n_2 &= \frac{2\lambda_1^3 m_2^*}{\lambda_1^{2,*} m_3^* m_1 + \lambda_1^2 |m_2|^2 + |\lambda_1|^2 m_1^* m_3} = \frac{2i\rho_1}{m_2} \\ n_3 &= \frac{2\lambda_1^3 m_1^*}{|\lambda_1|^2 m_3^* m_1 + \lambda_1^{2,*} |m_2|^2 + \lambda_1^2 m_1^* m_3} = \frac{2i\rho_1 m_1}{m_2^2}. \end{aligned} \qquad (16)$$

After putting $\lambda_1 = i\rho_1$ we obtain the 1-soliton solution of the first type for Tzitzeica eq.:

$$\phi_{1s}(x,t) = \frac{1}{2}\ln\left|\frac{|\mu_{01}|^2 e^{-3\mathcal{X}_1}\left(4\cos^2(\tilde{\Omega}_1) - 6\right) - 8|\mu_{01}|\mu_{02}\cos(\tilde{\Omega}_1) + \mu_{02}^2 e^{3\mathcal{X}_1}}{4|\mu_{01}|^2 e^{-3\mathcal{X}_1}\cos^2(\tilde{\Omega}_1) + 4|\mu_{01}|\mu_{02}\cos(\tilde{\Omega}_1) + \mu_{02}^2 e^{3\mathcal{X}_1}}\right| \tag{17}$$

where

$$\mathcal{X}_1 = \frac{1}{2}\left(\rho_1 x - \frac{t}{\rho_1}\right), \qquad \Omega_1 = \frac{\sqrt{3}}{2}\left(\rho_1 x + \frac{t}{\rho_1}\right). \tag{18}$$

Note: it is not traveling wave solution; it may have singularities! In the limit $\mu_{02} \to 0$ we obtain a traveling wave solution of the form

$$\phi(x,t) = \frac{1}{2}\ln\left[\frac{3}{2}\tanh^2\left(\frac{\sqrt{3}}{2}(\rho_1\xi + \rho_1^{-1}\eta) - \alpha_{01}\right) + \frac{1}{2}\right]. \tag{19}$$

One Soliton Solutions of Second Type

Now the anzatz for the dressing factor is

$$\begin{aligned} u(x,t,\lambda) = \mathbb{1} &+ \frac{1}{3}\left(\frac{A_1}{\lambda - \lambda_1} + \frac{J^{-1}A_1 J}{\lambda\omega^2 - \lambda_1} + \frac{J^{-2}A_1 J^2}{\lambda\omega - \lambda_1} \right) \\ &- \frac{1}{3}\left(\frac{A_1^*}{\lambda + \lambda_1^*} + \frac{J^{-1}A_1^* J}{\lambda\omega^2 + \lambda_1^*} + \frac{J^{-2}A_1^* J^2}{\lambda\omega + \lambda_1^*} \right) \end{aligned} \tag{20}$$

which obviously satisfies the $\mathbb{Z}_3$-reduction and the first $\mathbb{Z}_2$-reduction.

Again we obtain an algebraic relations between $n_j(x,t)$ in terms of $m_k(x,t)$ which are more complicated:

$$|\mu\rangle = \begin{pmatrix} m_3 \\ m_2 \\ m_1 \\ \hline m_3^* \\ m_2^* \\ m_1^* \end{pmatrix}, \qquad |\nu\rangle = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \hline n_1^* \\ n_2^* \\ n_3^* \end{pmatrix}, \qquad |\mu\rangle = \mathcal{M}|\nu\rangle \tag{21}$$

where

$$\mathcal{M} = \left(\begin{array}{ccc|ccc} c_1 P_1 & 0 & 0 & \zeta_1 K_1 & 0 & 0 \\ 0 & c_1 P_2 & 0 & 0 & \zeta_1 K_2 & 0 \\ 0 & 0 & c_1 P_3 & 0 & 0 & \zeta_1 K_3 \\ \hline \zeta_1 K_1^* & 0 & 0 & c_1 P_1^* & 0 & 0 \\ 0 & \zeta_1 K_2^* & 0 & 0 & c_1 P_2^* & 0 \\ 0 & 0 & \zeta_1 K_3^* & 0 & 0 & c_1 P_3^* \end{array}\right) \tag{22}$$

The result is

$$|\nu\rangle = \mathcal{M}^{-1} |\nu\rangle$$

$$\mathcal{M}^{-1} = \left(\begin{array}{ccc|ccc} -c_1^* \tilde{P}_1^* & 0 & 0 & \zeta_1 \tilde{K}_1 & 0 & 0 \\ 0 & -c_1^* \tilde{P}_2^* & 0 & 0 & \zeta_1 \tilde{K}_2 & 0 \\ 0 & 0 & -c_1^* \tilde{P}_3^* & 0 & 0 & \zeta_1 \tilde{K}_3 \\ \hline \zeta_1^* \tilde{K}_1^* & 0 & 0 & -c_1 \tilde{P}_1 & 0 & 0 \\ 0 & \zeta_1^* \tilde{K}_2^* & 0 & 0 & -c_1 \tilde{P}_2 & 0 \\ 0 & 0 & \zeta_1^* \tilde{K}_3^* & 0 & 0 & -c_1 \tilde{P}_3 \end{array}\right) \tag{23}$$

where

$$\tilde{P}_s^* = \frac{P_s^*}{d_s}, \qquad \tilde{P}_s = \frac{P_s}{d_s}, \qquad \tilde{K}_s = \frac{K_s}{d_s}, \qquad \tilde{K}_s^* = \frac{K_s^*}{d_1}$$

$$\begin{gathered} d_1 = \zeta_1\zeta_1^* K_1 K_1^* - c_1 c_1^* P_1 P_1^* \qquad d_2 = \zeta_1\zeta_1^* K_2 K_2^* - c_1 c_1^* P_2 P_2^* \\ d_3 = \zeta_1\zeta_1^* K_3 K_3^* - c_1 c_1^* P_3 P_3^*. \end{gathered} \tag{24}$$

From the above equations we obtain $|n\rangle$ in terms of $\langle m^T|$

$$\begin{gathered} n_1 = \frac{1}{d_1}(-c_1^* P_1^* m_3 + \zeta_1 K_1 m_3^*) \qquad n_2 = \frac{1}{d_2}(-c_1^* P_2^* m_2 + \zeta_1 K_2 m_2^*) \\ n_3 = \frac{1}{d_3}(-c_1^* P_3^* m_1 + \zeta_1 K_3 m_1^*). \end{gathered} \tag{25}$$

The explicit x, t dependence of $m_j(x,t)$ is

$$\begin{aligned} m_1 &= \omega^2 \mu_{01} e^{i\mathcal{X}_1 - \mathcal{Y}_1} + \mu_{02} e^{i\mathcal{X}_2 - \mathcal{Y}_2} + \omega \mu_{03} e^{i\mathcal{X}_3 - \mathcal{Y}_3} \\ m_2 &= \mu_{01} e^{i\mathcal{X}_1 - \mathcal{Y}_1} + \mu_{02} e^{i\mathcal{X}_2 - \mathcal{Y}_2} + \mu_{03} e^{i\mathcal{X}_3 - \mathcal{Y}_3} \\ m_3 &= \omega \mu_{01} e^{i\mathcal{X}_1 - \mathcal{Y}_1} + \mu_{02} e^{i\mathcal{X}_2 - \mathcal{Y}_2} + \omega^2 \mu_{03} e^{i\mathcal{X}_3 - \mathcal{Y}_3} \end{aligned} \tag{26}$$

where

$$
\begin{aligned}
\mathcal{X}_1 &= -\left(x\rho_1 + \frac{t}{\rho_1}\right)\cos\left(\beta_1 - \frac{2\pi}{3}\right), & \mathcal{Y}_1 &= -\left(x\rho_1 - \frac{t}{\rho_1}\right)\sin\left(\beta_1 - \frac{2\pi}{3}\right)\\
\mathcal{X}_2 &= -\left(x\rho_1 + \frac{t}{\rho_1}\right)\cos\left(\beta_1\right), & \mathcal{Y}_2 &= -\left(x\rho_1 - \frac{t}{\rho_1}\right)\sin\left(\beta_1\right)\\
\mathcal{X}_3 &= -\left(x\rho_1 + \frac{t}{\rho_1}\right)\cos\left(\beta_1 + \frac{2\pi}{3}\right), & \mathcal{Y}_3 &= -\left(x\rho_1 - \frac{t}{\rho_1}\right)\sin\left(\beta_1 + \frac{2\pi}{3}\right).
\end{aligned}
\tag{27}
$$

We determine the 1-soliton solution for the second kind of solitons using exactly the same technique

$$
\Phi = -\frac{1}{2}\ln\left|1 - \frac{1}{\lambda_1}n_1 m_1 - \frac{1}{\lambda_1^*}n_1^* m_1^*\right|. \tag{28}
$$

Multisoliton solutions $N = N_1 + N_2$ with N_1 solitons of first type and N_2 solitons of second type can also be derived: They would correspond to $6N_1 + 12N_2$ singularities of the RHP.

Reconstructing the potential $Q(x,t)$ from $u(x,t,\lambda)$

After constructing the dressing factor we use the fact that it satisfies the equation:

$$i\frac{\partial u}{\partial x} + (Q(x,t) - \lambda J)u(x,t,\lambda) - u(x,t,\lambda)(Q_0(x,t) - \lambda J) = 0, \qquad (29)$$

Take the limit $\lambda \to \infty$ and use that

$$\lim_{\lambda\to\infty} u(x,t,\lambda) = \mathbb{1},$$

and choose also $Q_0(x,t) = 0$. Then

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda(J - u(x,t,\lambda)J\hat{u}(x,t,\lambda)) \qquad (30)$$

which allows you to express $Q(x,t)$ in terms of the residue $A_1(x,t) = |\vec{n}_1\rangle\langle\vec{m}_1|$.

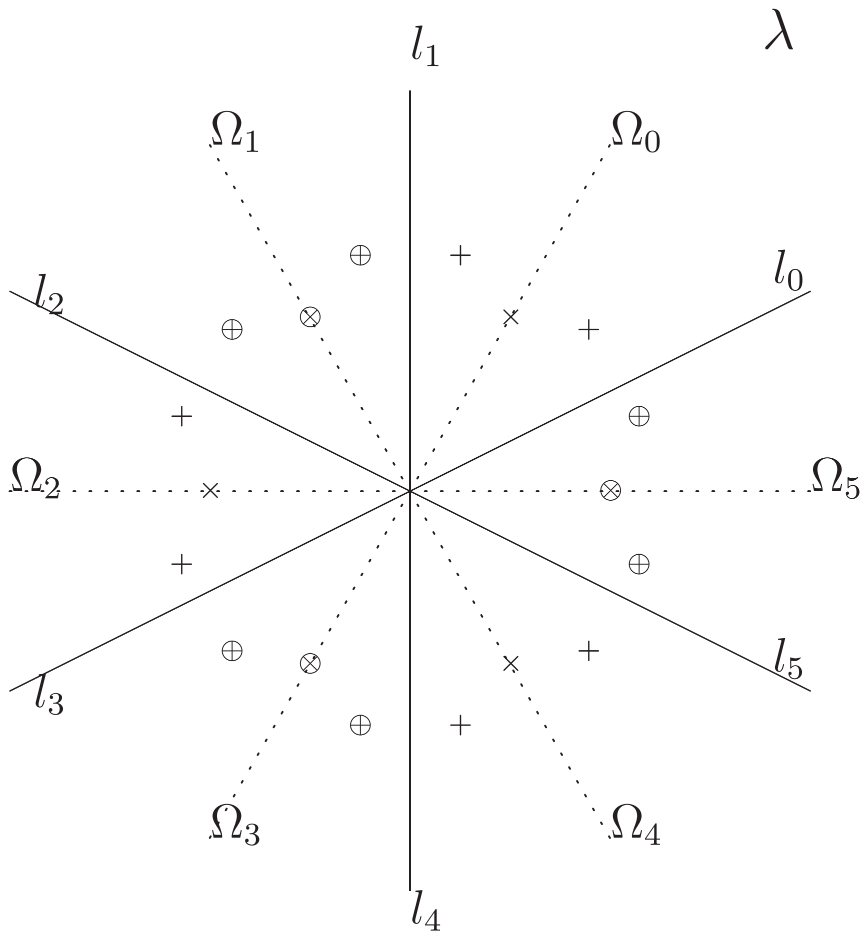

Figure 3: The discrete eigenvalues of L with $\mathbb{Z}_3$-symmetry and $\mathbb{Z}_2$-symmetries. Two types of discrete eigenvalues, two types of soliton solutions.

Conclusions and some open questions

- The mKdV eqs. are Hamiltonian. View the jets $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.
- Each of these eqs. has **two types** of soliton solutions. Find constraints on the soliton parameters that render them regular.
- One can derive their soliton interactions by evaluating the limits of the dressing factors for $x \to \pm\infty$.

Thank you for your attention!