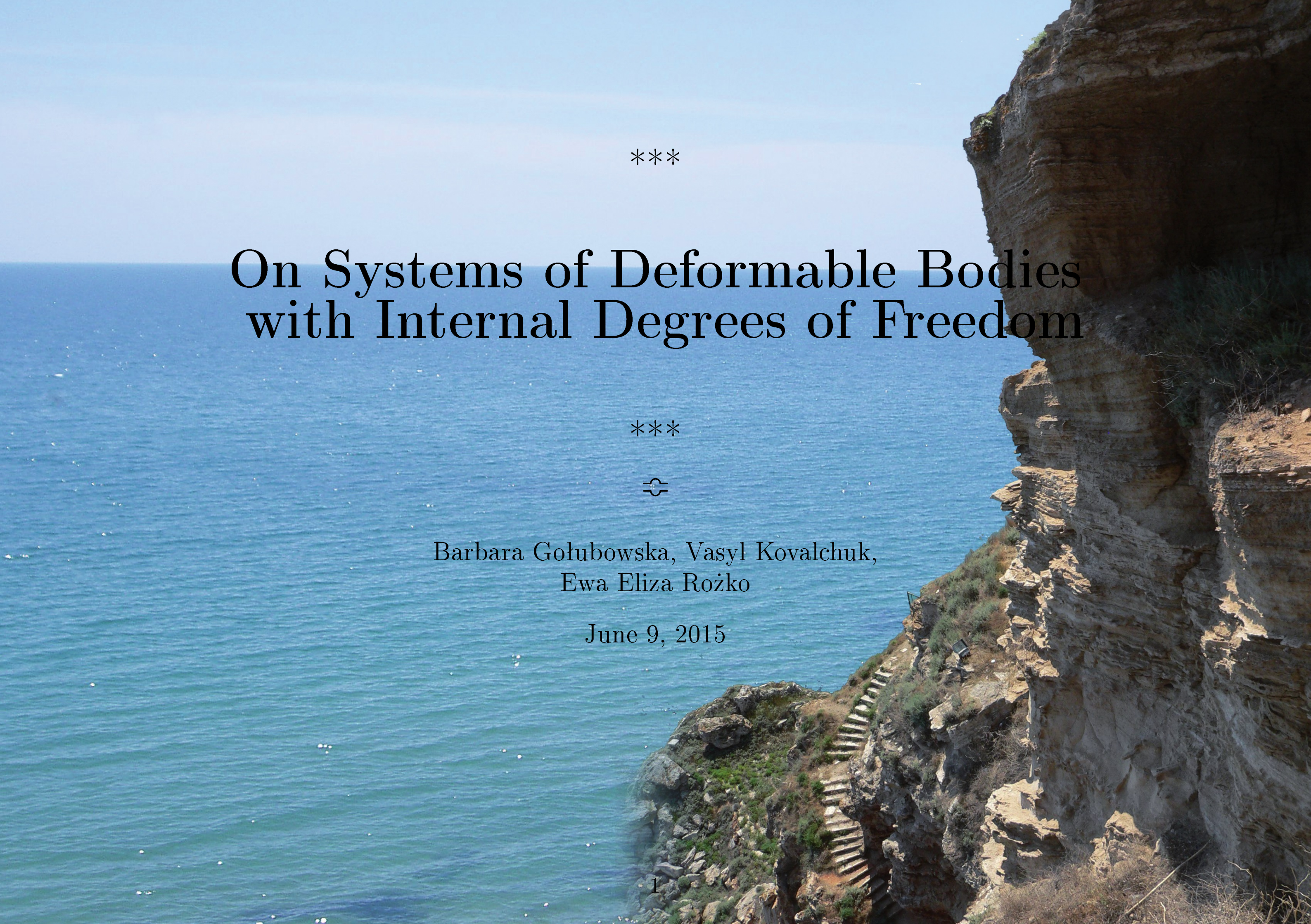

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On Systems of Deformable Bodies with Internal Degrees of Freedom

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1 Introduction

The special interest in the present work is devoted to the classical and quantum description of an affinely-rigid body.

Let $(M, V, \rightarrow)$ be an affine space and $(M, V, \rightarrow, g)$ be the corresponding Euclidean one, where M is a physical space in which the classical system of material points (discrete or continuous) is placed, V is a linear space of translations (free vectors) in M, and $g \in V^* \otimes V^*$ is the metric tensor.

Also let us introduce an affine $(N, U, \rightarrow)$ and the corresponding Euclidean $(N, U, \rightarrow, \eta)$ spaces, where N is the material space of labels which are assigned to every material point of our body in some way, U is the corresponding linear space of translations in N, and $\eta \in U^* \otimes U^*$ is the metric tensor.

Then the affine mapping from the material space into the physical one is as follows:

$$x^i(t, a) = r^i(t) + \varphi^i{}_A(t) a^A,$$

where $\varphi(t)$ is a linear part of the affine mapping (φ is non-singular for any time instant t), i.e., $\varphi(t) \in \mathrm{LI}(U, V)$, where $\mathrm{LI}(U, V)$ is a manifold of linear isomorphisms from the linear space U into the linear space V, $r(t)$ is the radius-vector of the centre of mass of our body if in the material space the position of the centre of mass is $a^A = 0$.

If the system is continuous, then the label a becomes the Lagrangian radius-vector (material variables) and x becomes the Eulerian radius-vector (physical variables).

At any fixed $t \in \mathbb{R}$ the configuration space Q of our problem is given by the following expression:

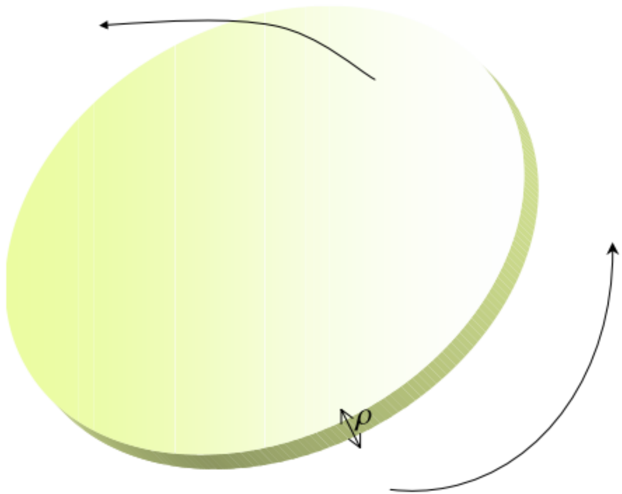
$$Q = \mathrm{AfI}(N, M) = Q_{\mathrm{tr}} \times Q_{\mathrm{int}} = M \times \mathrm{LI}(U, V),$$

where "tr" and "int" refer to the translational (spatial translations) and internal (rotations and homogeneous deformations) motions respectively.

The considered system is called an affinely-rigid body, i.e., during any admissible motion all affine relations between constituents of the body are invariant (the material straight lines remain straight lines, their parallelism is conserved, and all mutual ratios of segments placed on the same straight lines are constant). The concept of the affinely-rigid body is a generalization of the usual metrically-rigid body, in which during any admissible motion all distances (metric relations) between constituents of the body are constant.

We concentrate mainly on the case of such an affinely-rigid body that is subject to the additional constraints, i.e., it can deform homogeneously in the two-dimensional central plane of the body and simultaneously performs one-dimensional oscillations orthogonal to this central plane.

Then the material space N is presented as the Cartesian product $\mathbb{R}^{+} \times \mathbb{R}^{2}$ and the group of material transformations has the form $\mathbb{R}^{+} \times \mathrm{GL}\left(2, \mathbb{R}\right)$, where $\mathbb{R}^{+}$ is the dilatation group in the third dimension and the material transformations in $\mathbb{R}^{2}$ act as in the case of the usual affinely-rigid body with degenerate dimension.



We can identify configurations $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ with the pairs (ϱ, φ), where φ describes the immersion of the central plane in the physical space, i.e., analytically ${\varphi^i}_A$ is the 3×2 matrix. An element (k, B) acts on (ϱ, φ) as follows:

$$(k, B) \in \mathbb{R}^+ \times \mathrm{GL}(2, \mathbb{R}) : \quad (\varrho, \varphi) \mapsto (k\varrho, \varphi B).$$

The conservation of orthogonality of the direction of dilatations to the central plane means that the matrix

$$\Phi = \begin{bmatrix} \Phi^1{}_1 & \Phi^1{}_2 & \Phi^1{}_3 \\ \Phi^2{}_1 & \Phi^2{}_2 & \Phi^2{}_3 \\ \Phi^3{}_1 & \Phi^3{}_2 & \Phi^3{}_3 \end{bmatrix}$$

fulfils the condition that third column has to be proportional to the vector product of first and second ones. If we consider

$$\Phi^a{}_1,\ \Phi^b{}_2,\ a, b = 1, 2, 3,$$

as independent and arbitrary, then

$$\Phi^a{}_3 = \ell\ \varepsilon^a{}_{bc}\Phi^b{}_1\Phi^c{}_2,$$

where ε_{abc} is the completely antisymmetrical Levi-Civita (permutation) symbol,

ℓ is the parameter which depends both on the variable describing one-dimensional oscillations orthogonal to the central plane of the body and on the ones describing the state of deformation in this central plane, e.g., for the two-polar (singular value) decompositions we have

$$\ell_{\text{two}-\text{polar}} = \frac{\varrho}{\lambda\mu}$$

and for the polar decompositions we have respectively that

$$\ell_{\text{polar}} = \frac{\varrho}{\xi\zeta - \alpha^2},$$

where the meaning of variables λ, μ, α, ξ, ζ, ϱ will be described later.

The above-described orthogonality is well known in the theory of plates and shells as the Kirchhoff–Love condition.

2 Two-polar decomposition

In the language of the two-polar (singular value) decomposition the configurations are:

$$\Phi\left(\overline{k}; \lambda, \mu, \varrho; \theta\right) = R\left(\overline{k}\right) D\left(\lambda, \mu, \varrho\right) U\left(\theta\right)^{-1}, \qquad \lambda, \mu, \varrho > 0,$$

where $R, U \in \mathrm{SO}(3, \mathbb{R})$ are proper orthogonal matrices (whereas $\overline{k}$ is a rotation vector, i.e., a non-normalized vector codirectional with the rotation axis whose magnitude is equal to the rotation angle) and D is diagonal, i.e.,

$$D(\lambda, \mu, \varrho) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \varrho \end{bmatrix}, \qquad U(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the co-moving angular velocities for R- and U-tops are as follows:

$$\omega = R^{-1}\dot{R} = R^{T}\dot{R} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \qquad \omega^{T} = -\omega,$$

and

$$\vartheta = U^{-1}\dot{U} = U^{T}\dot{U} = \dot{\theta} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \vartheta^{T} = -\vartheta.$$

For $\dot{\Phi}$ and $\dot{\Phi}^{T}$ we have the following expressions:

$$\dot{\Phi} = R\big(\dot{D} + \omega D - D\vartheta\big)U^{-1}, \qquad \dot{\Phi}^{T} = U\big(\dot{D} + \vartheta D - D\omega\big)R^{T}.$$

The kinetic energy is assumed to have the usual form (we have only to substitute the constraints):

$$T = \frac{1}{2}\mathrm{Tr}\big(J\dot{\Phi}^{T}\dot{\Phi}\big) = \frac{1}{2}\mathrm{Tr}\Big(U^{-1}JU\big[\dot{D} + \vartheta D - D\omega\big]\big[\dot{D} + \omega D - D\vartheta\big]\Big),$$

where $J \in U \otimes U$ is the twice contravariant, symmetric, non-singular, positively-definite tensor describing the inertial properties of our affinely-rigid body.

If we take the tensor of inertia in the diagonal form, i.e., $J = \mathrm{Diag}\,(J_1, J_2, J_3)$, then the above kinetic energy can be rewritten as follows:

$$
\begin{aligned}
T = {} & \frac{J_1 \cos^2\theta + J_2 \sin^2\theta}{2} \left(\frac{d\lambda}{dt}\right)^2 + \frac{J_1 \sin^2\theta + J_2 \cos^2\theta}{2} \left(\frac{d\mu}{dt}\right)^2 \\
& + \frac{J_3}{2} \left(\frac{d\varrho}{dt}\right)^2 + \frac{\left(J_1 \sin^2\theta + J_2 \cos^2\theta\right)\mu^2 + J_3 \varrho^2}{2}\, \omega_1^2 \\
& + \frac{\left(J_1 \cos^2\theta + J_2 \sin^2\theta\right)\lambda^2 + J_3 \varrho^2}{2}\, \omega_2^2 + (J_1 + J_2)\,\lambda\mu\omega_3 \frac{d\theta}{dt} \\
& + (J_1 - J_2)\sin 2\theta \left[\left(\mu \frac{d\mu}{dt} - \lambda \frac{d\lambda}{dt}\right)\frac{d\theta}{dt} + \left(\lambda \frac{d\mu}{dt} - \mu \frac{d\lambda}{dt}\right)\omega_3 + \lambda\mu\omega_1\omega_2\right] \\
& + \frac{\left(J_1 \cos^2\theta + J_2 \sin^2\theta\right)\lambda^2 + \left(J_1 \sin^2\theta + J_2 \cos^2\theta\right)\mu^2}{2}\, \omega_3^2 \\
& + \frac{\left(J_1 \sin^2\theta + J_2 \cos^2\theta\right)\lambda^2 + \left(J_1 \cos^2\theta + J_2 \sin^2\theta\right)\mu^2}{2} \left(\frac{d\theta}{dt}\right)^2 .
\end{aligned}
$$

The above expressions significantly simplify when we consider the isotropic case in the central plane of the body, i.e., when we have $J_1 = J_2 = J$.

Then

$$T = \frac{J}{2}\left[\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{d\mu}{dt}\right)^2\right] + \frac{J_3}{2}\left(\frac{d\varrho}{dt}\right)^2 + \frac{J\mu^2 + J_3\varrho^2}{2}\omega_1^2$$
$$+ \frac{J\lambda^2 + J_3\varrho^2}{2}\omega_2^2 + 2J\lambda\mu\omega_3\frac{d\theta}{dt} + \frac{J}{2}\left(\lambda^2 + \mu^2\right)\left[\omega_3^2 + \left(\frac{d\theta}{dt}\right)^2\right].$$

We also remind here that the corresponding expression for the kinetic energy in the canonical variables has the following form:

$$\mathcal{T} = \frac{s_1^2}{2\left(J\mu^2 + J_3\varrho^2\right)} + \frac{s_2^2}{2\left(J\lambda^2 + J_3\varrho^2\right)}$$
$$+ \frac{\left(\lambda^2 + \mu^2\right)\left(s_3^2 + p_\theta^2\right) - 4\lambda\mu p_\theta s_3}{2J\left(\lambda^2 - \mu^2\right)^2} + \frac{p_\lambda^2 + p_\mu^2}{2J} + \frac{p_\varrho^2}{2J_3}.$$

Introducing some modelled potentials we obtained the Hamiltonian (total energy) and corresponding equations of motion for the isotropic case with the help of the Poisson brackets.

3 Polar decomposition

Later on we concentrate on the polar decomposition. The main feature of this decomposition is the more physically intuitive division on three main terms in the kinetic energy expression and the possibility to obtain the equations of motion in the quite simple form even for the general case, when the inertial tensor is not isotropic in the central plane ($J_1 \neq J_2$).

In the language of the polar decomposition we have

$$\Phi\left(\overline{\kappa}; \alpha, \xi, \zeta, \varrho\right) = L\left(\overline{\kappa}\right) S\left(\alpha, \xi, \zeta, \varrho\right),$$

where $L \in \mathrm{SO}(3, \mathbb{R})$ is a proper orthogonal matrix and $S \in \mathrm{Sym}(3, \mathbb{R})$ is symmetrical.

The connection between the polar and two-polar decompositions is given by:

$$L = RU^{-1},$$

$$\nu = L^{-1}\dot{L} = -\nu^T = \begin{bmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{bmatrix} = U\left(\omega - \vartheta\right)U^{-1}$$

$$= \begin{bmatrix} 0 & \omega_3 + \dot{\theta} & -\omega_1 \sin\theta - \omega_2 \cos\theta \\ -\omega_3 - \dot{\theta} & 0 & \omega_1 \cos\theta - \omega_2 \sin\theta \\ \omega_1 \sin\theta + \omega_2 \cos\theta & \omega_2 \sin\theta - \omega_1 \cos\theta & 0 \end{bmatrix},$$

$$
\begin{aligned}
S &= \begin{bmatrix} \xi & \alpha & 0 \\ \alpha & \zeta & 0 \\ 0 & 0 & \varrho \end{bmatrix} = UDU^{-1} \\
&= \begin{bmatrix} \lambda\cos^2\theta + \mu\sin^2\theta & (\lambda-\mu)\sin\theta\cos\theta & 0 \\ (\lambda-\mu)\sin\theta\cos\theta & \lambda\sin^2\theta + \mu\cos^2\theta & 0 \\ 0 & 0 & \varrho \end{bmatrix}.
\end{aligned}
$$

Let us consider the Lagrangian $L = T - V(\Phi)$ and Hamiltonian $H = \mathcal{T} + V(\Phi)$, where the kinetic energy in the polar decomposition is as follows:

$$
T = T_{\mathrm{rot}} + T_{\mathrm{rot-def}} + T_{\mathrm{def}},
$$

where

$$
\begin{aligned}
T_{\mathrm{rot}} = {} & \frac{J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2}{2}\nu_1^2 + \frac{J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2}{2}\nu_2^2 \\
& + \frac{J_1\xi^2 + J_2\zeta^2 + (J_1 + J_2)\,\alpha^2}{2}\nu_3^2 - (J_1\xi + J_2\zeta)\,\alpha\nu_1\nu_2
\end{aligned}
$$

describes coupling between the angular velocity ν of the L-top and deformation matrix S,

$$T_{\mathrm{rot-def}} = \left(J_1 \alpha \frac{d\xi}{dt} - J_2 \alpha \frac{d\zeta}{dt} - \left(J_1 \xi - J_2 \zeta \right) \frac{d\alpha}{dt} \right) \nu_3$$

describes the connection between the angular and deformation velocities,

and finally

$$T_{\mathrm{def}} = \frac{J_1 + J_2}{2} \left(\frac{d\alpha}{dt} \right)^2 + \frac{J_1}{2} \left(\frac{d\xi}{dt} \right)^2 + \frac{J_2}{2} \left(\frac{d\zeta}{dt} \right)^2 + \frac{J_3}{2} \left(\frac{d\varrho}{dt} \right)^2$$

describes the kinetic energy of the deformation oscillations.

The potential term $V(\Phi)$ depends on Φ only through the Green deformation tensor $G = S^2$, i.e., the potential term adapted to the polar decomposition is a function only of α, ξ, ζ, and ϱ.

Performing the Legendre transformation we obtain that

$$\pi_1 = \frac{\partial T}{\partial \nu_1} = \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\nu_1 - \left(J_1\xi + J_2\zeta\right)\alpha\nu_2,$$

$$\pi = \frac{\partial T}{\partial \nu_2} = \left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\nu_2 - \left(J_1\xi + J_2\zeta\right)\alpha\nu_1,$$

$$\pi = \frac{\partial T}{\partial \nu_3} = \left(J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right)\nu_3 + J_1\alpha\dot{\xi} - J_2\alpha\dot{\zeta} - \left(J_1\xi - J_2\zeta\right)\dot{\alpha},$$

$$p_\alpha = \frac{\partial T}{\partial \dot{\alpha}} = \left(J_1 + J_2\right)\dot{\alpha} - \left(J_1\xi - J_2\zeta\right)\nu_3,$$

$$p = \frac{\partial T}{\partial \dot{\xi}} = J_1\left(\dot{\xi} + \alpha\nu_3\right),$$

$$p_\zeta = \frac{\partial T}{\partial \dot{\zeta}} = J_2\left(\dot{\zeta} - \alpha\nu_3\right),$$

$$p_\varrho = \frac{\partial T}{\partial \dot{\varrho}} = J_3\dot{\varrho},$$

where π_i are canonical "spin" variables conjugate to angular velocities ν_i.

Therefore after inverting the above dependencies, i.e.,

$$\nu_1 = \frac{\left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\pi_1 + \left(J_1\xi + J_2\zeta\right)\alpha\pi_2}{J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4},$$

$$\nu_2 = \frac{\left(J_1\xi + J_2\zeta\right)\alpha\pi_1 + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\pi_2}{J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4},$$

$$\nu_3 = \frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},$$

$$\frac{d\alpha}{dt} = \frac{\left(J_1\xi - J_2\zeta\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi^2 + J_2\zeta^2\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},$$

$$\frac{d\xi}{dt} = \frac{p_\xi}{J_1} - \alpha\frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},$$

$$\frac{d\zeta}{dt} = \frac{p_\zeta}{J_2} + \alpha\frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},$$

$$\frac{d\varrho}{dt} = \frac{p_\varrho}{J_3},$$

we obtain the kinetic energy in the canonical variables as follows:

$$\mathcal{T} = \frac{\left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\pi_1^2 + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\pi_2^2}{2\left(J_1 J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4\right)}$$
$$+ \frac{\left(J_1\xi + J_2\zeta\right)\alpha\pi_1\pi_2}{J_1 J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4}$$
$$+ \frac{J_1 + J_2}{2J_1 J_2\left(\xi + \zeta\right)^2}\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]^2 + \frac{J_1\xi^2 + J_2\zeta^2}{2J_1 J_2\left(\xi + \zeta\right)^2}p_\alpha^2$$
$$+ \frac{J_1\xi - J_2\zeta}{J_1 J_2\left(\xi + \zeta\right)^2}\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]p_\alpha + \frac{p_\xi^2}{2J_1} + \frac{p_\zeta^2}{2J_2} + \frac{p_\varrho^2}{2J_3}.$$

We can see that the generalized velocities $\dot{\alpha}$, $\dot{\xi}$, $\dot{\zeta}$ corresponding to α, ξ, ζ and other variables describing the motion in the central plane of the body are separated from the generalized velocity $\dot{\varrho}$ describing the one-dimensional oscillations orthogonal to this central plane.

The same can be said also about the above expression in the canonical variables, i.e., the momentum p_ϱ conjugated to ϱ is orthogonal (in the sense of metrics encoded in the kinetic energy expression) to the other canonical momenta.

Hence, the most simple are those dynamical models in which also the isotropic potential will have the separated form:

$$V\left(\alpha, \xi, \zeta, \varrho\right) = V_{\text{plane}}\left(\alpha, \xi, \zeta\right) + V_\varrho\left(\varrho\right),$$

where the potential V_ϱ can describe the nonlinear oscillations

$$V_\varrho(\varrho) = \frac{a}{\varrho} + \frac{b}{2}\varrho^2, \qquad a, b > 0,$$

where the first term prevents from the unlimited compressing of the body, whereas the second one restricts the motion for large values of ϱ, i.e., prevents from the non-physical unlimited stretching of the body.

So, the Hamiltonian (total energy) can be written as follows:

$$H = \mathcal{T} + V_{\text{plane}}\left(\alpha, \xi, \zeta\right) + V_\varrho(\varrho).$$

Then the equations of motion can be calculated with the help of the following Poisson brackets:

$$\frac{d\pi_i}{dt} = \left\{\pi_i, H\right\}, \qquad \frac{dp_\alpha}{dt} = \left\{p_\alpha, H\right\}, \qquad \frac{dp_\xi}{dt} = \left\{p_\xi, H\right\},$$
$$\frac{dp_\zeta}{dt} = \left\{p_\zeta, H\right\}, \qquad \frac{dp_\varrho}{dt} = \left\{p_\varrho, H\right\}.$$

The only non-zero basic Poisson brackets are

$$\left\{\alpha, p_\alpha\right\} = \left\{\xi, p_\xi\right\} = \left\{\zeta, p_\zeta\right\} = \left\{\varrho, p_\varrho\right\} = 1, \qquad \left\{\pi_i, \pi_j\right\} = -\varepsilon_{ij}{}^{k}\pi_k,$$

and they are based on the structure constants of the special orthogonal group $\mathrm{SO}\left(3, \mathbb{R}\right)$.

The kinetic energy can be written in a more symbolic way:

$$\mathcal{T} = \frac{\Omega\left(\pi_1, \pi_2\right)}{2\Xi} + \frac{\Upsilon\left(\pi_3 + \alpha\left(p_\zeta - p_\xi\right), p_\alpha\right)}{2J_1 J_2 \left(\xi + \zeta\right)^2} + \frac{p_\xi^2}{2J_1} + \frac{p_\zeta^2}{2J_2} + \frac{p_\varrho^2}{2J_3},$$

where

$$\Xi = J_1 J_2 \left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1 \xi^2 + J_2 \zeta^2 + \left(J_1 + J_2\right)\alpha^2\right] J_3 \varrho^2 + J_3^2 \varrho^4,$$

and two expressions built of the canonical momenta are as follows:

$$\begin{aligned}\Omega\left(\pi_1, \pi_2\right) = {} & \left(J_1 \xi^2 + J_2 \alpha^2 + J_3 \rho^2\right)\pi_1^2 + 2\left(J_1 \xi + J_2 \zeta\right)\alpha \pi_1 \pi_2 \\ & + \left(J_1 \alpha^2 + J_2 \zeta^2 + J_3 \rho^2\right)\pi_2^2, \\ \Upsilon\left(\pi_3 + \alpha\left(p_\zeta - p_\xi\right), p_\alpha\right) = {} & \left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]^2 + \left(J_1 \xi^2 + J_2 \zeta^2\right) p_\alpha^2 \\ & + 2\left(J_1 \xi - J_2 \zeta\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] p_\alpha.\end{aligned}$$

Then we obtain the following equations of motion:

$$\frac{d\pi_1}{dt} = -\frac{\left[\left(J_1\xi + J_2\zeta\right)\alpha\pi_1 + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\pi_2\right]\pi_3}{\Xi} + \frac{\pi_2\left[\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha\right]}{J_1 J_2\left(\xi + \zeta\right)^2},$$

$$\frac{d\pi_2}{dt} = \frac{\left[\left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\pi_1 + \left(J_1\xi + J_2\zeta\right)\alpha\pi_2\right]\pi_3}{\Xi} - \frac{\pi_1\left[\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha\right]}{J_1 J_2\left(\xi + \zeta\right)^2},$$

$$\frac{d\pi_3}{dt} = \frac{\left(J_1\xi + J_2\zeta\right)\alpha\left(\pi_1^2 - \pi_2^2\right) + \left[J_1\left(\alpha^2 - \xi^2\right) + J_2\left(\zeta^2 - \alpha^2\right)\right]\pi_1\pi_2}{\Xi},$$

$$\frac{d\alpha}{dt} = -\frac{\partial V_{\text{plane}}}{\partial\alpha} - \frac{\left(J_2\pi_1^2 + J_1\pi_2^2\right)\alpha + \left(J_1\xi + J_2\zeta\right)\pi_1\pi_2}{\Xi} + \frac{2J_1J_2\alpha\left(\alpha^2 - \xi\zeta\right) + \left(J_1 + J_2\right)\alpha J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1, \pi_2\right) - \frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1 J_2\left(\xi + \zeta\right)^2}\left(p_\zeta - p_\xi\right),$$

$$\frac{d\xi}{dt} = -\frac{\partial V_{\text{plane}}}{\partial \xi} - \frac{J_1 \xi \pi_1^2 + J_1 \alpha \pi_1 \pi_2}{\Xi} + \frac{J_1 J_2 \zeta \left(\xi\zeta - \alpha^2\right) + J_1 \xi J_3 \varrho^2}{\Xi^2} \Omega\left(\pi_1, \pi_2\right)$$
$$- \frac{J_1 \xi p_\alpha^2 + J_1 \left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] p_\alpha}{J_1 J_2 \left(\xi + \zeta\right)^2} + \frac{\Upsilon\left(\pi_3 + \alpha\left(p_\zeta - p_\xi\right), p_\alpha\right)}{J_1 J_2 \left(\xi + \zeta\right)^3},$$
$$\frac{d\zeta}{dt} = -\frac{\partial V_{\text{plane}}}{\partial \zeta} - \frac{J_2 \zeta \pi_2^2 + J_2 \alpha \pi_1 \pi_2}{\Xi} + \frac{J_1 \xi J_2 \left(\xi\zeta - \alpha^2\right) + J_2 \zeta J_3 \varrho^2}{\Xi^2} \Omega\left(\pi_1, \pi_2\right)$$
$$- \frac{J_2 \zeta p_\alpha^2 - J_2 \left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] p_\alpha}{J_1 J_2 \left(\xi + \zeta\right)^2} + \frac{\Upsilon\left(\pi_3 + \alpha\left(p_\zeta - p_\xi\right), p_\alpha\right)}{J_1 J_2 \left(\xi + \zeta\right)^3},$$
$$\frac{d\varrho}{dt} = -\frac{dV_\varrho}{d\varrho} - \frac{J_3 \rho}{\Xi}\left(\pi_1^2 + \pi_2^2\right) +$$
$$+ \frac{J_3 \varrho}{\Xi^2}\left[J_1 \xi^2 + J_2 \zeta^2 + \left(J_1 + J_2\right)\alpha^2 + 2 J_3 \rho^2\right] \Omega\left(\pi_1, \pi_2\right).$$

The structure of the above expressions implies that even in the simplest case of the completely separated potential the dynamical coupling between the parameter describing one-dimensional oscillations orthogonal to the central plane of the body and the variables living in this central plane is present.

4 Stationary ellipsoids as special solutions

The above equations of motion are strongly nonlinear and in a general case there is hardly a hope to solve them analytically.

Nevertheless, there exists a way for imaging some features of the phase portrait of such a dynamical system, i.e., we have to find some special solutions, namely, the stationary ellipsoids, which are analogous to the ellipsoidal figures of equilibrium well known in astro- and geophysics, e.g., in the theory of the Earth's shape.

In the case of the two-polar (singular value) decomposition we obtain the special solutions just putting the deformation invariants λ, μ, ϱ and the angular velocities ω, ϑ equal to some constant values.

In the case of the polar decomposition, the Green deformation tensor G, therefore the deformation matrix S, and the angular velocity ν of the L-top have to be constant:

$$\frac{dG}{dt} = \frac{d}{dt}\left(\Phi^T \Phi\right) = \frac{d}{dt}\left(S^2\right) = 0, \qquad \frac{d\nu}{dt} = \frac{d}{dt}\left(L^{-1}\dot{L}\right) = 0.$$

This means that the L-top performs the stationary rotation, i.e., if at the initial time $t = 0$ we have that the configuration of the body is L_0, then at the time instant t the configuration will be as follows:

$$L_0 \circ e^{\nu t},$$

where $\circ$ is the function composition symbol.

The whole affinely-rigid body, which at the initial time $t = 0$ has the internal configuration $\Phi_0 = L_0 \circ S$, at the time instant t will be in the following configuration:

$$\Phi(t) = L_0 \circ e^{\nu t} \circ S = e^{\widehat{\nu} t} \circ L_0 \circ S = e^{\widehat{\nu} t} \circ \Phi_0,$$

where $\widehat{\nu} = L_0 \circ \nu \circ L_0^{-1}$.

While the affinely-rigid body rotates in the stationary way around the axis fixed in the physical and material spaces, the deformation and the angular velocity of rotation are not independent and related by some algebraic expressions.

5 Summary

It is interesting to note that the special solutions obtained for the polar decomposition case are conceptually different from those obtained for the two-polar one because for the polar decomposition the Green deformation tensor $G = S^2$ has a constant value (i.e., $\dot{G} = 2S\dot{S} = 0$) contrary to the two-polar case when the Green deformation tensor $G = \Phi^T \Phi = UD^2U^{-1}$, as well as the Cauchy one $C = \Phi^{-1T}\Phi^{-1} = RD^2R^{-1}$, depended on time explicitly through the time dependence of U and R respectively, i.e.,

$$\frac{dG}{dt} = U\left(\vartheta D^2 - D^2\vartheta\right)U^{-1} \neq 0, \qquad \frac{dC}{dt} = R\left(\omega D^2 - D^2\omega\right)R^{-1} \neq 0,$$

and performed the stationary rotations around their principal axes, whereas the deformation invariants λ, μ, ϱ had the constant values.

So, if we additionally keep in mind that for the two-polar decomposition we obtain the stationary solutions only for the isotropic model $J_1 = J_2 = J$ and for the polar one the general situation $J_1 \neq J_2$ is allowed, then we can compare the four studied cases according to the following scheme:

- The only degrees of freedom we can manipulate are the rotational degrees of R- and U-tops, because the deformation matrix D is constant for this type of stationary solutions.

- To achieve the constant behaviour of the Green deformation tensor $G = S^2 = \left(UDU^{-1}\right)^2 = UD^2U^{-1}$ we have to suppose that the U-top is fixed and does not rotate at all. If U is constant, then the principal axes of the R- and $L\left(= RU^{-1}\right)$-tops (for the two-polar and polar decompositions respectively) rotate in the same manner, i.e., at any moment ones can be obtained from others with the help of applying some constant orthogonal transformation. This situation corresponds to the three cases describing the stationary rotations of the L-top around its three principal axes.

- If U-top is not fixed, then the Green deformation is not constant and we have to consider three branches of the stationary motion for R- and U-tops when they rotate not independently but in the correlated manner, i.e., either both around their first principal axes or both around the second ones or both around the third ones. Nevertheless, for our affinely-rigid body subject to the Kirchhoff–Love constraints only the third case is possible.

Quantized version of the theory is based on the formulas:

$$\mathbf{H} = \mathbf{T} + V(\lambda, \mu),$$

where

$$\begin{aligned}
\mathbf{T} &= \frac{{\mathbf{S}_1}^2}{2J\mu^2} + \frac{{\mathbf{S}_2}^2}{2J\lambda^2} + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)}{\mathbf{S}_3}^2 + \frac{\lambda^2+\mu^2}{2J\left(\lambda^2-\mu^2\right)}{\mathbf{p}_\theta}^2 \\
&- \frac{2\lambda\mu}{J\left(\lambda^2-\mu^2\right)^2}\mathbf{p}_\theta\mathbf{S}_3 - \frac{\hbar^2}{2J}\frac{\partial^2}{\partial\lambda^2} - \frac{\hbar^2}{2J}\frac{\partial\ln\mathcal{P}}{\partial\lambda}\frac{\partial}{\partial\lambda} \\
&- \frac{\hbar^2}{2J}\frac{\partial^2}{\partial\mu^2} - \frac{\hbar^2}{2J}\frac{\partial\ln\mathcal{P}}{\partial\mu}\frac{\partial}{\partial\mu}.
\end{aligned}$$

$$\mathcal{P}(\lambda, \mu) = \lambda\mu\left(\lambda^2 - \mu^2\right),$$

$\mathbf{S}_a, \mathbf{p}_\theta$ are co-moving components of spin and vorticity (with respect to the bases R- and θ-co-moving).

Wave functions are substituted as series of matrix elements of irreducible representations:

$$\Psi(R;\lambda,\mu;\theta) = \sum_{j,m,m',k} f^{j,k}_{m',m}(\lambda,\mu)\ \mathfrak{D}^{j}_{m\ m'}(R)\ e^{ik\theta}.$$

Substituting this to the Schrödinger equation

$$\mathbf{H}\Psi = E\Psi$$

one obtains the system of equation for functions ${f^{ij}}_{mm'}$ defending only on two variables (λ, μ). This is the far-reaching reduction of variables from $(R, \theta, \lambda, \mu)$.

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Thank You for Your attention
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