Bour surface companions in non-Euclidean space forms

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Introduction

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- see Güler [22], Güler, Yaylı and Hacısalihoğlu [23], Güler and Yaylı [24].

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- see Güler [22], Güler, Yaylı and Hacısalihoğlu [23], Güler and Yaylı [24].
- also Özgür, Arslan and Murathan, [25].

Introduction

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- They are called Bour's minimal surfaces $\mathfrak{B}_m$ of value m.

Introduction

- Furthermore, when m is an integer greater than 1, $\mathfrak{B}_m$ become algebraic, that is, there is an implicit polynomial equation satisfied by the three coordinates of $\mathfrak{B}_m$, see also Gray [7], Nitsche [15], Whittemore [21].

Introduction

- Kobayashi [11] gave an analogous Weierstrass-type representation for conformal spacelike surfaces with mean curvature identically 0, called maximal surfaces, in 3-dimensional Minkowski space $\mathbb{R}^{2,1}$.

Introduction

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- However, unlike the case of minimal surfaces in $\mathbb{R}^3$, maximal surfaces generally have singularities.
- Details about singularities of maximal surfaces can be found in Fujimori et al [6], Umehara and Yamada [20].

Introduction

- We remark that Magid [14] gave a Weierstrass-type representation for timelike surfaces with mean curvature identically 0, called timelike minimal surfaces, in $\mathbb{R}^{2,1}$, see also Inoguchi and Lee [10].

Introduction

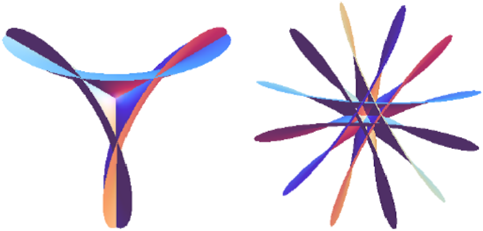

Figure 1. Bour's minimal surfaces of value 3 and 6 in $\mathbb{R}^3$.

Introduction

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- and Palmer [16] showed that there is an analogous correspondence between spacelike CMC surfaces in Lorentzian space forms.

Introduction

- In particular, minimal surfaces in $\mathbb{R}^3$ correspond to CMC 1 surfaces in 3-dimensional hyperbolic space $\mathbb{H}^3$, and maximal surfaces in $\mathbb{R}^{2,1}$ correspond to CMC 1 surfaces in 3-dimensional de Sitter space $\mathbb{S}^{2,1}$.

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- Thus it is natural to expect existence of corresponding Weierstrass-type representations in these cases. Bryant [3] gave such a representation formula for CMC 1 surfaces in $\mathbb{H}^3$, and Umehara, Yamada [18] applied it.

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- Thus it is natural to expect existence of corresponding Weierstrass-type representations in these cases. Bryant [3] gave such a representation formula for CMC 1 surfaces in $\mathbb{H}^3$, and Umehara, Yamada [18] applied it.
- Similarly, Aiyama, Akutagawa [1] gave a representation formula for CMC 1 surfaces in $\mathbb{S}^{2,1}$.

Introduction

- However, analogues of Bour's surfaces in other 3-dimensional space forms had not yet been studied.

Introduction

- In Sections 2 and 3 of this talk, in order to show that several maximal and timelike minimal Bour's surfaces of value m are algebraic, we review Weierstrass-type representations for maximal surfaces and timelike minimal surfaces in $\mathbb{R}^{2,1}$, and give explicit parametrizations for spacelike and timelike minimal Bour's surfaces of value m.

Introduction

- In Section 4, we introduce Bour type CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, and show several properties of those surfaces.

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- In Section 4, we introduce Bour type CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, and show several properties of those surfaces.
- Finally, in Section 5, we calculate the degrees, classes, implicit equations of the maximal and timelike minimal Bour's surfaces of values 2, 3, 4 in $\mathbb{R}^{2,1}$ in terms of their coordinates.

Introduction

- We remark that in the cases of $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, all surfaces are algebraic in some sense, because the Lorentz ($\mathbb{R}^{3,1}$) norm of all elements in $\mathbb{H}^3 \subset \mathbb{R}^{3,1}$ or $\mathbb{S}^{2,1} \subset \mathbb{R}^{3,1}$ is constant.

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- However, we have the following remaining problems:
- What is the class of maximal and timelike minimal Bour's surfaces of general value m in $\mathbb{R}^{2,1}$?
- Are there any other implicit equations for CMC 1 Bour type surfaces? If there exist implicit equations, what are the corresponding degrees and classes?

Spacelike maximal Bour type surfaces

Let

$$\mathbb{R}^{n,1} := \left(\left\{x = (x_1, \cdots, x_n, x_0)^t \middle| x_i \in \mathbb{R}\right\}, \langle\cdot,\cdot\rangle\right)$$

be the $(n+1)$-dimensional Lorentz-Minkowski (for short, Minkowski) space with Lorentz metric

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n - x_0 y_0.$$

Spacelike maximal Bour type surfaces

Then the 3-dimensional hyperbolic space $\mathbb{H}^3$ and 3-dimensional de Sitter space $\mathbb{S}^{2,1}$ are defined as follows:

$$\mathbb{H}^3 := \{x \in \mathbb{R}^{3,1} | \langle x, x \rangle = -1,\ x_0 > 0\} \cong \left\{F\bar{F}^t | F \in \mathrm{SL}_2\mathbb{C}\right\},$$
$$\mathbb{S}^{2,1} := \{x \in \mathbb{R}^{3,1} | \langle x, x \rangle = 1\} \cong \left\{F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t | F \in \mathrm{SL}_2\mathbb{C}\right\}.$$

Spacelike maximal Bour type surfaces

- A vector $x \in \mathbb{R}^{n,1}$ is called spacelike if $\langle x, x \rangle > 0$, timelike if $\langle x, x \rangle < 0$, and lightlike if $x \neq 0$ and $\langle x, x \rangle = 0$.

Spacelike maximal Bour type surfaces

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- A surface in $\mathbb{R}^{n,1}$ is called spacelike (resp. timelike, lightlike) if the induced metric on the tangent planes is a positive definite Riemannian (resp. Lorentzian, degenerate) metric.

Spacelike maximal Bour type surfaces

- Kobayashi [11] found a Weierstrass-type representation for spacelike conformal maximal surfaces in $\mathbb{R}^{2,1}$.

Spacelike maximal Bour type surfaces

Theorem (1)

Let g, ω be holomorphic functions defined on a simply connected open subset $\mathcal{U} \subset \mathbb{C}$ such that ω does not vanish on $\mathcal{U}$. Then

$$f(z) = Re \int \begin{pmatrix} \left(1+g^2\right)\omega \\ i\left(1-g^2\right)\omega \\ 2g\omega \end{pmatrix} dz$$

is a spacelike conformal immersion with mean curvature identically 0 *(i.e. spacelike conformal maximal surface). Conversely, any spacelike conformal maximal surface can be described in this manner.*

Spacelike maximal Bour type surfaces

Remark (1). A pair of a holomorphic function g and a holomorphic function ω, (g, ω) is called Weierstrass data for a maximal surface. In Section 4, we also call (g, ω) the Weierstrass data for CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$.

Spacelike maximal Bour type surfaces

- We call maximal surfaces $\mathfrak{B}_m$ $(m \in \mathbb{Z}_{\geq 2} := \{n \in \mathbb{Z} | n \geq 2\})$ given by $(g, \omega) = (z, z^{m-2})$ the spacelike Bour's maximal surfaces $\mathfrak{B}_m$ of value m (spacelike $\mathfrak{B}_m$, for short).

Spacelike maximal Bour type surfaces

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- Several properties of spacelike $\mathfrak{B}_m$ can be found in Güler [8].

Spacelike maximal Bour type surfaces

The parametrization of spacelike $\mathfrak{B}_m(u,v)$ is

$$\mathrm{Re}\begin{pmatrix} \frac{1}{m-1}\sum_{k=0}^{m-1}\binom{m-1}{k}u^{m-1-k}(iv)^k + \frac{1}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(iv)^k \\ \frac{i}{m-1}\sum_{k=0}^{m-1}\binom{m-1}{k}u^{m-1-k}(iv)^k - \frac{i}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(iv)^k \\ \frac{2}{m}\sum_{k=0}^{m}\binom{m}{k}u^{m-k}(iv)^k \end{pmatrix} \quad (1)$$

Spacelike maximal Bour type surfaces

with Gauss map

$$n = \left(\frac{2u}{u^2+v^2-1}, \frac{2v}{u^2+v^2-1}, \frac{u^2+v^2+1}{u^2+v^2-1} \right),$$

where $z = u + iv$.

Timelike minimal Bour type surfaces

Next, we give the Weierstrass-type representation for timelike minimal surfaces in $\mathbb{R}^{2,1}$, which was obtained by M. Magid [14] (see also Inoguchi and Lee [10]).

Timelike minimal Bour type surfaces

Theorem (2)

Let $g_1(u)$, $\omega_1(u)$ (resp. $g_2(v)$, $\omega_2(v)$) be smooth functions depending on only u (resp. v) on a connected orientable 2-manifold with local coordinates u, v. Then

$$\hat{f}(u,v) = \int \begin{pmatrix} 2g_1\omega_1 \\ \left(1-g_1^2\right)\omega_1 \\ -\left(1+g_1^2\right)\omega_1 \end{pmatrix} du + \int \begin{pmatrix} 2g_2\omega_2 \\ \left(1-g_2^2\right)\omega_2 \\ \left(1+g_2^2\right)\omega_2 \end{pmatrix} dv.$$

is a timelike surface with mean curvature identically 0 (i.e. timelike minimal surface). Conversely, any timelike minimal surface can be described in this manner.

Timelike minimal Bour type surfaces

The timelike minimal surfaces given by $(g_1(u), \omega_1(u)) = \left(u, u^{m-2}\right)$, $(g_2(v), \omega_2(v)) = \left(v, v^{m-2}\right)$ are called timelike Bour surfaces $\mathfrak{B}_m$ of value m (timelike $\mathfrak{B}_m$, for short) in $\mathbb{R}^{2,1}$, where $m \in \mathbb{Z}_{\geq 2}$.

Timelike minimal Bour type surfaces

The parametrization of timelike $\mathfrak{B}_m$ is

$$\mathfrak{B}_m(u,v) = \begin{pmatrix} \frac{2}{m}\left(u^m + v^m\right) \\ \frac{1}{m-1}\left(u^{m-1} + v^{m-1}\right) - \frac{1}{m+1}\left(u^{m+1} + v^{m+1}\right) \\ -\frac{1}{m-1}\left(u^{m-1} - v^{m-1}\right) - \frac{1}{m+1}\left(u^{m+1} - v^{m+1}\right) \end{pmatrix}, \tag{2}$$

Timelike minimal Bour type surfaces

with Gauss map

$$n = \left(\frac{uv - 1}{1 + uv}, \frac{u + v}{1 + uv}, \frac{u - v}{1 + uv} \right).$$

Timelike minimal Bour type surfaces

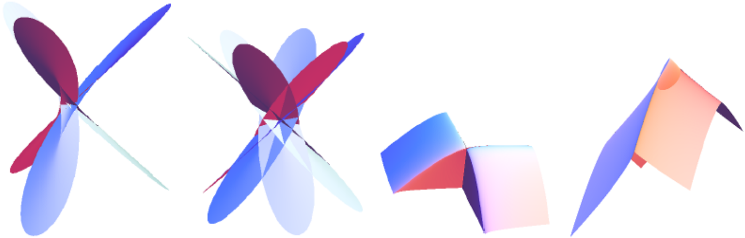

Figure 2. Left two pictures: spacelike $\mathfrak{B}_3$ and $\mathfrak{B}_6$ in $\mathbb{R}^{2,1}$, right two pictures: timelike $\mathfrak{B}_3$ and $\mathfrak{B}_6$ in $\mathbb{R}^{2,1}$

CMC 1 Bour type surfaces

- In this section we consider CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. Here we identify elements in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ with $SL_2\mathbb{C}$ matrix forms as in Section 2.

CMC 1 Bour type surfaces

- In this section we consider CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. Here we identify elements in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ with $SL_2\mathbb{C}$ matrix forms as in Section 2.
- In this setting Bryant [3] showed the following representation formula for CMC 1 surfaces in $\mathbb{H}^3$:

CMC 1 Bour type surfaces

Theorem (3)

Let $F \in \mathrm{SL}_2\mathbb{C}$ *be a solution of the equation*

$$dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega, \;\; F|_{z=z_0} \in \mathrm{SL}_2\mathbb{C} \tag{3}$$

for some z_0 *in a given domain, where* (g, ω) *is Weierstrass data. Then the surface* $f = F\bar{F}^t$ *is a conformal CMC 1 immersion into* $\mathbb{H}^3$*. Conversely, any conformal CMC 1 immersion in* $\mathbb{H}^3$ *can be described in this way. The metric of f is* $(1 + |g|^2)^2|\omega|^2$.

CMC 1 Bour type surfaces

Similarly, Aiyama and Akutagawa [1] showed the following Bryant-type representation formula for CMC 1 surfaces in $\mathbb{S}^{2,1}$:

CMC 1 Bour type surfaces

Theorem (4)

Let $\hat{F} \in \mathrm{SL}_2\mathbb{C}$ *be a solution of Equation* (3), *where* (g, ω) *is Weierstrass data. Then the surface* $f = F \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{F}^t$ *is a spacelike conformal CMC* 1 *immersion into* $\mathbb{S}^{2,1}$. *Conversely, any spacelike conformal CMC* 1 *immersion in* $\mathbb{S}^{2,1}$ *is described in this way. The metric of* f *is* $(1 - |g|^2)^2 |\omega|^2$.

CMC 1 Bour type surfaces

Note that, unlike in $\mathbb{H}^3$, CMC 1 surfaces in $\mathbb{S}^{2,1}$ generally have singularities. Their singularities have been investigated Fujimori et al [6], Umehara and Yamada [20].

CMC 1 Bour type surfaces

We call CMC 1 surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ given by the Weierstrass data $(g, \omega) = (z, z^{m-2})$ the Bour type CMC 1 cousins $\mathfrak{B}_m$ of value m ($\mathfrak{B}_m$ cousin, for short).

CMC 1 Bour type surfaces

We now describe F explicitly:

CMC 1 Bour type surfaces

Theorem (5)

Let $F(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathrm{SL}_2\mathbb{C}$ *be a solution of Equation (3) with* $(g, \omega) = (z, z^{m-2}dz)$ *and with initial condition* $F(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

CMC 1 Bour type surfaces

Theorem (5)

(Cont.) Then

$$\begin{aligned}
a(z) &= m^{\frac{1}{m}}\Gamma\left(\frac{m+1}{m}\right) z^{\frac{m-1}{2}}\,\mathrm{Bessel\,I}\left(-\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right), \\
b(z) &= -m^{\frac{1}{m}}\Gamma\left(\frac{m+1}{m}\right) z^{\frac{m+1}{2}}\,\mathrm{Bessel\,I}\left(\frac{m+1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right), \\
c(z) &= m^{\frac{-1}{m}}\Gamma\left(\frac{m-1}{m}\right) z^{\frac{m-1}{2}}\,\mathrm{Bessel\,I}\left(\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right), \\
d(z) &= -m^{\frac{-1}{m}}\Gamma\left(\frac{m-1}{m}\right) z^{\frac{m+1}{2}}\,\mathrm{Bessel\,I}\left(-\frac{m+1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right),
\end{aligned} \tag{4}$$

CMC 1 Bour type surfaces

Theorem (5)

(Cont.) where Γ denotes the Gamma function and $\mathrm{Bessel\,I}$ *represents the modified Bessel function.*

- The definition of $\mathrm{Bessel\,I}$ can be found in standard textbooks, for example, see [9].

CMC 1 Bour type surfaces

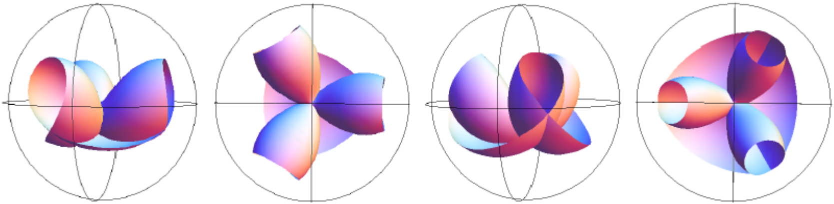

Figure 3. Left two pictures: $\mathfrak{B}_3$ cousin in $\mathbb{H}^3$, right two pictures: its dual cousin in $\mathbb{H}^3$ (in the Poincare ball model for $\mathbb{H}^3$)

CMC 1 Bour type surfaces

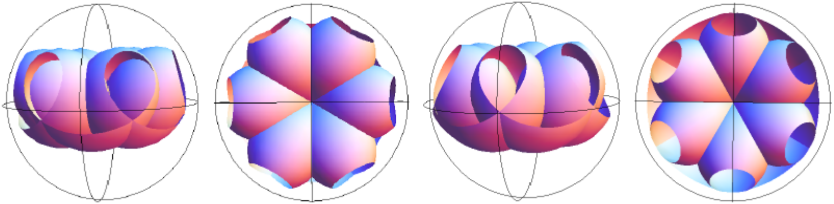

Figure 4. Left two pictures: $\mathfrak{B}_6$ cousin in $\mathbb{H}^3$, right two pictures: its dual cousin in $\mathbb{H}^3$

CMC 1 Bour type surfaces

Proof.
Equation (3) gives

$$X'' - \frac{\omega'}{\omega}X' - g'\omega X = 0, \quad (X = a(z),\ c(z)) \qquad (5)$$

$$Y'' - \frac{(g^2\omega)'}{g^2\omega}Y' - g'\omega Y = 0 \quad (Y = b(z),\ d(z)), \qquad (6)$$

which are given by Umehara and K. Yamada [18]. Here we solve Equation (5).

CMC 1 Bour type surfaces

Proof. (cont.)
Inserting $(g, \omega) = (z, z^{m-2})$ into Equation (5), we have

$$X'' - \frac{m-2}{z} X' - z^{m-2} X = 0. \quad (m \in \mathbb{Z}_{\geq 2}) \tag{7}$$

CMC 1 Bour type surfaces

Proof. (cont.)
We give two independent power series solutions of the differential equation (7) by the Frobenius method. The indicial equation at $z = 0$ is $\rho(\rho - 1) - (m - 2)\rho = 0$. So we see that the characteristic exponents of the equation (7) are 0 and $m - 1$.

CMC 1 Bour type surfaces

Proof. (cont.)
Then we have a solution of the form

$$z^{m-1}\sum_{p=0}^{\infty} a_p z^p,$$

where the coefficients a_p are inductively given by

$$a_{mk+l} = 0 \quad (l = 0, \cdots, m),$$
$$a_{mk+m+1} = \frac{a_{m(k-1)+m-1}}{(m-2)k(mk+m-1)}$$
$$= \frac{\Gamma\left(\frac{m-1}{m}+k\right)}{m^2\Gamma\left(\frac{m-1}{m}+k+1\right)} a_{m(k-1)+m-1} \quad (l \geq m+1).$$

CMC 1 Bour type surfaces

Proof. (cont.)
Therefore we obtain a solution of the differential equation (7):

$$\begin{aligned} & z^{\frac{m-1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\frac{m-1}{m}+k+1)} \left(\frac{z^{\frac{m}{2}}}{m}\right)^{2k+\frac{m-1}{m}} \\ = \quad & z^{\frac{m-1}{2}} \operatorname{Bessel I}\left(\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right). \end{aligned}$$

CMC 1 Bour type surfaces

Proof. (cont.)
Similarly, we obtain another independent solution as

$$
\begin{aligned}
& z^{\frac{m-1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-\frac{m-1}{m}+k+1)} \left(\frac{z^{\frac{m}{2}}}{m}\right)^{2k-\frac{m-1}{m}} \\
= \quad & z^{\frac{m-1}{2}} \mathrm{Bessel\,I}\left(-\frac{m-1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right).
\end{aligned}
$$

CMC 1 Bour type surfaces

Proof. (cont.)
So we have two independent solutions of Equation (5). Next, we find two independent solutions of Equation (6).

CMC 1 Bour type surfaces

Proof. (cont.)
Inserting $(g, \omega) = (z, z^{m-2})$ into Equation (6), we have

$$Y'' - \frac{m}{z}Y' - z^{m-2}Y = 0. \quad (m \in \mathbb{Z}_{\geq 2})$$

CMC 1 Bour type surfaces

Proof. (cont.)
Similarly to the way we solved Equation (5), we have two independent solutions

$$z^{\frac{m+1}{2}} \operatorname{Bessel I}\left(\frac{m+1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right), \quad z^{\frac{m+1}{2}} \operatorname{Bessel I}\left(-\frac{m+1}{m}, \frac{2}{m} z^{\frac{m}{2}}\right).$$

CMC 1 Bour type surfaces

Proof. (cont.)
Using the initial conditions, we have the solution F as in Equations (4).

CMC 1 Bour type surfaces

Remark (2). If F is a solution of Equation (3), the surface

$$f^\sharp = (F^{-1})\overline{(F^{-1})}^t \ \left(\text{resp. } f^\sharp = (F^{-1})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\overline{(F^{-1})}^t\right)$$

is also a CMC 1 surface in $\mathbb{H}^3$ (resp. $\mathbb{S}^{2,1}$).
This was proven by Umehara and Yamada [19] (resp. Lee [13]).
The surface $f^\sharp$ is called the CMC 1 dual of f.

CMC 1 Bour type surfaces

Using the explicit parametrization of the $\mathfrak{B}_m$ cousin, we can easily show the following corollary, which implies the rotational symmetric property of the $\mathfrak{B}_m$ cousins in $\mathbb{H}^3$, $\mathbb{S}^{2,1}$.

CMC 1 Bour type surfaces

Corollary (1)

Let $F(z) \in SL_2\mathbb{C}$ be the form as in Theorem 5 with complex coordinate z. Then

$$F(e^{i\frac{2\pi}{m}} \cdot z) = \begin{pmatrix} a(z) & e^{i\frac{2\pi}{m}} \cdot b(z) \\ e^{-i\frac{2\pi}{m}} \cdot c(z) & d(z) \end{pmatrix}.$$

CMC 1 Bour type surfaces

Writing $\mathfrak{B}_m$ cousin in $\mathbb{H}^3$ or $\mathbb{S}^{2,1}$ as $f(z) = (x_1(z), x_2(z), x_3(z), x_0(z))^t$, given by Theorem 5, and setting $f\left(\mathrm{e}^{i\frac{2\pi}{m}} \cdot z\right) = (\hat{x}_1(z), \hat{x}_2(z), \hat{x}_3(z), \hat{x}_0(z))^t$.

CMC 1 Bour type surfaces

By Corollary (1), we have

$$\hat{x}_1(z) = \cos\left(\frac{2\pi}{m}\right) x_1(z) - \sin\left(\frac{2\pi}{m}\right) x_2(z),$$
$$\hat{x}_2(z) = \sin\left(\frac{2\pi}{m}\right) x_1(z) + \cos\left(\frac{2\pi}{m}\right) x_2(z),$$
$$\hat{x}_3(z) = x_3(z), \ \hat{x}_0(z) = x_0(z),$$

that is, by rotating z by angle $\frac{2\pi}{m}$, the first and second coordinates are also rotated by the same angle.

CMC 1 Bour type surfaces

So like in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, $\mathfrak{B}_m$ has symmetry with respect to rotation by angle $\frac{2\pi}{m}$. Its dual $(\mathfrak{B}_m)^\sharp$ also has the same symmetry.

CMC 1 Bour type surfaces

- In order to see CMC 1 surfaces in $\mathbb{H}^3$, we use a stereographic projection.

CMC 1 Bour type surfaces

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- Consider the map

$$\mathbb{H}^3 \ni (x_1, x_2, x_3, x_0)^t \mapsto \left(\frac{x_1}{1+x_0}, \frac{x_2}{1+x_0}, \frac{x_3}{1+x_0} \right)^t \in \mathbb{B}^3,$$

where $\mathbb{B}^3$ denotes the 3-dimensional unit ball.

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where $\mathbb{B}^3$ denotes the 3-dimensional unit ball.

- This is the Poincaré ball model for $\mathbb{H}^3$.

CMC 1 Bour type surfaces

- In order to show graphics of CMC 1 surfaces in $\mathbb{S}^{2,1}$, the hollow ball model is used, see Fujimori [4] for example.

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- In order to show graphics of CMC 1 surfaces in $\mathbb{S}^{2,1}$, the hollow ball model is used, see Fujimori [4] for example.
- Consider the map

$$\begin{aligned} \mathbb{S}^{2,1} &\ni (x_1, x_2, x_3, x_0)^t \\ &\mapsto \left(\frac{e^{\arctan(x_0)} \cdot x_1}{\sqrt{1+x_0^2}}, \frac{e^{\arctan(x_0)} \cdot x_2}{\sqrt{1+x_0^2}}, \frac{e^{\arctan(x_0)} \cdot x_3}{\sqrt{1+x_0^2}} \right)^t \\ &\in \mathbb{B}^3_{(-\pi,\pi)}, \end{aligned}$$

where
$$\mathbb{B}^3_{(-\pi,\pi)} := \{(y_1, y_2, y_3)^t \in \mathbb{R}^3 \mid e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}\}.$$

CMC 1 Bour type surfaces

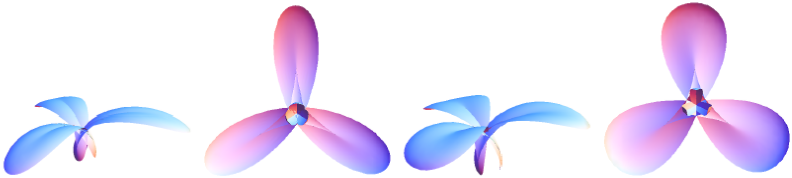

Figure 5. Left two pictures: $\mathfrak{B}_3$ cousin in $\mathbb{S}^{2,1}$, right two pictures: its dual cousin in $\mathbb{S}^{2,1}$

CMC 1 Bour type surfaces

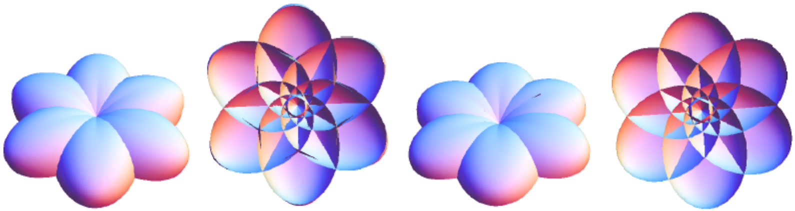

Figure 6. Left two pictures: $\mathfrak{B}_6$ cousin in $\mathbb{S}^{2,1}$, right two pictures: its dual cousin in $\mathbb{S}^{2,1}$

Degree and class of Bour type surfaces

- For $\mathbb{R}^{2,1}$, the set of roots of a polynomial $Q(x, y, z) = 0$ gives an algebraic surface.

Degree and class of Bour type surfaces

- For $\mathbb{R}^{2,1}$, the set of roots of a polynomial $Q(x, y, z) = 0$ gives an algebraic surface.
- An algebraic surface f is said to be of *degree* (or *order*) n when $n = \deg(f)$.

Degree and class of Bour type surfaces

The tangent plane at a point (u, v) on a surface $f(u, v) = (x(u, v), y(u, v), z(u, v))$ is given by

$$Xx + Yy - Zz + P = 0, \tag{8}$$

where the Gauss map is $n = (X(u, v), Y(u, v), Z(u, v))$ and $P = P(u, v)$.

Degree and class of Bour type surfaces

We have inhomogeneous tangential coordinates $a = X/P$, $b = Y/P$, and $c = Z/P$.

Degree and class of Bour type surfaces

When we can obtain an implicit equation $\hat{Q}(a, b, c) = 0$ of $f\,(u, v)$ in tangential coordinates, the maximum degree of the equation gives the *class* of $f\,(u, v)$.

Degree and class of Bour type surfaces

Next, using polynomial elimination methods (in Maple software), we calculate the implicit equations, degrees and classes of spacelike and timelike $\mathfrak{B}_2$, $\mathfrak{B}_3$ and $\mathfrak{B}_4$.

Degree and class of spacelike Bour of value 2,3,4

From (1), the parametrization of $\mathfrak{B}_2$ (maximal Enneper surface) is

$$\mathfrak{B}_2(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + u \\ u^2 v - \frac{1}{3}v^3 - v \\ u^2 - v^2 \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix},$$

where $u, v \in \mathbb{R}$.

Degree and class of spacelike Bour of value 2,3,4

- In this section, $Q_m(x, y, z) = 0$ denotes the irreducible implicit equation that spacelike or timelike $\mathfrak{B}_m$ will satisfy.

Degree and class of spacelike Bour of value 2,3,4

Then

$$
\begin{aligned}
Q_2(x,y,z) &= -64z^9+432x^2z^6-432y^2z^6+1215x^4z^3\\
&+6318x^2y^2z^3-3888x^2z^5+1215y^4z^3-3888y^2z^5\\
&+1152z^7+729x^6-2187x^4y^2-4374x^4z^2+2187x^2y^4\\
&+6480x^2z^4-729y^6+4374y^4z^2-6480y^2z^4\\
&-729x^4z+1458x^2y^2z+3888x^2z^3-729y^4z\\
&+3888y^2z^3-5184z^5,
\end{aligned}
$$

Degree and class of spacelike Bour of value 2,3,4

- Its degree is $\deg(\mathfrak{B}_2) = 9$.

Degree and class of spacelike Bour of value 2,3,4

- Its degree is $\deg(\mathfrak{B}_2) = 9$.
- Therefore, $\mathfrak{B}_2$ is an algebraic maximal surface.

Degree and class of spacelike Bour of value 2,3,4

- To find the class of the surface $\mathfrak{B}_2$, we obtain

$$P_2(u,v) = \frac{(u^2+v^2-3)(u-v)(u+v)}{3(u^2+v^2-1)},$$

where $P_m(u,v)$ denotes the function as in Equation (8) for spacelike or timelike $\mathfrak{B}_m$.

Degree and class of spacelike Bour of value 2,3,4

- To find the class of the surface $\mathfrak{B}_2$, we obtain

$$P_2(u,v) = \frac{(u^2+v^2-3)(u-v)(u+v)}{3(u^2+v^2-1)},$$

where $P_m(u,v)$ denotes the function as in Equation (8) for spacelike or timelike $\mathfrak{B}_m$.

- The inhomogeneous tangential coordinates are

$$a = \frac{6u}{\alpha(u,v)},\ b = \frac{6v}{\alpha(u,v)},\ c = \frac{6(u^2+v^2+1)}{\alpha(u,v)},$$

where $\alpha(u,v) = (u^2+v^2-3)(u-v)(u+v)$.

Degree and class of spacelike Bour of value 2,3,4

- In the tangential coordinates a, b, c,

$$\hat{Q}_2(a,b,c) = 4a^6 + 9a^4 + 9b^4 + 6a^2b^2c^2 + 12b^2c^3$$
$$-3b^4c^2 - 18b^4c - 4a^4b^2 + 18a^4c - 12a^2c^3$$
$$-4a^2b^4 - 3a^4c^2 + 18a^2b^2 - 4a^2b^4 + 4b^6,$$

where $\hat{Q}_m(a,b,c) = 0$ denotes the irreducible implicit equation for spacelike or timelike $\mathfrak{B}_m$ in terms of tangential coordinates.

Degree and class of spacelike Bour of value 2,3,4

- In the tangential coordinates a, b, c,

$$\hat{Q}_2(a,b,c) = 4a^6 + 9a^4 + 9b^4 + 6a^2b^2c^2 + 12b^2c^3$$
$$-3b^4c^2 - 18b^4c - 4a^4b^2 + 18a^4c - 12a^2c^3$$
$$-4a^2b^4 - 3a^4c^2 + 18a^2b^2 - 4a^2b^4 + 4b^6,$$

  where $\hat{Q}_m(a,b,c) = 0$ denotes the irreducible implicit equation for spacelike or timelike $\mathfrak{B}_m$ in terms of tangential coordinates.

- Therefore, the class of the spacelike $\mathfrak{B}_2$ is $cl(\mathfrak{B}_2) = 6$.

Degree and class of spacelike Bour of value 2,3,4

Similarly,

$$\mathfrak{B}_3(u,v) = \begin{pmatrix} \frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ u^3v - uv^3 - uv \\ \frac{2}{3}u^3 - 2uv^2 \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix},$$

$$\mathfrak{B}_4(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + \frac{1}{5}u^5 - 2u^3v^2 + uv^4 \\ -u^2v + \frac{1}{3}v^3 + u^4v - 2u^2v^3 + \frac{1}{5}v^5 \\ \frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4 \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix},$$

Degree and class of spacelike Bour of value 2,3,4

and

$$Q_3(x,y,z) = -43046721z^{16} + 272097792x^3z^{12}$$
$$-816293376xy^2z^{12} + 3009871872x^6z^8$$
$$+14834368512x^4y^2z^8 + (69 \text{ other lower order terms}),$$
$$Q_4(x,y,z) = -1514571848868138319872z^{25}$$
$$+92442129447518208000000x^4z^{20}$$
$$-241927616557617187500000000x^4y^{12}z^5$$
$$-55465277668510924800000x^2y^2z^{20}$$
$$-30652572326660156250000000x^{12}y^6z^2$$
$$+(233 \text{ other lower order terms}),$$

and their degrees are $\deg(\mathfrak{B}_3) = 16$, $\deg(\mathfrak{B}_4) = 25$.

Degree and class of spacelike Bour of value 2,3,4

Therefore, $\mathfrak{B}_3$ and $\mathfrak{B}_4$ are algebraic spacelike maximal surfaces. Furthermore,

$$P_3(u,v) = \frac{u(u^2+v^2-2)(u^2-3v^2)}{(u^2+v^2-1)},$$

$$P_4(u,v) = \frac{(3u^2+3v^2-5)(u^2-2uv-v^2)(u^2+2uv-v^2)}{30(u^2+v^2-1)},$$

Degree and class of spacelike Bour of value 2,3,4

and the inhomogeneous tangential coordinates are

$$a = \frac{12}{\beta(u,v)}, \quad b = \frac{12v}{u\beta(u,v)}, \quad c = \frac{6(u^2+v^2+1)}{u\beta(u,v)} \quad (m=3),$$

$$a = \frac{60u}{\gamma(u,v)}, \quad b = \frac{60v}{\gamma(u,v)}, \quad c = \frac{30(u^2+v^2+1)}{\gamma(u,v)} \quad (m=4),$$

where $\beta(u,v) = (u^2+v^2-2)(u^2-3v^2)$,
$\gamma(u,v) = (3u^2+3v^2-5)(u^2-2uv-v^2)(u^2+2uv-v^2)$.
Then

Degree and class of spacelike Bour of value 2,3,4

$$
\begin{aligned}
\hat{Q}_3(a,b,c) &= 9a^8 + 72a^6b^2 - 8a^6c^2 + 144a^4b^4 - 168a^4b^2c^2 \\
&-96a^2b^4c^2 + 96a^2b^2c^4 + 64b^6c^2 - 48b^4c^4 - 72a^7 \\
&-288a^5b^2 + 288a^5c^2 + 288a^3b^2c^2 - 192a^3c^4 + 144a^6, \\
\hat{Q}_4(a,b,c) &= -16a^{10} - 8640a^2b^2c^5 - 9000a^4b^4c - 3600a^2b^6c \\
&+12000a^2b^4c^3 + 570a^4b^4c^2 - 180a^2b^6c^2 + 15b^8c^2 - 900b^8 \\
&+1440a^4c^5 + 1440b^4c^5 - 5400a^4b^4 - 3600a^2b^6 + 900b^8c \\
&-2400b^6c^3 - 416a^6b^4 - 416a^4b^6 + 176a^2b^8 - 16b^{10} \\
&+12000a^4b^2c^3 - 3600a^6b^2c - 180a^6b^2c^2 - 3600a^6b^2 \\
&+176a^8b^2 - 2400a^6c^3 + 900a^8c + 15a^8c^2 - 900a^8.
\end{aligned}
$$

Degree and class of spacelike Bour of value 2,3,4

Therefore,

$$cl(\mathfrak{B}_3) = 8 \text{ and } cl(\mathfrak{B}_4) = 10.$$

Degree and class of timelike Bour of value 2,3,4

From (2), the parametrization of $\mathfrak{B}_2$ (timelike Enneper surface) is

$$\mathfrak{B}_2\left(u,v\right)=\begin{pmatrix} u^2+v^2 \\ u+v-\frac{1}{3}\left(u^3+v^3\right) \\ -u+v-\frac{1}{3}\left(u^3-v^3\right) \end{pmatrix}=\begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}.$$

where $u,v\in\mathbb{R}$. Then

Degree and class of timelike Bour of value 2,3,4

$$Q_2(x,y,z) = -16z^9 - 2916y^4z + 4374x^4y^2 - 6318y2x^2z^3$$
$$+4374x^2y^4 - 15552y^2z^3 - 2916x^4z - 5832x^2y^2z - 20736z^5$$
$$+1152z^7 - 8748x^4z^2 + 8748y^4z^2 + 3888y^2z^5 - 3888x^2z^5$$
$$+15552x^2z^3 + 1215x^4z^3 + 1458x^6 + 216x^2z^6 + 1458y^6$$
$$+1215y^4z^3 + 216y^2z^6 + 12960y^2z^4 + 12960x^2z^4.$$

Degree and class of timelike Bour of value 2,3,4

- Its degree is $\deg(\mathfrak{B}_2) = 9$.

Degree and class of timelike Bour of value 2,3,4

- Its degree is $\deg(\mathfrak{B}_2) = 9$.
- Hence, $\mathfrak{B}_2$ is an algebraic timelike minimal surface.

Degree and class of timelike Bour of value 2,3,4

To find the class of surface $\mathfrak{B}_2$ we obtain

$$P_2(u,v) = \frac{(uv+3)(u^2+v^2)}{3(uv+1)},$$

and the inhomogeneous tangential coordinates are

Degree and class of timelike Bour of value 2,3,4

$$
\begin{aligned}
a &= -\frac{(uv-1)(3uv+3)}{\hat{\alpha}(u,v)}, \\
b &= -\frac{(u+v)(3uv+3)}{\hat{\alpha}(u,v)}, \\
c &= -\frac{(u-v)(3uv+3)}{\hat{\alpha}(u,v)},
\end{aligned}
$$

where $\hat{\alpha}(u,v) = (uv+1)(uv+3)(u^2+v^2)$.

Degree and class of timelike Bour of value 2,3,4

Then

$$\hat{Q}_2(a,b,c) = 16a^6 + 9a^4 + 36b^4c + 24a^2c^3$$
$$+24b^2c^3 - 24a^2b^2c^2 - 12a^4c^2 - 16a^2b^4 - 12b^4c^2$$
$$-36a^4c + 16a^4b^2 + 9b^4 - 16b^6 - 18a^2b^2.$$

Hence, $cl(\mathfrak{B}_2) = 6$.

Degree and class of timelike Bour of value 2,3,4

Similarly,

$$\mathfrak{B}_3(u,v) = \begin{pmatrix} \frac{2}{3}\left(u^3+v^3\right) \\ \frac{1}{2}\left(u^2+v^2\right)-\frac{1}{4}\left(u^4+v^4\right) \\ -\frac{1}{2}\left(u^2-v^2\right)-\frac{1}{4}\left(u^4-v^4\right) \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix},$$

$$\mathfrak{B}_4(u,v) = \begin{pmatrix} \frac{1}{2}\left(u^4+v^4\right) \\ \frac{1}{3}\left(u^3+v^3\right)-\frac{1}{5}\left(u^5+v^5\right) \\ -\frac{1}{3}\left(u^3-v^3\right)-\frac{1}{5}\left(u^5-v^5\right) \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix},$$

and

Degree and class of timelike Bour of value 2,3,4

$$
\begin{aligned}
&Q_3(x,y,z) = 43046721z^{16} - 1836660096z^{14} \\
&+5435817984x^6z^4 + 602404356096x^4z^8 \\
&+165112971264x^2z^8 + (69 \text{ other lower order terms}), \\
&Q_4(x,y,z) = 3118369126021466283345445989415649 28z^{25} \\
&-380660293703792270916192137379840000 0x^4z^{20} \\
&-228396176222275362549715282427904000 00x^2y^2z^{20} \\
&-380660293703792270916192137379840000 0y^4z^{20} \\
&-271833827901267673933071777792000000 0000x^8z^{15} \\
&+(233 \text{ other lower order terms}).
\end{aligned}
$$

Degree and class of timelike Bour of value 2,3,4

So

- $\deg(\mathfrak{B}_3) = 16$, $\deg(\mathfrak{B}_4) = 25$.

Degree and class of timelike Bour of value 2,3,4

In the tangential coordinates a, b, c,

$$\begin{aligned}\hat{Q}_3(a,b,c) &= 81a^6b^2 - 27a^4b^4 - 72a^4b^2c^2 - 45a^2b^6 \\ &-48a^2b^4c^2 - 9b^8 - 8b^6c^2 - 108a^6b + 180a^4b^3 \\ &+432a^4bc^2 - 36a^2b^5 - 288a2b^3c^2 - 288a^2bc^4 \\ &-36b^7 - 144b^5c^2 - 96b^3c^4 + 36a^6 - 108a^4b^2 \\ &+108a^2b^4 - 36b^6,\end{aligned}$$

Degree and class of timelike Bour of value 2,3,4

$$
\begin{aligned}
\hat{Q}_4(a,b,c) &= -16a^{10} + 16b^{10} - 450a^8c + 15b^8c^2 \\
&-225b^8 - 720a^4c^5 - 1350a^4b^4 + 900a^2b^6 - 450b^8c \\
&-1200b^6c^3 - 416a^6b^4 + 416a^4b^6 + 176a^2b^8 \\
&-4320a^2b^2c^5 + 4500a^4b^4c - 1800a^2b^6c \\
&-6000a^2b^4c^3 + 570a^4b^4c^2 + 180a^2b^6c^2 \\
&+6000a^4b^2c^3 - 1800a^6b^2c + 180a^6b^2c^2 \\
&-225a^8 - 720b^4c^5 + 900a^6b^2 - 176a^8b^2 \\
&+1200a^6c^3 + 15a^8c^2.
\end{aligned}
$$

Degree and class of timelike Bour of value 2,3,4

Therefore,

$$cl(\mathfrak{B}_3) = 8, \ cl(\mathfrak{B}_4) = 10.$$

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Thank you