Toroidal Surfaces

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Summary

1. We show that the 2-torus in $\mathbb{R}^3$ is a critical point of a sequence of functionals $\mathcal{F}_n$ $(n = 1, 2, 3, \cdots)$ defined over compact 2-surfaces (closed membranes) in $\mathbb{R}^3$.
2. When the Lagrange function $\mathcal{E}$ is a polynomial of degree n of the mean curvature H of the surface, the radii (a, r) of the 2-torus are related as $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}, \quad n \geq 2.$
3. A simple generalization of 2-torus in $\mathbb{R}^3$ is a tube of radius r along a curve α which we call it toroidal surface (TS). We show that toroidal surfaces with non-circular curve α do not provide minimal energy surfaces of the functionals $\mathcal{F}_n$ $(n = 2, 3)$ on closed surfaces.
4. We discuss possible applications of the functionals discussed in this work on cell membranes.

Basic References

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Introduction

In the history of differential geometry there are some special subclasses of 2-surfaces, such as surfaces of constant Gaussian curvature, surfaces of constant mean curvature, minimal surfaces and the Willmore surfaces. These surfaces arise in many different branches of sciences; in particular, in various parts of theoretical physics (string theory, general theory of relativity), cell-biology and differential geometry . All these special surfaces constitute critical points of certain functionals. Euler-Lagrange equations of these functionals are very complicated and difficult. There are some techniques developed to solve them, such as using the deformation of the Lax equations of the integrable equations.

Introduction

The main objective in our work is to investigate 2-surfaces derivable from a variational principle, such as the minimal and Willmore surfaces and surfaces solving the shape equation. All these surfaces are critical points of a functional where the Lagrange function is a polynomial of degree two in the mean curvature of the surface. It is natural to ask whether there are surfaces solving the Euler-Lagrange equations corresponding to more general Lagrange functions depending on the mean and Gaussian curvatures of the surface (Tu-Yang). It is the purpose of this work to give an answer to such a question.

Introduction

The quadratic Helfrich functional for a theoretical model of a closed cell-membrane is

$$\mathcal{F} = \frac{1}{2} \int_S [k_c (2H + c_0)^2 + 2w] \, dA + p \int_V dV$$

where k_c is the elasticity constant, H and c_0 are the mean and the spontaneous curvatures, w is the surface tension and p is the pressure difference between in and out of the surface. First variation of the above functional gives the shape equation

$$p - 2wH + 2 k_c \nabla^2 H + k_c (2H + c_0)(2H^2 - c_0 H - 2K) = 0.$$

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Sphere with an arbitrary radius is an exact solution of this equation. The radius of the sphere is related to the model parameters k_c, c_0, w and p through the shape equation . Stability of this solution has been studied in Tu-Yang (1987). A special 2-torus, called the Clifford torus, is also an exact solution of the shape equation above. We might consider that Helfrich's functional as an approximation of some higher order functionals $\mathcal{F}_n = \int_S \mathcal{E}_n \, dA + p \int_V dV, \quad (n \geq 2)$ where

$$\begin{aligned} \mathcal{E}_n &= \sum_{k=0}^{n} a_{n+1-k} \, H^k, \quad n = 1, 2, \cdots \\ &= a_1 \, H^n + a_2 \, H^{n-1} + \cdots + a_n \, H + a_{n+1}. \end{aligned} \tag{4.1}$$

Introduction

Here a_i's are constants describing the parameters of the cell-membrane model. Hence it is worthwhile to study such functionals and search for possible critical points. This is another motivation of this work on functionals on closed surfaces. In this work we consider only surfaces which are diffeomorphic to 2-torus. We call such surfaces as toroidal surfaces. The first example we consider is the 2-torus itself. The second example is the tube of radius r about a closed planar curve α.

Surfaces from a variational principle

Let S be a regular closed 2-surface in $\mathbb{R}^3$ with Gaussian (K) and mean (H) curvatures. A functional $\mathcal{F}$ is defined by

$$\mathcal{F} = \int_S \mathcal{E}(H, K) dA + p \int_V dV,$$

where $\mathcal{E}$ is the Lagrange function depending on H and K. Functional $\mathcal{F}$ is also called curvature energy or shape energy. Here p is a constant which play the role of Lagrange multiplier and V is the volume enclosed within the surface S. We obtain the Euler-Lagrange equations corresponding to the above functional from its first variation. Let $\mathcal{E}$ be a twice differentiable function of H and K. Then the first variation of $\mathcal{F}$ is given by

$$\delta \mathcal{F} = \int_S E(\mathcal{E})\, \Omega\, dA,$$

where Ω is an arbitrary smooth function on S.

Surfaces from a variational principle

Then the Euler-Lagrange equation $E(\mathcal{E}) = 0$ for $\mathcal{F}$ reduces to (Tu-Yang)

$$E(\mathcal{E}) = (\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0.$$

Here and in what follows, $\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right)$and $\nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} K h^{ij} \frac{\partial}{\partial x^j} \right)$, $g = \det(g_{ij})$, g^{ij} and h^{ij} are inverse components of the first and second fundamental forms; $x^i = (u, v)$ and we assume Einstein's summation convention on repeated indices over their ranges.

Surfaces from a variational principle

Weingarten surfaces are the surfaces where the mean and Gauss curvatures satisfy certain algebraic relations. Surfaces are called linear Weingarten surfaces if $\alpha H + \beta K + \gamma = 0$ relation holds for any constants α, β and γ. Here we have a nice theorem on such surfaces.

Theorem 1. *Let S be linear Weingarten surface ,i.e., $\alpha H + \beta K + \gamma$, where α, β and γ are constants. Then S is a critical point of the functional $\mathcal{F}$ with a Lagrange function $\mathcal{E} = \frac{\beta}{2} H + \frac{\alpha}{2}$ and $p = -\gamma$*

Proof. Inserting $\mathcal{E} = \frac{\beta}{2} H + \frac{\alpha}{2}$ into the Euler-Lagrange equation we simply obtain the linear Weingarten relation with $\gamma = -p$.

Surfaces from a variational principle

For the second variation of the functional we assume that $\mathcal{E}$ depends only on H. In this case the expression is much simpler (Tu-Yang)

$$\begin{aligned}\delta^2\,\mathcal{F} &= \int_S \Bigg(\mathcal{E}_1\,\Omega^2 + \mathcal{E}_2\,\Omega\,\nabla^2\,\Omega - 2\frac{\partial\mathcal{E}}{\partial H}\,\Omega\nabla\cdot\tilde{\nabla}\,\Omega + \frac{1}{4}\,\frac{\partial^2\mathcal{E}}{\partial H^2}\,(\nabla^2\Omega)^2 \\ &\quad + \frac{\partial\mathcal{E}}{\partial H}\,[\nabla(H\Omega)\cdot\nabla\Omega - \nabla\Omega\cdot\tilde{\nabla}\Omega]\Bigg)\,dA\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_1 &= (2H^2-K)^2\,\frac{\partial^2\mathcal{E}}{\partial H^2} - 2HK\frac{\partial\mathcal{E}}{\partial H} + 2K\mathcal{E} - 2Hp, \\ \mathcal{E}_2 &= (2H^2-K)\,\frac{\partial^2\mathcal{E}}{\partial H^2} + 2H\frac{\partial\mathcal{E}}{\partial H} - \mathcal{E},\end{aligned}$$

where Ω is an arbitrary function over the closed surface. To have minimal energy solutions of it is expected that the second variation $\delta^2\mathcal{F} > 0$.

Surfaces from a variational principle

We have the following classical examples:

i) Minimal surfaces: $\mathcal{E} = 1, \quad p = 0$.

ii) Constant mean curvature surfaces: $\mathcal{E} = 1$.

iii) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants.

iv) Willmore surfaces: $\mathcal{E} = H^2$.

v) Surfaces solving the shape equation of lipid bilayer cell membranes: $\mathcal{E} = \frac{1}{2}\, k_c\, (2H + c_0)^2 + w$, where k_c, c_0 and w are constants.

Surfaces from a variational principle

2-sphere in $\mathbb{R}^3$ has constant mean and Gaussian curvatures. Hence it is a critical point of the most general functional $\mathcal{E}(H, K)$. Eq. Euler-Lagrange equation $E(\mathcal{E}) = 0$ gives a relation between p, radius of the sphere and other parameters in model.

Another compact surface in $\mathbb{R}^3$ is the 2-torus, T. It has been shown (Yang-can) that a special kind of torus, known as the Clifford torus, solves the shape equation. In this work we shall show that, T is not only a critical point of quadratic functional but it is also critical point of functional with Lagrange function $\mathcal{E}$ is any polynomial function of the mean curvature H, provided that the radii of the torus satisfies certain relations.

2-Torus

Definition 1 (2-Torus T). *2-Torus in $\mathbb{R}^3$ is defined as $X : U \to \mathbb{R}^3$ where*

$$X(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u),$$
$$0 < u < 2\pi, 0 < v < 2\pi$$

The first and second fundamental forms of T are

$$ds_1^2 = g_{ij}\, dx^i\, dx^j = r^2\, du^2 + (a + r\cos u)^2\, dv^2,$$
$$ds_2^2 = h_{ij}\, dx^i\, dx^j = r\, du^2 + (a + r\cos u)\cos u dv^2$$

The Gaussian, K, and mean, H, curvatures of T are

$$K = \frac{\cos u}{r(a + r\cos u)}, \quad H = \frac{1}{2}\left(\frac{1}{r} + \frac{\cos u}{(a + r\cos u)}\right),$$

where a and r $(a > r)$ are the radii of the torus.

It is interesting that K and H satisfy a linear equation $r^2K - 2rH + 1 = 0$. Hence torus T is a linear Weingarten surface.

Functionals with Mean Curvature

In this section we shall consider the Lagrange function $\mathcal{E}$ depending only on the mean curvature H of the surface. Furthermore we shall assume that $\mathcal{E}$ is a polynomial function of H. Let the degree of the polynomial be n, then we write

$$\begin{aligned} \mathcal{E}_n &= \sum_{k=0}^{n} a_{n+1-k}\, H^k, \quad n = 1, 2, \cdots \\ &= a_1\, H^n + a_2\, H^{n-1} + \cdots + a_n\, H + a_{n+1}, \end{aligned} \tag{7.1}$$

where a_k, $(k = 1, 2, \cdots)$ are constants to be determined. Assuming that the torus is a critical point of the functional $\mathcal{F}$ we shall determine the coefficients a_i of the polynomial expansion of $\mathcal{E}$ and p in terms of the torus radii a and r. We shall give three examples here in this section. In all examples in this section S is the 2-torus and H is the mean curvature of the 2-torus.

Functionals with Mean Curvature

Example 1: First Order Functional. Since torus T is a linear Weingarten surfaces then by Theorem 1 it is a critical point of the functional where the Lagrange function is a linear function of H, i.e., $\mathcal{E}_1 = a_1 H + a_2$. Euler-Lagrange equation is exactly solved, provided

$$p = -\frac{a_1}{r^2}, \quad a_2 = -\frac{a_1}{r}.$$

There is no restriction on the radii a and r. Torus T is a critical point of the functional $\mathcal{F}$ for all values of r and a.

Functionals with Mean Curvature

Example 2: Second Order Functional. Lagrange function is a quadratic function of H, i.e., $\mathcal{E}_2 = a_1 H^2 + a_2 H + a_3$. Euler-Lagrange equation is exactly solved, provided

$$p = -\frac{a_2}{r^2}, \quad a_3 = -\frac{a_2}{r}, \quad a^2 = 2r^2.$$

This is the Clifford Torus. We find that (for $p = 0$)

$$\mathcal{F}_2 = \int_T \mathcal{E}_2 dS = 2\pi^2 a_1.$$

2-torus with $a^2 = 2r^2$ minimizes the functional $\int_S \mathcal{E}_2 dS$ which is the minimum energy . Willmore conjecture states that ($a_1 \neq 0$)

$$\frac{1}{a_1} \int_S \mathcal{E}_2 dA \geq 2\pi^2$$

for all compact surfaces S with genus $g > 0$.

Functionals with Mean Curvature

The proof of this conjecture has been given very recently (Marques and Neves). In terms of the Helfrich's functional we have $a_1 = 2\,k_c$, $a_2 = 2k_c c_0$ and $a_3 = \frac{1}{2}\,k_c c_0^2 + w$. The parameters p, w, c_0 must satisfy

$$p = \frac{2k_c c_0}{r^2}, \quad w = p\,r\,(1 + \frac{1}{4} r c_0).$$

Remark: Since K is a topological invariant and total curvature for torus is zero, there will be no contribution of adding linear K terms to $\mathcal{E}$.

Functionals with Mean Curvature

Example 3: Third Order Functional. Lagrange function is a polynomial of H of degree three, i.e., $\mathcal{E}_3 = a_1 H^3 + a_2 H^2 + a_3 H + a_4$. Euler-Lagrange equation is exactly solved, provided

$$\begin{aligned} p &= -\frac{3a_1 - a_3 r^2}{r^4}, \quad a_4 = \frac{2a_1 - a_3 r^2}{r^3}, \quad a_2 = \frac{15 a_1}{2r}, \\ a^2 &= (6/5) r^2. \end{aligned}$$

We also find that (for $p = 0$)

$$\mathcal{F}_3 = \int_T \mathcal{E}_3 dA = 9\sqrt{5}\,\pi^2\,(a_1/r).$$

2-torus with $a^2 = (6/5)r^2$ minimizes the functional $\int_S \mathcal{E}_3 dA$ and it is minimum energy. We expect that $(a_1 \neq 0)$

$$(a_1/r)^{-1} \int_S \mathcal{E}_3 dA \geq 9\sqrt{5}\,\pi^2$$

for all compact surfaces S with genus $g > 0$. This is the Willmore conjecture for $n = 3$.

Functionals with Mean Curvature

Definition 2 (Torii T_n). *2-Torus T with radii (a, r) satisfying the relation*

$$\frac{a^2}{r^2} = \frac{n^2 - n}{n^2 - n - 1}, \quad n = 2, 3, 4, \cdots$$

are special and symbolized by T_n. For all these surfaces $1 < \frac{a^2}{r^2} \leq 2$.

It is possible to continue on finding critical points of higher order functionals with $\mathcal{E}_n$, $(n = 4, 5, 6, \cdots)$. We observe that critical points are the special torii T_n $(n = 2, 3, \cdots)$. We have the following theorem.

Functionals with Mean Curvature

Theorem 2: *2-torus T in $\mathbb{R}^3$ is a critical point of functionals $\mathcal{F}_n = \int_S \mathcal{E}_n \, dA + p \int_V dV$ where $\mathcal{E}_n$ are the nth degree polynomials of the mean curvature H of the surface S if $T = T_n$ for all $n \geq 2$.*

Proof: The shape equation with $\mathcal{E} = \mathcal{E}(H)$ is given by

$$E(\mathcal{E}) = (\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} - 4H\mathcal{E} + 2p = 0.$$

Using metrics of the 2-torus we obtain

$$\begin{aligned}
\nabla^2 H &= \frac{1}{a^2 r^3}\,[4r^3(-a^2 + r^2)H^3 + 2r^2(5a^2 - 6r^2)H^2 \\
&\quad + 4r(-2a^2 + 3r^2)H + 2(a^2 - 2r^2)], \\
(\nabla H)\cdot(\nabla H) &= \frac{1}{a^2 r^4}\,[4r^4(-a^2 + r^2)\,H^4 + 4r^3(3a^2 - 4r^2)\,H^3 \\
&\quad + r^2(-13a^2 + 24r^2)\,H^2 + 2r(3a^2 - 8r^2)\,H + 4r^2]
\end{aligned}$$

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Using these equations for the 2-torus for all $n \geq 2$ we get

$$\nabla^2 H^n = \frac{4n^2}{a^2}(-a^2+r^2)H^{n+2} + \frac{2}{a^2 r}[(6n^2-n)a^2-(8n^2-2n)r^2]H^{n+1}+\cdots \tag{7.2}$$

Hence inserting

$$\mathcal{E}_n = a_1\, H^n + a_2\, H^{n-1} + \cdots + a_n\, H + a_{n+1},$$

into the general shape equation we get

$$\begin{aligned} E(\mathcal{E}) &= (na_1\nabla^2 H^{n-1} + (n-1)a_2\nabla^2 H^{n-2} + \cdots) \\ &\quad +(4(n-1)a_1 H^{n+1} + 4(n-2)a_2 H^n + \cdots) \\ &\quad -\frac{2}{r^2}(2rH-1)(na_1 H^{n-1} + (n-1)a_2 H^{n-2} + \cdots) = 0. \end{aligned}$$

Functionals with Mean Curvature

Using the identity (7.2) for the 2-torus and collecting the coefficients of the powers of H we get equations for a_i's. The coefficient of the highest power H^{n+1} in $E(\mathcal{E})$ can be calculated exactly

$$a_1 \, [\frac{4n(n-1)^2\,(-a^2+r^2)}{a^2} + 4(n-1)]\, H^{n+1} + \cdots = 0.$$

Then coefficient of H^{n+1} must vanish which leads to the constraint equations

$$\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1},$$

for all $n \geq 2$. This completes the proof Theorem 1.

The remaining $n+1$ number of equations are linear algebraic equations for $a_i, \quad (i = 1, 2, \cdots, n+1)$ and p. In general one can solve them in terms of one arbitrary parameter, for instance a_1. In examples 1-3 above, the solutions contain two arbitrary coefficients. This means that one of the remaining equations vanishes identically. This is due to the following property

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Theorem 3: *Let the torus T_n be a critical point of the functional $\mathcal{F}_n$. T_n is left invariant under the change of the Lagrange function $\bar{\mathcal{E}}_n = \mathcal{E}_n + b_1 H + b_0$ where b_1 and b_0 are constants satisfying*

$$\bar{p} = p - \frac{b_1}{r^2}, \quad b_0 = -\frac{b_1}{r}.$$

Proof: It is straightforward to show that

$$E(\bar{\mathcal{E}}_n) = E(\mathcal{E}_n) + 2\bar{p} - 2p - 4H(b_0 + \frac{b_1}{r}) + \frac{2b_1}{r^2}.$$

Since $E(\bar{\mathcal{E}}_n) = E(\mathcal{E}_n) = 0$ we obtain equations in the theorem.

Here b_1 is left arbitrary. This is the reason why the coefficients of the linear H terms are arbitrary in the first, second and third order functionals studied above. It is left arbitrary in all $\mathcal{E}_n$.

Toroidal Surfaces

Definition 3 (Toroidal Surfaces TS**).** *Let* $\alpha(v)$ *be a simple and a regular closed curve in* $\mathbb{R}^3$ *with the unit tangent vector* $\mathbf{t}(v)$*, the unit normal vector* $\mathbf{n}(v)$ *and the bi-normal vector* $\mathbf{b}(v)$*. Here* $v \in I = [v_1, v_2]$ *is the arclength parameter of the curve. A parametrization* $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *of this surface is given as*

$$X(u, v) = a\,\alpha(v) + r[-\cos u\,\mathbf{n}(v) + \sin u\,\mathbf{b}(v)],$$

where a *and* r *are constants. Here* $u \in [0, 2\pi]$ *and* $v \in I$ *such that* $\alpha(v_1) = \alpha(v_2)$*. This is a tube of radius* r *around the closed curve* α*. The radius* r *is so chosen that tube has no self intersections. We call these surfaces as toroidal surface (TS). The first and second fundamental forms of this surface are*

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$$
\begin{aligned}
ds_1^2 &= g_{ij}\, dx^i\, dx^j \\
&= r^2\, du^2 - 2r^2\, \tau(v)\, dudv + [(a + rk(v)\, \cos u)^2 + (\tau(v))^2\, r^2]\, dv^2, \\
ds_2^2 &= h_{ij}\, dx^i\, dx^j \\
&= r\, du^2 - 2r\tau(v) dudv + [(a + rk(v)\, \cos u)\, k(v)\, \cos u + r(\tau(v))^2]\, dv^2
\end{aligned}
$$

The Gaussian (K) and mean (H) curvatures are

$$
K = \frac{k(v)\, \cos u}{r(a + rk(v)\, \cos u)}, \quad H = \frac{1}{2} \left(\frac{1}{r} + \frac{k(v)\, \cos u}{(a + rk(v)\, \cos u)} \right),
$$

where $k(v)$ and $\tau(v)$ are the curvature and torsion of the closed curve α.

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It is simple to show that K and H satisfy the same linear equation $r^2K - 2rH + 1 = 0$. Hence TS is also a linear Weingarten surface. When $k = 1$ this surface becomes the the 2-Torus we discussed in section 2. Below we shall assume that the Lagrange function $\mathcal{E}$ is a polynomial of the mean curvature H.

Example 4: Linear functional. Since toroidal surface TS is linear Weingarten surface then by Theorem 1 it is a critical point of the functional with the Lagrange function $\mathcal{E}_1 = a_1\, H + a_2$, provided that

$$p = -\frac{a_1}{r^2}\,, \quad a_2 = -\frac{a_1}{r} \tag{8.1}$$

Hence any toroidal surface TS with arbitrary closed curve $\alpha(v)$ in $\mathbb{R}^3$ is a critical point of the corresponding functional $\mathcal{F}_1$.

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Example 5: Quadratic functional. The Lagrange function is a quadratic function of H, i.e., $\mathcal{E}_2 = a_1 H^2 + a_2 H + a_3$. The toroidal surface TS is a critical point of the corresponding functional $\mathcal{F}_2$ provided that $\tau = 0$, $k = k_0$ a constant and

$$p = -\frac{a_2}{r^2}, \quad a_3 = -\frac{a_2}{r}, a^2 = 2k_0^2 r^2 \tag{8.2}$$

Without loosing any generality we take $k_0 = 1$. Hence TS is the 2-torus T_2, i.e., Clifford Torus.

Example 6: Qubic functional. The Lagrange function is a cubic polynomial of H, i.e.,$\mathcal{E}_3 = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4$. The toroidal surface TS is a critical point of the corresponding functional $\mathcal{F}_3$ provided that $\tau = 0$, $k = k_0$ a constant and

$$p = \frac{3a_1 - a_3}{r^4}\,, \quad a_2 = \frac{15a_1}{2r}, \quad a_4 = -\frac{2a_1 - a_3 r^2}{r^3},\; a^2 = (6/5)k_0^2 r^2 \quad (8.3)$$

Again we take $k_0 = 1$. Hence TS is the 2-torus T_3.

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We may continue finding solutions of the Euler-Lagrange equations for $\mathcal{E}_n$ with $n \geq 4$. We observe that, except $n = 1$, for all $n \geq 2$ toroidal surfaces reduce to 2-torus. We claim that this is true in general. Toroidal surface TS with non-vanishing torsion τ, non-constant curvature k is not a critical point of the functional $\mathcal{F}_n$ $(n = 2, 3, \cdots)$ where the Lagrange function $\mathcal{E}_n$ is a polynomial of the mean curvature H.

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Theorem 4. *Let S be a toroidal surface TS given in definition 3. Let the Lagrange function function $\mathcal{E}$ be a polynomial of the mean curvature H of degree $n \geq 2$ given in (4.1). Then the critical points of the functional (7.1) on TS are the surfaces T_n, $n \geq 2$.*

Proof: The Euler-Lagrange equation for the Lagrange function

$$\mathcal{E}_n = \sum_{k=0}^{n} a_{n+1-k}\, H^k, \quad n = 1, 2, \cdots \tag{8.4}$$

$$= a_1\, H^n + a_2\, H^{n-1} + \cdots + a_n\, H + a_{n+1}, \tag{8.5}$$

takes the form

$$E(\mathcal{E}) = (na_1 \nabla^2 H^{n-1} + (n-1)a_2 \nabla^2 H^{n-2} + \cdots) + (4(n-1)a_1 H^{n+1} + 4(n-2)a_2 H^n + \cdots) - \frac{2}{r^2}(2rH - 1)(na_1 H^{n-1} + (n-1)a_2 H^{n-2} = 0.$$

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We can write $\nabla^2 H^{n-1},\ \nabla^2 H^{n-2},\ \cdots$ by using $\nabla^2 H$ and $\nabla H \cdot \nabla H$ given in the Appendix. We collect the terms $\sin u\ H^n$ and H^n for all $n = 0, 1, 2, \cdots$ and equate the coefficients of these terms to zero. It is clear from the expression of $\nabla^2 H$ the highest order term $\sin u H^{n+4}$ gives $\tau = 0$. This simplifies the remaining equations considerably. Equating the coefficient of the highest order factor H^{n+4} to zero in the remaining equations we get $k' = 0$ .With this result α reduces to a plane curve with constant curvature. Since it is a closed curve then α is the circle with $k = 1$ and TS is the 2-torus. From the theorem 3 we know that critical points of functional (8.4) on TS are T_n $(n \geq 2)$.

Concluding Remarks

1. Clifford torus is a critical point of Willmore and also Helfrich's functional where the Lagrange function is a quadratic polynomial in the mean curvature of a closed surface in $\mathbb{R}^3$. One of the main contributions of this work is that the sequence torus surfaces $\{T_n\}$ where radii a and r restricted to satisfy $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ for all $n \geq 2$ are the critical points of the functionals $\mathcal{F}_n$ where the Lagrange function $\mathcal{E}_n$ are polynomial of degree n in the mean curvature H of the surface. We have given 3 examples $n = 1, 2, 3$ and proved this assertion in section 3.

Concluding Remarks

2. A simple generalization of 2-torus in $\mathbb{R}^3$ is the tube around a closed curve α. We call such surfaces as toroidal surfaces which are topologically diffeomorphic to 2-torus. Except the linear case we showed that these surfaces with non-vanishing torsion τ and nonconstant curvature k are not critical points of the functionals $\mathcal{F}_n$. Euler-Lagrange equations force the torsion of the curve α to vanish and the curvature be a constant.

3. In section 3, for each solution with $n = 2, 3$ and $p = 0$ we have calculated the curvature energy $\mathcal{F}_n$. As in the case of the Willmore energy functional $(n = 2)$ it is expected the torus surfaces for $n \geq 3$ with the constraints are minimal energy surfaces. To support this assertion, second variation of the functionals on these surfaces must be nonnegative. Another point to be examined is the stability of these minimal energy surfaces. These points will be clarified in a forthcoming communication.

Concluding Remarks

4. The constraints $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ can be utilized to select the correct functional for the toroidal fluid membranes. These functionals are used to minimize the energy of the lipid membranes. The ratio a/r of toroidal configuration can be measured experimentally. Comparing the measured value of this ratio with $\frac{a^2}{r^2} = \frac{n^2-n}{n^2-n-1}$ we can identify the degree of the polynomial function $\mathcal{E}_n$ from (7.2), hence finding the functional for the corresponding closed membrane. As an example, for vesicle membranes such a measurement had been done by Mutz and Bensimon (Phys. Rev. A **43**, 4525 (1991)). They measured the value of this ratio approximately as $\frac{a}{r} = 1.43$, or $\frac{a^2}{r^2} = 2.04$. Hence for vesicle membranes the correct functional should be the quadratic one which was first introduced by W. Helfrich several years ago (Z. Naturforssch, **28**, 693 (1973)).. For other closed fluid membranes the functionals might be different.

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