Slant Curves in $\mathcal{S}-$Space Forms

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Introduction

J. S. Kim, M. K. Dwivedi and M. M. Tripathi obtained the Ricci curvature of integral submanifolds of an $\mathcal{S}$-space form in [KDT-2007]. On the other hand, D. Fetcu and C. Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [Fetcu-2008] and [Fetcu-2009]. We studied biharmonic Legendre curves of $\mathcal{S}-$space forms in [OG-2014]. J. T. Cho, J. Inoguchi and J.E. Lee defined and studied slant curves in Sasakian 3-manifolds in [CIL-2006].

Motivated by these studies, in the present talk, we focus our interest on biharmonic slant curves in $\mathcal{S}-$space forms. We find curvature characterizations of these special curves in four cases.

Let (M,g) and (N,h) be two Riemannian manifolds and $\phi:(M,g)\to(N,h)$ a smooth map. The energy functional of ϕ is defined by

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2\, v_g.$$

The critical points of the energy functional $E(\phi)$ are called harmonic [Eells-Sampson-1964]. The Euler-Lagrange equation gives the harmonic map equation

$$\tau(\phi)=trace\nabla d\phi=0,$$

where $\tau(\phi)$ is called the tension field of ϕ .

The bienergy functional of ϕ is given by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g.$$

A biharmonic map is a critical point of $E_2(\phi)$. The Euler-Lagrange equation of $E_2(\phi)$ gives the biharmonic map equation

$$\tau_2(\phi) = -J^{\phi}(\tau(\phi)) = -\Delta\tau(\phi) - traceR^N(d\phi, \tau(\phi))d\phi = 0,$$

where J^{ϕ} is the Jacobi operator of ϕ. $\tau_2(\phi)$ is called the bitension field of ϕ [Jiang-1986].

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In a different setting, in [Chen-1996], B.Y. Chen defined a biharmonic submanifold $M \subset \mathbb{E}^n$ of the Euclidean space as its mean curvature vector field H satisfies $\Delta H = 0$, where Δ is the Laplacian.

$\mathcal{S}-$space form and its submanifolds

Let (M, g) be a $(2m+s)$-dimensional framed metric manifold [Yano-Kon-1984] with a framed metric structure $(f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, that is, f is a $(1,1)$ tensor field defining an f-structure of rank $2m$; $\xi_1, ..., \xi_s$ are vector fields; $\eta^1, ..., \eta^s$ are 1-forms and g is a Riemannian metric on M such that for all $X, Y \in TM$ and $\alpha, \beta \in \{1, ..., s\}$,

$$f^2 = -I + \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0 \tag{1}$$

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2}$$

$$d\eta^\alpha(X, Y) = g(X, fY) = -d\eta^\alpha(Y, X), \quad \eta^\alpha(X) = g(X, \xi). \tag{3}$$

$(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$ is also called framed f-manifold [Nakagawa-1966] or almost r-contact metric manifold [Vanzura-1972].

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If the Nijenhuis tensor of f equals $-2d\eta^\alpha \otimes \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$, then $(f, \xi_\alpha, \eta^\alpha, g)$ is called $\mathcal{S}$-structure [Blair-1970].

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If a framed metric structure on M is an $\mathcal{S}$-structure, then the following equations hold [Blair-1970]:

$$(\nabla_X f)Y = \sum_{\alpha=1}^{s} \left\{ g(fX, fY)\xi_\alpha + \eta^\alpha(Y)f^2X \right\}, \quad (4)$$

$$\nabla \xi_\alpha = -f, \ \alpha \in \{1, ..., s\}. \quad (5)$$

A plane section in T_pM is an f-section if there exist a vector $X \in T_pM$ orthogonal to $\xi_1, ..., \xi_s$ such that $\{X, fX\}$ span the section. The sectional curvature of an f-section is called an f-sectional curvature. In an $\mathcal{S}$-manifold of constant f-sectional curvature, the curvature tensor R of M is of the form

$$\begin{aligned} R(X,Y)Z = \sum_{\alpha,\beta} \{ &\eta^\alpha(X)\eta^\beta(Z)f^2Y - \eta^\alpha(Y)\eta^\beta(Z)f^2X \\ &-g(fX,fZ)\eta^\alpha(Y)\xi_\beta + g(fY,fZ)\eta^\alpha(X)\xi_\beta\} \\ &+\tfrac{c+3s}{4}\{-g(fY,fZ)f^2X + g(fX,fZ)f^2Y\} \\ &\tfrac{c-s}{4}\{g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ\}, \end{aligned} \tag{6}$$

for all $X, Y, Z \in TM$ [CFF-1993]. An $\mathcal{S}$-manifold of constant f-sectional curvature c is called an $\mathcal{S}$-space form which is denoted by $M(c)$.

When $s = 1$, an $\mathcal{S}$-space form becomes a Sasakian space form [Blair-2002].

A submanifold of an $\mathcal{S}$-manifold is called an integral submanifold if $\eta^{\alpha}(X) = 0$, $\alpha = 1, ..., s$, for every tangent vector X [KDT-2007]. We call a 1-dimensional integral submanifold of an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_{\alpha}, \eta^{\alpha}, g)$ a Legendre curve of M. In other words, a curve $\gamma : I \to M = (M^{2m+s}, f, \xi_{\alpha}, \eta^{\alpha}, g)$ is called a Legendre curve if $\eta^{\alpha}(T) = 0$, for every $\alpha = 1, ...s$, where T is the tangent vector field of γ.

Let γ be a unit-speed curve in an $\mathcal{S}$-manifold $(M^{2m+s}, f, \xi_{\alpha}, \eta^{\alpha}, g)$. We call γ a slant curve, if there exists a constant angle θ such that $\eta^{\alpha}(T) = \cos\theta$, for all $\alpha = 1, ...s$. Here, θ is called the contact angle of γ. Every Legendre curve is slant with contact angle $\frac{\pi}{2}$.

We can give the following essential proposition for slant curves:

Proposition 1

Let $M = (M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$ be an $\mathcal{S}$-manifold. If θ is the contact angle of a non-geodesic unit-speed slant curve in M, then

$$\frac{-1}{\sqrt{s}} < \cos\theta < \frac{1}{\sqrt{s}}.$$

Biharmonic Slant curves in $\mathcal{S}$-Space Forms

Let $\gamma : I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). If there exists orthonormal vector fields $E_1, E_2, ..., E_r$ along γ such that

$$\begin{aligned} E_1 &= \gamma' = T, \\ \nabla_T E_1 &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\ &\dots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned} \tag{7}$$

then γ is called a Frenet curve of osculating order r, where $\kappa_1, ..., \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$.

A Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 is called a circle if κ_1 is a non-zero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a helix of order r if $\kappa_1, ..., \kappa_{r-1}$ are non-zero positive constants; a helix of order 3 is shortly called a helix.

Now let $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$ be an $\mathcal{S}$-space form and $\gamma : I \to M$ a slant curve of osculating order r. Differentiating

$$\eta^\alpha(T) = \cos\theta \tag{8}$$

and using (7), we find

$$\eta^\alpha(E_2) = 0, \;\; \alpha \in \{1, ..., s\}. \tag{9}$$

Then, (1) and (9) give us

$$f^2 E_2 = -E_2. \tag{10}$$

By the use of (1), (2), (3), (6), (7), (9) and (10), it can be seen that

$$\nabla_T \nabla_T T = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T &= -3\kappa_1 \kappa_1' E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2\right) E_2 \\ &\quad + \left(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2'\right) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4, \end{aligned}$$

$$\begin{aligned} R(T, \nabla_T T)T &= -\kappa_1 \left[s^2 \cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) \right] E_2 \\ &\quad -3\kappa_1 \frac{(c-s)}{4} g(fT, E_2) fT. \end{aligned}$$

So we have

$$\begin{aligned}\tau_2(\gamma) &= \nabla_T\nabla_T\nabla_T T - R(T,\nabla_T T)T \\ &= -3\kappa_1\kappa_1' E_1 \\ &\quad + \left\{\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 \right. \\ &\qquad \left. + \kappa_1\left[s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta)\right]\right\} E_2 \\ &\quad +(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 \\ &\quad +3\kappa_1\frac{(c-s)}{4}g(fT,E_2)fT. \end{aligned} \tag{11}$$

So we have

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Let $k = \min\{r, 4\}$. From (11), the curve γ is proper biharmonic if and only if $\kappa_1 > 0$ and
(1) $c = s$ or $fT \perp E_2$ or $fT \in span\{E_2, ..., E_k\}$; and
(2) $g(\tau_2(\gamma), E_i) = 0$, for any $i = 1, ..., k$.
So we can state the following theorem:

Theorem 2

Let γ be a slant curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$ and $k = \min\{r, 4\}$. Then γ is proper biharmonic if and only if

(1) *$c = s$ or $fT \perp E_2$ or $fT \in span\{E_2, ..., E_k\}$; and*
(2) *the first k of the following equations are satisfied (replacing $\kappa_k = 0$):*

$$\kappa_1 = constant > 0,$$
$$\kappa_1^2 + \kappa_2^2 = s^2 \cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) + \frac{3(c-s)}{4}\left[g(fT, E_2)\right]^2,$$
$$\kappa_2' + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_3) = 0,$$
$$\kappa_2\kappa_3 + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_4) = 0.$$

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(1) $c = s$ or $fT \perp E_2$ or $fT \in span\{E_2, ..., E_k\}$; and
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$$\kappa_2\kappa_3 + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_4) = 0.$$

Now we give the interpretations of Theorem 2.

Case I: $c = s$.

In this case γ is proper biharmonic if and only if

$$\kappa_1 = \text{constant} > 0,$$
$$\kappa_1^2 + \kappa_2^2 = s,$$
$$\kappa_2 = \text{constant},$$
$$\kappa_2 \kappa_3 = 0.$$

Theorem 3

Let γ be a slant curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c = s$. Then γ is proper biharmonic if and only if either γ is a circle with $\kappa_1 = \sqrt{s}$, or a helix with $\kappa_1^2 + \kappa_2^2 = s$. Moreover, if γ is Legendre, then $2m + s > 3$.

Remark 4

If $2m + s = 3$, then $m = s = 1$. So M is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [CB − 1994]), we can write $\kappa_1 > 0$ and $\kappa_2 = 1$, which contradicts $\kappa_1^2 + \kappa_2^2 = s = 1$. Hence γ cannot be proper biharmonic.

Case II: $c \neq s$, $fT \perp E_2$.

In this case, $g(fT, E_2) = 0$. From the main Theorem, we obtain

$$\begin{gathered}\kappa_1 = \text{constant} > 0, \\ \kappa_1^2 + \kappa_2^2 = s^2 \cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta), \\ \kappa_2 = \text{constant}, \\ \kappa_2\kappa_3 = 0.\end{gathered} \tag{12}$$

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Proposition 5

Let γ be a slant curve of osculating order 3 in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$ and $fT \perp E_2$. Then $\{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent at any point of γ. Therefore $m \geq 3$.

Now we can state the following Theorem:

Theorem 6

Let γ be a slant curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c \neq s$ and $fT \perp E_2$. Then γ is proper biharmonic if and only if either
(1) $m \geq 2$ and γ is a circle with $\kappa_1 = \frac{1}{2}\sqrt{c + 3s - (c - s)s\cos^2\theta}$, where $c > -3s + (c - s)s\cos^2\theta$ and $\{T = E_1, E_2, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent; or
(2) $m \geq 3$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = \frac{c+3s-(c-s)s\cos^2\theta}{4}$, where $c > -3s + (c - s)s\cos^2\theta$ and $\{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent.

Case III: $c \neq s$, $fT \parallel E_2$

In this case, $fT = \pm\sqrt{1 - s\cos^2\theta}E_2, g(fT, E_2) = \pm(1 - s\cos^2\theta)$, $g(fT, E_3) = 0$ and $g(fT, E_4) = 0$. From Theorem 2, γ is biharmonic if and only if

$$\begin{aligned}
\kappa_1 &= \text{constant} > 0, \\
\kappa_1^2 + \kappa_2^2 &= c - s\cos^2\theta(c - s), \\
\kappa_2 &= \text{constant}, \\
\kappa_2\kappa_3 &= 0.
\end{aligned}$$

Case III: $c \neq s$, $fT \parallel E_2$

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We can assume that $fT = \sqrt{1 - s\cos^2\theta}E_2$. From equation (1), we get

$$\sqrt{1 - s\cos^2\theta}fE_2 = f^2T = -T + \sum_{\alpha=1}^{s}\eta^\alpha(T)\xi_\alpha = -T + \cos\theta\sum_{\alpha=1}^{s}\xi_\alpha. \tag{13}$$

From (13), we find

$$\begin{aligned}\nabla_T fT &= -s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha \\ &\quad +\kappa_1\left[\frac{-1}{\sqrt{1-s\cos^2\theta}}T + \frac{\cos\theta}{\sqrt{1-s\cos^2\theta}}\sum_{\alpha=1}^{s}\xi_\alpha\right] \quad (14) \\ &= \sqrt{1-s\cos^2\theta}(-\kappa_1 T + \kappa_2 E_3).\end{aligned}$$

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$$\begin{aligned} \nabla_T fT &= -s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha \\ &\quad +\kappa_1\left[\frac{-1}{\sqrt{1-s\cos^2\theta}}T + \frac{\cos\theta}{\sqrt{1-s\cos^2\theta}}\sum_{\alpha=1}^{s}\xi_\alpha\right] \quad (14) \\ &= \sqrt{1-s\cos^2\theta}(-\kappa_1 T + \kappa_2 E_3). \end{aligned}$$

Using (14), we can write

$$\left(1+\frac{\kappa_1\cos\theta}{\sqrt{1-s\cos^2\theta}}\right)\left(-s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha\right) = \kappa_2\sqrt{1-s\cos^2\theta}E_3, \quad (15)$$

which gives us the following Theorem:

Theorem 7

Let γ be a slant curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c \neq s$ and $fT \parallel E_2$. Then γ is proper biharmonic if and only if it is one of the following:
i) a Legendre helix with the Frenet frame field

$$\left\{ T, fT, \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha \right\}$$

and $\kappa_1 = \sqrt{c-s}$ and $\kappa_2 = \sqrt{s}$, where $c > s$;

ii) a non-Legendre slant circle with the Frenet frame field

$$\left\{T, \frac{fT}{\sqrt{1-s\cos^2\theta}}\right\}$$

and

$$\kappa_1 = \frac{-\sqrt{1-s\cos^2\theta}}{\cos\theta} = \sqrt{c-s\cos^2\theta(c-s)};$$

iii) a non-Legendre slant helix with the Frenet frame field

$$\left\{T, \frac{fT}{\sqrt{1-s\cos^2\theta}}, \frac{1}{\sqrt{s}\sqrt{s\cos^2\theta-\cos(2\theta)}}\left(\sum_{\alpha=1}^{s}\xi_\alpha - s\cos\theta\, T\right)\right\}$$

and

$$\kappa_1^2+\kappa_2^2 = c-s\cos^2\theta(c-s).$$

Thus, we can give the following corollary for Legendre curves:

Corollary 8

Let γ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, $c \neq s$ and $fT \parallel E_2$. Then

$$\left\{ T, fT, \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha \right\}$$

is the Frenet frame field of γ and γ is proper biharmonic if and only if it is a helix with $\kappa_1 = \sqrt{c-s}$ and $\kappa_2 = \sqrt{s}$, where $c > s$. If $c \leq s$, then γ is biharmonic if and only if it is a geodesic [OG – 2014].

Case IV: $c \neq s$, $fT \nparallel E_2$ and $g(fT, E_2) \neq 0$.

Now, let $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$ be an $\mathcal{S}$-space form, $\alpha \in \{1, ..., s\}$ and $\gamma : I \to M$ a slant curve of osculating order r, where $4 \leq r \leq 2m + s$ and $m \geq 2$. If γ is biharmonic, then $fT \in span\{E_2, E_3, E_4\}$. Let $\mu(t)$ denote the angle function between fT and E_2, that is, $g(fT, E_2) = \sqrt{1 - s\cos^2\theta}\cos\mu(t)$. Differentiating $g(fT, E_2)$ along γ and using (1), (3), (7), we find

$$\begin{aligned}
-\sqrt{1 - s\cos^2\theta}\mu'(t)\sin\mu(t) &= \nabla_T g(fT, E_2) \\
&= g(\nabla_T fT, E_2) + g(fT, \nabla_T E_2) \\
&= g(-s\cos\theta T + \sum_{\alpha=1}^{s}\xi_\alpha + \kappa_1 fE_2, E_2) \\
&\quad + g(fT, -\kappa_1 T + \kappa_2 E_3) \\
&= \kappa_2 g(fT, E_3). \qquad (16)
\end{aligned}$$

If we write $fT = g(fT, E_2)E_2 + g(fT, E_3)E_3 + g(fT, E_4)E_4$, Theorem 2 gives us

$$\kappa_1 = \text{constant} > 0,$$
$$\kappa_1^2 + \kappa_2^2 = s^2 \cos^2\theta + \tfrac{c+3s}{4}(1 - s\cos^2\theta) + \tfrac{3(c-s)}{4}\left[g(fT, E_2)\right]^2,$$
$$\kappa_2' + \tfrac{3(c-s)}{4} g(fT, E_2) g(fT, E_3) = 0,$$
$$\kappa_2\kappa_3 + \tfrac{3(c-s)}{4} g(fT, E_2) g(fT, E_4) = 0.$$

If we multiply the third equation of the above system with $2\kappa_2$, using (16), we obtain

$$2\kappa_2\kappa_2' + \sqrt{1 - s\cos^2\theta}\frac{3(c-s)}{4}(-2\mu'\cos\mu\sin\mu) = 0,$$

which is equivalent to

$$\kappa_2^2 = -\sqrt{1 - s\cos^2\theta}\frac{3(c-s)}{4}\cos^2\mu + \omega_0, \tag{17}$$

where ω_0 is a constant. If we write (17) in the second equation, we have

$$\begin{aligned} \kappa_1^2 &= s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) \\ &+ \frac{3(c-s)}{4}\left(1 - s\cos^2\theta + \sqrt{1 - s\cos^2\theta}\right)\cos^2\mu + \omega_0. \end{aligned}$$

Thus μ is a constant. From (16) and (17), we find $g(fT, E_3) = 0$ and κ_2 =constant> 0. Since $\|fT\| = \sqrt{1 - s\cos^2\theta}$ and $fT = \sqrt{1 - s\cos^2\theta}\cos\mu E_2 + g(fT, E_4)E_4$, we get $g(fT, E_4) = \sqrt{1 - s\cos^2\theta}\sin\mu$. From the assumption $fT \nparallel E_2$ and $g(fT, E_2) \neq 0$, it is clear that $\mu \in (0, 2\pi)\setminus\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Now we can state the following Theorem:

Theorem 9

Let $\gamma : I \to M$ be a slant curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, where $r \geq 4$, $m \geq 2$, $c \neq s$, $fT \nparallel E_2$ and $g(fT, E_2) \neq 0$. Then γ is proper biharmonic if and only if

$$\begin{aligned}
\kappa_i &= constant > 0, \;\; i \in \{1, 2, 3\}, \\
\kappa_1^2 + \kappa_2^2 &= s^2 \cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) \\
&\quad + \frac{3(c-s)}{4}(1 - s\cos^2\theta)\cos^2\mu, \\
\kappa_2\kappa_3 &= \frac{3(s-c)}{8}(1 - s\cos^2\theta)\sin 2\mu,
\end{aligned}$$

where $fT = \sqrt{1 - s\cos^2\theta}\cos\mu E_2 + \sqrt{1 - s\cos^2\theta}\sin\mu E_4$, $\mu \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant.

Corollary 10

Let $\gamma : I \to M$ be a Legendre curve of osculating order r in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, ..., s\}$, where $r \geq 4$, $m \geq 2$, $c \neq s$, $g(fT, E_2)$ is not constant 0, 1 or -1. Then γ is proper biharmonic if and only if

$$\begin{aligned} \kappa_i &= constant > 0, \ i \in \{1, 2, 3\}, \\ \kappa_1^2 + \kappa_2^2 &= \frac{1}{4}\left[c + 3s + 3(c - s)\cos^2 \mu\right], \\ \kappa_2\kappa_3 &= \frac{3(s - c)\sin 2\mu}{8}, \end{aligned}$$

where $c > -3s$, $fT = \cos \mu E_2 + \sin \mu E_4$, $\mu \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant such that $c + 3s + 3(c - s)\cos^2 \mu > 0$ and $3(s - c)\sin 2\mu > 0$. If $c \leq -3s$, then γ is biharmonic if and only if it is a geodesic [OG-2014].

Slant Curves in $\mathbb{R}^{2n+s}(-3s)$

Let us consider $M=\mathbb{R}^{2n+s}$ with coordinate functions $\{x_1, ...x_n, y_1, ..., y_n, z_1, ..., z_s\}$ and define

$$\xi_\alpha = 2\frac{\partial}{\partial z_\alpha},\ \alpha = 1, ..., s,$$

$$\eta^\alpha = \frac{1}{2}\left(dz_\alpha - \sum_{i=1}^{n} y_i dx_i\right),\ \alpha = 1, ..., s,$$

$$fX = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} X_i \frac{\partial}{\partial y_i} + \left(\sum_{i=1}^{n} Y_i y_i\right)\left(\sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}\right),$$

$$g=\sum_{\alpha=1}^{s}\eta^{\alpha}\otimes\eta^{\alpha}+\frac{1}{4}\sum_{i=1}^{n}\left(dx_i\otimes dx_i+dy_i\otimes dy_i\right),$$

where

$$X=\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right)+\sum_{\alpha=1}^{s}\left(Z_\alpha\frac{\partial}{\partial z_\alpha}\right)\in\chi(M).$$

It is known that $\left(\mathbb{R}^{2n+s},f,\xi_\alpha,\eta^\alpha,g\right)$ is an $\mathcal{S}$-space form with constant f-sectional curvature $c=-3s$ and it is denoted by $\mathbb{R}^{2n+s}(-3s)$ [Hasegawa-1986].

The vector fields

$$X_i = 2\frac{\partial}{\partial y_i},\ X_{n+i} = fX_i = 2(\frac{\partial}{\partial x_i} + y_i \sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}),\ \xi_\alpha = 2\frac{\partial}{\partial z_\alpha}$$

form a g-orthonormal basis and the Levi-Civita connection is calculated as

$$\nabla_{X_i} X_j = \nabla_{X_{n+i}} X_{n+j} = 0, \nabla_{X_i} X_{n+j} = \delta_{ij} \sum_{\alpha=1}^{s} \xi_\alpha, \nabla_{X_{n+i}} X_j = -\delta_{ij} \sum_{\alpha=1}^{s} \xi_\alpha,$$

$$\nabla_{X_i} \xi_\alpha = \nabla_{\xi_\alpha} X_i = -X_{n+i}, \nabla_{X_{n+i}} \xi_\alpha = \nabla_{\xi_\alpha} X_{n+i} = X_i.$$

(see [Hasegawa-1986]).

Let $\gamma : I \to \mathbb{R}^{2n+s}(-3s)$ be a slant curve with contact angle θ. Let us denote

$$\gamma(t) = (\gamma_1(t), ..., \gamma_n(t), \gamma_{n+1}(t), ..., \gamma_{2n}(t), \gamma_{2n+1}(t), ..., \gamma_{2n+s}(t)),$$

where t is the arc-length parameter. The tangent vector field of γ is

$$\begin{aligned} T &= \gamma_1' \frac{\partial}{\partial x_1} + ... + \gamma_n' \frac{\partial}{\partial x_n} + \gamma_{n+1}' \frac{\partial}{\partial y_1} + ... + \gamma_{2n}' \frac{\partial}{\partial y_n} \\ &\quad + \gamma_{2n+1}' \frac{\partial}{\partial z_1} + ... + \gamma_{2n+s}' \frac{\partial}{\partial z_\alpha}. \end{aligned}$$

In terms of the g-orthonormal basis, T can be written as

$$\begin{aligned} T &= \frac{1}{2}\left[\gamma'_{n+1}X_1 + \ldots + \gamma'_{2n}X_n + \gamma'_1 X_{n+1} + \ldots + \gamma'_n X_{2n}\right. \\ &\quad + \left(\gamma'_{2n+1} - \gamma'_1\gamma_{n+1} - \ldots - \gamma'_n\gamma_{2n}\right)\xi_1 + \ldots \\ &\quad \left. + \left(\gamma'_{2n+s} - \gamma'_1\gamma_{n+1} - \ldots - \gamma'_n\gamma_{2n}\right)\xi_s\right]. \end{aligned}$$

In terms of the g-orthonormal basis, T can be written as

$$\begin{aligned} T &= \frac{1}{2}\left[\gamma'_{n+1}X_1 + \ldots + \gamma'_{2n}X_n + \gamma'_1 X_{n+1} + \ldots + \gamma'_n X_{2n}\right. \\ &\quad + \left(\gamma'_{2n+1} - \gamma'_1\gamma_{n+1} - \ldots - \gamma'_n\gamma_{2n}\right)\xi_1 + \ldots \\ &\quad \left. + \left(\gamma'_{2n+s} - \gamma'_1\gamma_{n+1} - \ldots - \gamma'_n\gamma_{2n}\right)\xi_s\right]. \end{aligned}$$

Since γ is slant, we obtain

$$\eta^{\alpha}(T) = \frac{1}{2}\left(\gamma'_{2n+\alpha} - \gamma'_1\gamma_{n+1} - \ldots - \gamma'_n\gamma_{2n}\right) = \cos\theta$$

for all $\alpha = 1, \ldots, s$. Thus, we have

$$\gamma'_{2n+1} = \ldots = \gamma'_{2n+s} = \gamma'_1\gamma_{n+1} + \ldots + \gamma'_n\gamma_{2n} + 2\cos\theta.$$

Since γ is a unit-speed curve, we can write

$$\left(\gamma_1'\right)^2 + ... + \left(\gamma_{2n}'\right)^2 = 4\left(1 - s\cos^2\theta\right).$$

Now we can give the following examples:

Since γ is a unit-speed curve, we can write

$$\left(\gamma_1'\right)^2 + ... + \left(\gamma_{2n}'\right)^2 = 4\left(1 - s\cos^2\theta\right).$$

Now we can give the following examples:

Example 1

Let $n = 1$ and $s = 2$. Then, $\gamma : I \to \mathbb{R}^4(-6)$, $\gamma(t) = \left(\sqrt{2}t, 0, t, t\right)$ is a slant circle with contact angle $\frac{\pi}{3}$.

Example 2

The curve $\gamma : I \to \mathbb{R}^4(-6)$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ is a slant curve with contact angle θ, where

$$\gamma_1(t) = c_1 + 2\sqrt{-\cos 2\theta} \int_{t_0}^{t} \cos u(p)dp,$$

$$\gamma_2(t) = c_2 + 2\sqrt{-\cos 2\theta} \int_{t_0}^{t} \sin u(p)dp,$$

$$\gamma_3(t) = \gamma_4(t) + c_3 = c_4 + 2t\cos\theta$$

$$+ 2\sqrt{-\cos 2\theta} \int_{t_0}^{t} \cos u(q) \left(c_2 + 2\sqrt{-\cos 2\theta} \int_{t_0}^{q} \sin u(p)dp \right) dq,$$

$$\cos\theta \in \left(-1/\sqrt{2}, 1/\sqrt{2}\right),$$

$t_0 \in I$, c_1, c_2, c_3 and c_4 are arbitrary constants.

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Thank you...