Transverse conformal Killing forms on foliated manifolds

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Abstract & Keyword

- **Abstract**
  We study transverse Killing vector, forms on foliations and prove some vanishing theorem for foliations.
- **Keyword**
  Transverse Killing vector field, Transverse coformal vector field, Transverse Killing form, Transverse conformal form

Foliation

Definition

A **codimension** q **foliation** $\mathcal{F}$ on M is given by an open cover (U_j), submersion $f_j : U_j \to N$ over a q-dimensional transverse manifold N and, for $U_i \cap U_j \neq \emptyset$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \subset N \to f_j(U_i \cap U_j) \subset N$ satisfying

$$f_j(x) = \gamma_{ij} \circ f_i(x) \quad x \in U_i \cap U_j.$$

We say that $\{U_j, f_j, N, \gamma_{ij}\}$ is a **foliated cocycle** defining $\mathcal{F}$.

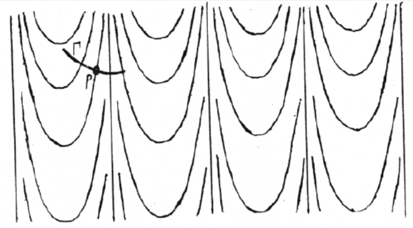
- Roughly speaking, a foliation corresponds to a decomposition of a manifold into a union of connected submanifolds, which are called **leaves** .

Examples

(1) $M = \mathbb{R}^2 - \{0\}$ and $L_r = \{(x,y)|x^2+y^2=r^2\}$. Then $\mathcal{F} = \{L_f\}$.

(2) $M = \mathbb{R}^2$ and $L_a = \{(x,y)|y = x^2 + a\}$.

(3) $M = \mathbb{R}^2$ and $L_a = \{(x,y)|y = \ln|\sec x| + a\}$ together with the vertical lines $\cos x = 0$. Equivalently, the solution of $\frac{dy}{dx} = \tan x$ is a foliation L_a.



(4) Let $M = D^2 \times S^1$ $(D^2 = \{(x, y) | x^2 + y^2 \leqslant 1\})$ and for $0 \leqslant a < 1$,

$$L_a = \{(x, e^{i2\pi(a+f(|x|))}) | x \in \mathrm{Int}(D^2)\},$$
$$\partial(D^2 \times S^1) = S^1 \times S^1 = T^2.$$

Then $\mathcal{F} \equiv \{L_a, T^2\}$ is a codimension 1 foliation of $D^2 \times S^1$. In this case, L_a is diffeomorphic to $\mathbb{R}^2$ and T^2 is the only compact leaf. This is called a **Reeb foliation** of the solid torus $D^2 \times S^1$.

(5) Let $S^3 = \{(z,w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1\}$. Let two solid torus be

$$S^3_+ = \{(z,w) \in S^3 | |z|^2 \geqslant \frac{1}{2}\} \cong D^2 \times S^1,$$
$$S^3_- = \{(z,w) \in S^3 | |z|^2 \leqslant \frac{1}{2}\} \cong D^2 \times S^1.$$

Then $S^3 = S^3_+ \cup S^3_- \cong (D^2 \times S^1) \cup (D^2 \times S^1)$ by pasting the boundaries $\partial(D^2 \times S^1)$. And $S^3_+ \cap S^3_- = T^2$. A foliation on S^3 is obtained from **Reeb foliations** $\{L_a\}$ in (9) and one compact leaf T^2.

(6) (**Submersion**) A smooth submersion $f : M \to B$ is a map of manifolds with a surjective derivative map at every point of M.

(7) An ordinary manifold can be considered as a foliated manifold with the point foliation.

Known facts

- A nowhere zero differential 1-form ω defines a codimension one foliation on M if L is integrable, where $L_x = Ker\omega_x$, i.e. $\omega \wedge d\omega = 0$ (integrable condition). (For example, Level hypersurfaces)

- Any compact manifold M admits a one dimensional foliation if and only if the Euler characteristic $\chi(M) = 0$.

- Every closed manifold M with $\chi(M) = 0$ admits a codimension one foliation (Thurston,1974).

Leaf space

- Define $x \sim y$ in $M \iff x$ and y are in the same leaf.
- Then $M/\mathcal{F} := M/\sim$, endowed with the quotient topology. This is called as the **leaf space** of $\mathcal{F}$.
- Generally, $M/\mathcal{F}$ is not a manifold. But we can define on $M/\mathcal{F}$ many geometrical objects like functions, diffenential forms, differential operators etc. They correspond to their analogues on M invariant along the leaves.
- The **tangential geometry** is infinitesimally modeled by the leaves. And the **transversal geometry** is infinitesimally modeled by the leaf space, which plays a central role in the current research.

Riemannian foliation

- Let $T\mathcal{F}$ be the tangent bundle of $\mathcal{F}$ and $Q = TM/T\mathcal{F}$ the normal bundle of $\mathcal{F}$. Then we have the exact sequence of vector bundles

$$0 \to T\mathcal{F} \to TM \stackrel{\pi}{\to} Q \to 0. \tag{1}$$

- $\mathcal{F}$ is a **Riemannian foliation** if there exists a metric g_Q on Q satisfying $\overset{\circ}{\nabla}_X g_Q = 0$ for any $X \in T\mathcal{F}$. where $\overset{\circ}{\nabla}$ is the partial Bott connection in Q.

- The property $\mathcal{F}$ is Riemannian means that the leaf space $M/\mathcal{F}$ is a Riemannian manifold even if $M/\mathcal{F}$ does not support any diffenentiable structure.

Bundle-like metric

- Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a Riemannian metric g_M.
- g_M is a **bundle-like metric** $\iff$ All geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.
- A Riemannian foliation admits a bundle-like metric.
- Let M be a foliated manifold and complete in a bundle-like metric. Let $\mathcal{F}$ be a codimension 1-foliation. If one leaf is compact, then every leaf is compact.
- Not all foliations have bundle-like metrics.

Transversal vector fields

- A vector field Y on M is an **transversal infinitesimal automorphism** if its flow preserves the leaves. That is, $[Y, Z] \in T\mathcal{F}$ for all $Z \in T\mathcal{F}$.

- An infinitesimal automorphism Y is called a **transversal Killing field** (or **transversal conformal field**) if Y satisfies $\theta(Y)g_Q = 0$ (or $\theta(Y)g_Q = 2f_Y g_Q$ for a basic function f_Y depending on Y).

- A transversal Killing (or conformal) field Y preserves the transverse metric, i.e., transversally isometric (or the conformal class of the transverse metric).

Basic forms

- A differential form $\omega \in \Omega^r(M)$ is **basic**, if

$$i(X)\omega = 0, \ \theta(X)\omega = 0 \ \forall X \in \Gamma L.$$

- Let $\Omega_B^*(\mathcal{F})$ be the space of all basic forms on M. Then $d : \Omega_B^r \to \Omega_B^{r+1}$ and $d^2 = 0$. So the **basic cohomology** is given by

$$H_B^r(\mathcal{F}) = H(\Omega_B(\mathcal{F}), d_B), \quad d_B = d|_{\Omega_B}.$$

- $H_B^1(\mathcal{F}) \to H_{DR}^1(M)$: injective (Tondeur, 1977).

- The basic cohomology plays the role of the De Rham cohomology of the leaf space of the foliation.

Basic Laplacian

- Let δ_B the formal adjoint of $d_B = d|_{\Omega_B}$. Generally, $\delta_B \neq \delta|_{\Omega_B}$, but for any $\phi \in \Omega_B^1$, $\delta_B \phi = \delta\phi$.

- The **basic Laplacian** is given by $\Delta_B = d_B \delta_B + \delta_B d_B$.

- (El Kacimi-Hector-Sergiescu, 1985) Let M be a closed manifold. Then

$$\Omega_B^r(\mathcal{F}) \cong \mathcal{H}_B^r \oplus \mathrm{im} d_B \oplus \mathrm{im} \delta_B,$$

  with finite dimensional $\mathcal{H}_B^r = \{\phi \in \Omega_B^r | \Delta_B \phi = 0\}$.

- (Kamber-Tondeur, 1997) $H_B^r(\mathcal{F}) \cong \mathcal{H}_B^r$.

Transverse conformal Killing forms

- A basic r-form ϕ is said to be a **transverse conformal Killing form** if

$$\nabla_X \phi = \frac{1}{r+1} i(X) d_B \phi - \frac{1}{q-r+1} X^* \wedge \delta_T \phi \qquad (2)$$

for any $X \in T\mathcal{F}^\perp$, where $\delta_T = \delta_B - i(\kappa^\sharp)$.

- A basic r-form ϕ is a **transverse Killing form** if

$$\nabla_X \phi = \frac{1}{r+1} i(X) d_B \phi \qquad (3)$$

for any $X \in T\mathcal{F}^\perp$.

Remarks

- Note that a transverse conformal Killing 1-form (or Killing 1-form) is a dual form of a transversal conformal (or Killing) vector field.
- Also, on a transverse spin foliation, transverse conformal Killing forms (or Killing forms) are related to transversal twistor spinors, i.e., $\nabla_X\psi = -\frac{1}{q}X\cdot D_b\psi$ (or Killing spinors, i.e., $\nabla_X\psi = \mu X\cdot\psi$). Here D_b is a basic Dirac operator on $(M,\mathcal{F})$.

The curvature operator

- Let F be the curvature endomorphism, which is defined by

$$F(\phi) = \sum_{a,b=1}^{q} \theta^a \wedge i(E_b) R^Q(E_b, E_a)\phi, \tag{4}$$

where R^Q is the curvature tensor on $\Omega_B^r(\mathcal{F})$ induced by the connection on Q.

- For any basic 1-form ϕ, $F(\phi)^\sharp = \mathrm{Ric}^Q(\phi^\sharp)$.
- The operator A_Y is defined by

$$A_Y\phi = \theta(Y)\phi - \nabla_Y\phi. \tag{5}$$

Generalized Weitzenbock formula

- The **generalized Weitzenböck formula** is given by

$$\Delta_B \phi = \nabla_{tr}^* \nabla_{tr} \phi + F(\phi) + A_{\kappa^\sharp} \phi, \tag{6}$$

where $\nabla_{tr}^* \nabla_{tr} = -\sum_{a=1}^{q} \nabla^2_{E_a, E_a} + \nabla_{\kappa^\sharp}$ and κ is the **mean curvature form** of $\mathcal{F}$.

- Assume that F is positive definite. Then

$$H_B^r(\mathcal{F}) = 0.$$

- Assume that the transversal Ricci curvature Ric^Q is positive definite. Then $H_B^1(\mathcal{F}) = 0$.

Formulas

- Let ϕ be a transverse conformal Killing r-form. Then

$$F(\phi) = \frac{r}{r+1}\delta_T d_B \phi + \frac{r^*}{r^*+1} d_B \delta_T \phi, \tag{7}$$

where $r^* = q - r$.

- If ϕ is a transverse Killing r-form, then

$$F(\phi) = \frac{r}{r+1}\delta_T d_B \phi, \tag{8}$$

or

$$\Delta_B \phi = \frac{r+1}{r} F(\phi) + \theta(\kappa^\sharp)\phi. \tag{9}$$

Vanishing theorems

Theorem(Jung-Richardson, 2012)

Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold.
(i) Assume that $F \leqslant 0$. Then any transverse conformal Killing r $(1 \leqslant r \leqslant q-1)$-forms are parallel.
(ii) In addition, if $F < 0$ at some point, then there are no transverse conformal Killing r-forms on M.

Corollary. Assume the transversal Ricci curvature Ric^Q is negative definite. Then there are no transversal conformal fields (of course, Killing fields) on M.

Kahler foliation

- **Kähler foliation** $\mathcal{F}$ satisfies the following three conditions;
  (i) $\mathcal{F}$ is Riemannian,
  (ii) there is an almost complex structure $J : Q \to Q$ such that

$$g_Q(JX, JY) = g_Q(X, Y)\ \forall X, Y \in Q. \tag{10}$$

  (iii) $\nabla J = 0$.

  Examples. (1) Sasakian manifold (M^{2n+1}, g) is a Kähler foliation with one dimensional foliation generated by the structure vector.
  (2) The generalized Hopf-fiberation $S^{2n+1} \to \mathbb{C}P^n$ is an example of a Kähler foliation with constant (transversal) holomorphic sectional curvature.

Kähler form

- Note that
$$\Omega(X,Y) = g_Q(X, JY) \tag{11}$$
defines a basic Kähler 2-form Ω, which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$.

- Then Ω is given by
$$\Omega = \sum_{k=1}^{n} \theta^{2k-1} \wedge \theta^{2k} = -\frac{1}{2}\sum_{k=1}^{2n} \theta^k \wedge J\theta^k, \tag{12}$$
where θ^a is a g_Q-dual 1-form to E_a on M.

Operators on Kähler foliation

- Let $L : \Omega_B^r \to \Omega_B^{r+2}$ and $\Lambda : \Omega_B^r \to \Omega_B^{r-2}$ be given by

$$L(\phi) = \Omega \wedge \phi, \quad \Lambda(\phi) = -\frac{1}{2}\sum_{a=1}^{2m} i(JE_a)i(E_a)\phi. \qquad (13)$$

- Let $J : \Omega_B^r \to \Omega_B^r$ and $S : \Omega_B^r \to \Omega_B^r$ be

$$J(\phi) = \sum_{a=1}^{2m} J\theta^a \wedge i(E_a)\phi, \qquad (14)$$

$$S(\phi) = \sum_{a=1}^{2m} J\theta^a \wedge i(Ric^Q(E_a))\phi. \qquad (15)$$

- $[J, L] = [J, \Lambda] = [F, J] = [F, \Lambda] = [S, J] = [S, \Lambda] = [S, L] = 0.$

Lemmas on Kähler foliations (Jung, 2015)

- On a Kähler foliation $(\mathcal{F}, J)$, a transverse conformal Killing form ϕ satisfies

$$(q + r^2 - qr)S(\phi) = F(J\phi), \tag{16}$$

$$(q + r^2 - qr)S(\phi) = (1 - r)F(J\phi). \tag{17}$$

- On a Kähler foliation $(\mathcal{F}, J)$, if ϕ is a transverse conformal Killing form, then

$$F(J\phi) = 0 \tag{18}$$

Vanishing theorem (Killing forms)

- If $\kappa^\sharp$ is transversally holomorphic, i.e., $\theta(\kappa)J = 0$, then

$$[\Delta_B, \Lambda] = [A_{\kappa^\sharp}, \Lambda] = \delta_T i(J\kappa^\sharp) + i(J\kappa^\sharp)\delta_T.$$

Theorem (Jung-Jung, 2012)

Let $(\mathcal{F}, J)$ be a Kähler foliation in a compact Riemannian manifold M. Assume that $\kappa^\sharp$ is transversally holomorphic. Then any transverse Killing r-form $(2 \leqslant r \leqslant q)$ is parallel.

- Note that on a Kähler foliation, we prove vanishing theorem without the conditions of the transversal Ricci curvature.
- Open when $r = 1$.

Vanishing theorem (Conformal forms)

Theorem (Jung, 2015)

Let $(\mathcal{F}, J)$ be a Kähler foliation with a codimension $q = 2m$ in a closed, connected Riemanian manifold M. Let ϕ be a transverse conformal Killing $\frac{q}{2}$-form. Then
(i) If $q \neq 4$, then $J\phi$ is parallel.
(ii) If $q = 4$ and $\mathcal{F}$ is minimal, then $J\phi$ is parallel.

Proof. (i) If $q \neq 4$ or $m \neq 2$, then

$$\Delta_B J\phi = \theta(\kappa) J\phi.$$

So by the generalized Weitzenbock formula,

$$\frac{1}{2}(\Delta_B - \kappa)|J\phi|^2 = -|\nabla_{tr} J\phi|^2 \leqslant 0.$$

By the generalized maximum principle, it is proved.

(ii) If $q = 4$, then

$$\Delta_B J\phi = -2\delta_B i(\kappa)L\phi + di(\kappa)J\phi.$$

Hence if $\mathcal{F}$ is minimal, then by the generalized Weitzenbock formula,

$$\nabla_{tr}^{*}\nabla_{tr}J\phi = 0.$$

The proof is completed. □

Vanishing theorem (Conformal forms)

Theorem (Jung, 2015)

Let $(\mathcal{F}, J)$ be a minimal Kähler foliation on a compact manifold. Then for a transverse conformal Killing r $(2 \leqslant r \leqslant q-2)$-form ϕ, $J\phi$ is parallel.

Proof. First, note that $F(J\Lambda\phi) = \Lambda F(J\phi) = 0$. Since $\mathcal{F}$ is minimal, $\Delta_B(J\Lambda\phi) = 0$. By the generalized Weitzenbock formula,

$$\nabla_{tr}^*\nabla_{tr} J\Lambda\phi = 0,$$

which means that $J\Lambda\phi$ is parallel. Similarly, $JL\phi$ is parallel. Note that $(m-r)J\phi = [\Lambda, L]J\phi$ and $[\nabla, L] = [\nabla, \Lambda] = 0$. Hence if $r \neq m$, then $J\phi$ is parallel. For $r = m$, see before Theorem. $\square$

Relations between vector fields

- Riemannian manifold

  - Conformal field $\Longleftarrow$ Killing field

  - Conformal field and $d(\text{div}Y) = 0 \Longrightarrow$ Killing field

- Riemannian foliation

  - Transversal conformal field $\Longleftarrow$ Transversal Killing field

  - Transversal conformal field and $d_B(\text{div}_\nabla \bar{Y}) = 0$;
$\int < A_Y\bar{Y} + A_Y^t\bar{Y}, \kappa > \geqslant 0 \Longrightarrow$ Transversal Killing field

Relations between vector fields

- Kähler manifold

  - Conformal field $\Longleftrightarrow$ Killing field

- Kähler foliation

  - Transversal Killing field $\Longrightarrow$ Transversal conformal field

  - Transversal conformal field and $\sigma^\nabla \neq 0$; constant $\Longrightarrow$ Transversal Killing field.
  Here σ^∇ is the transversal scalr curvature of $\mathcal{F}$.

Vanishing results (Vector fields)

- Riemannian manifold

  - If $\mathrm{Ric} \leqslant 0$ and $\mathrm{Ric} < 0$ at some point, then $\nexists$ Killing vector and conformal vector.

- Riemannian foliation

  - If $\mathrm{Ric}^Q \leqslant 0$ and $\mathrm{Ric}^Q < 0$ at some point, then $\nexists$ transversal Killing field.

  - If $\mathrm{Ric}^Q \leqslant 0$ and $\mathrm{Ric} < 0$ at some point and $\delta_B \kappa = 0$, then $\nexists$ transversal conformal field.

General forms on Riemannian case

- Riemannian manifold

  - If $F \leqslant 0$, then every Killing (conformal) r-forms are parallel.
  - In addition, if $F < 0$, then $\nexists$ Killing (conformal) r-forms.

- Riemannian foliation (Jung-Richardson, 2012)

  - The results are same in case of transverse Killing forms.
  - If $F \leqslant 0$ and $\delta_B \kappa = 0$, then transverse conformal r-forms are parallel.
  - In addition, if $F < 0$ at some point, then $\nexists$ transverse conformal r-forms.

General forms on Kähler case

- Kähler manifold
  - Any Killing r $(2 \leqslant r \leqslant 2m)$-forms are parallel.
  - For any conformal r $(2 \leqslant r \leqslant 2m-2)$-form ϕ, $J\Lambda\phi$ is parallel. If $r \neq m$, then $J\phi$ is parallel (Moroianu-Semmelmann, 2003).

- Kähler foliation
  - Any transverse Killing r $(2 \leqslant r \leqslant q)$-forms are parallel (Jung-Jung, 2012).
  - For any transverse conformal Killing r $(2 \leqslant r \leqslant q-2)$-form, if $\mathcal{F}$ is minimal, then $J\phi$ is parallel. (Jung, 2015)

  - Open when $\mathcal{F}$ is not minimal !!!

References

[1] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39(2001), 253-264.
[2] S. D. Jung, *Transversal twistor spinors on a Riemannian foliation*, Foliations 2005, World Scientific, 2006, 215-228.
[3] S. D. Jung, *Transverse conformal Killing forms on Kähler foliations*, J. Geom. Phys. 90 (2015), 29-41.
[4] M. J. Jung and S. D. Jung, *Riemannian foliations admitting transversal conformal fields*, Geometriae Dedicata 133(2008), 155-168.
[5] S. D. Jung and K. Richardson, *Transverse conformal Killing forms and a Gallot-Meyer theorem for foliations*, Math. Z. 270(2012), 337-350

[6] S. D. Jung and M. J. Jung, Transverse Killing forms on a Kahler foliation, Bull. Korean Math. Soc. 49(2012), 445-454
[7] T. Kashiwada, *On conformal Killing tensor*, Natur. Sci. Rep. Ochanomizu Univ. 19(1968), 67-74.
[8] T. Kashiwada and S. Tachibana, *On the integrability of Killing-Yano's equation*, J. Math. Soc. Japan 21(1969), 259-265.
[9] S.D. Jung, K. R. Lee and K. Richardson, *Generalized Obata theorem and its applications for foliations*, J. Math. Anal. Appl. 376(2011), 129-135.
[10] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Math. Z. 245(2003), 503-527.

[11] S. Tachibana, *On conformal Killing tensor in a Riemannian space*, Tôhoku Math. J. 21(1969), 56-64.
[12] S. Tachibana, *On Killing tensors in Riemannian manifolds of positive curvature operator*, Tôhoku Math. J. 28(1976), 177-184.
[13] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New-York, 1988.

Thank You for your attention