Gauss map of real hypersurfaces in complex projective space and submanifolds in complex 2-plane Grassmannians

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Gauss map of hypersurfaces in sphere

For an immersion $x : M^n \to S^{n+1} \subset \mathbb{R}^{n+1}$, let $x(p) \in S^{n+1} \subset \mathbb{R}^{n+1}$ be the position vector at $p \in M^n$, and let $N_p$ be a unit normal vector of oriented hypersurface $M \subset S^{n+1}$ at $p \in M^n$.

Then the Gauss map $\gamma : M^n \to \mathbb{E}^{2n+2} \sim = Q_n$ is defined by $\gamma(p) = x(p) \wedge N_p$ (B. Palmer, 1997).
For an immersion $x : M^n \rightarrow S^{n+1} \subset \mathbb{R}^{n+1}$, let $x(p) \in S^{n+1} \subset \mathbb{R}^{n+2}$ be the position vector at $p \in M$, and...
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Then the Gauss map \( \gamma : M \to \tilde{G}_2(\mathbb{R}^{n+2}) \cong Q^n \) is defined by
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Then the Gauss map $\gamma : M \to \tilde{G}_2(\mathbb{R}^{n+2}) \cong Q^n$ is defined by
$\gamma(p) = x(p) \wedge N_p$ (B. Palmer, 1997).
Then the image of the Gauss map $\gamma(M)$ is a Lagrangian submanifold of complex quadric $Q^n$. 

Moreover, if $M_n \subset S^{n+1}$ is either isoparametric or austere, then $\gamma(M) \subset Q^n$ is a minimal Lagrangian submanifold.

For parallel hypersurface $M_r := \cos rx + \sin rN$ of $M$, the Gauss image is not changed: $\gamma(M) = \gamma(M_r)$.
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We define Gauss map $\gamma : M^{2n-1} \to \mathbb{G}_2(\mathbb{C}^{n+1})$ for real hypersurface $M^{2n-1}$ in $\mathbb{CP}^n$. 
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\downarrow & & \downarrow & & \downarrow & & \pi \\
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\end{array}
\]

For $p \in M$, take a point $z_p \in \pi^{-1}(p) \subset \pi^{-1}(M)$ and let $N'_p$ be a horizontal lift of unit normal of $M \subset \mathbb{CP}^n$ at $z_p$. 
If we put $\gamma(p) = \text{span}_C \{z_p, N'_p\}$, then the map $
abla : M \to G_2(C^{n+1})$ is well-defined.
If we put $\gamma(p) = \text{span}_\mathbb{C}\{z_p, N'_p\}$, then the map $\gamma: M \rightarrow G_2(\mathbb{C}^{n+1})$ is well-defined.

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Note that for parallel hypersurface $M_r := \pi(\cos r z_p + \sin r N'_p)$ of $M$, image of the Gauss map $\gamma : M^{2n-1} \to \mathbb{CP}^n$ is not changed: $\gamma(M) = \gamma(M_r)$. 
For a real hypersurface $M^{2n-1}$ in Kähler manifold $(\tilde{M}^n, J)$ and a unit normal vector $N$, a vector $\xi := -JN$ tangent to $M$ is called the structure vector of $M$. And when $\xi$ is an eigenvector of the shape operator $A$ of $M$, we call $M$ a Hopf hypersurface in $fM^n$. 
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And when $\xi$ is an eigenvector of the shape operator $A$ of $M$, we call $M$ a Hopf hypersurface in $\tilde{M}$. 
A real hypersurface which lies on a tube over a complex submanifold $\Sigma$ in $\mathbb{CP}^n$ is Hopf.
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Conversely, if a Hopf hypersurface $M$ in $\mathbb{CP}^n(4)$ satisfies $A\xi = \mu\xi$ ($\mu$ is necessarily constant), and for $r \in (0, \pi/2)$ with $\mu = 2\cot 2r$, $r \in (0, \pi/2)$, if rank of the focal map $\phi_r : M \to \mathbb{CP}^n$ is constant, then
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In this talk, we will give a characterization of Hopf hypersurface $M$ in $\mathbb{CP}^n$ by using the Gauss map $\gamma : M \rightarrow G_2(\mathbb{C}^{n+2})$. 
Complex 2-plane Grassmann manifold $\tilde{M} = \mathbb{G}_2(\mathbb{C}^{n+1})$ has two important geometric structures, (i) Kähler and (ii) quaternionic Kähler structure $(\tilde{g}, Q)$:
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Here, $\tilde{g}$ is a Riemannian metric of $\tilde{M}$, $Q$ is a subbundle of $\text{End}(T\tilde{M})$ with rank 3, satisfying:

For each $p \in \tilde{M}$, there exists a neighborhood $U \ni p$, such that there exists local frame field $\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ of $Q$. 
\[ \tilde{I}_1^2 = \tilde{I}_2^2 = \tilde{I}_3^2 = -1, \quad \tilde{I}_1\tilde{I}_2 = -\tilde{I}_2\tilde{I}_1 = \tilde{I}_3, \]
\[ \tilde{I}_2\tilde{I}_3 = -\tilde{I}_3\tilde{I}_2 = \tilde{I}_1, \quad \tilde{I}_3\tilde{I}_1 = -\tilde{I}_1\tilde{I}_3 = \tilde{I}_2. \]
Quaternionic Kähler manifold

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For each \(L \in Q_p\), \(\tilde{g}\) is invariant, i.e.,
\[\tilde{g}_p(LX, Y) + \tilde{g}_p(X, LY) = 0\] for \(X, Y \in T_p\tilde{M}\), \(p \in \tilde{M}\).
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\]

Vector bundle \( Q \) is parallel with respect to the Levi-Civita connection of \( \tilde{g} \) at \( \text{End} \, T\tilde{M} \).
A submanifold $M^{2m}$ in quaternionic Kähler manifold $\tilde{M}$ is called **almost Hermitian submanifold**, if there exists a section $\tilde{I}$ of vector bundle $Q|_M$ over $M$ such that

1. $\tilde{I}^2 = -1$,
2. $\tilde{I}TM = TM$.

If we write the almost complex structure on $M$ which is induced by $\tilde{I}$ as $I$, then with respect to the induced metric, $(M, I)$ is an almost Hermitian manifold.
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In particular, when almost Hermitian submanifold \((M, \bar{g}, I)\) is Kähler, we call \(M\) a Kähler submanifold of quaternionic Kähler manifold \(\tilde{M}\).
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Similarly, an almost Hermitian submanifold \((M, \bar{g}, I)\) is called **totally complex submanifold** if at each point \(p \in M\), with respect to \(\tilde{L} \in Q_p\) which anti-commute with \(\tilde{I}_p, \tilde{L}T_pM \perp T_pM\) hold.
In particular, when almost Hermitian submanifold $(M, \bar{g}, I)$ is Kähler, we call $M$ a Kähler submanifold of quaternionic Kähler manifold $\tilde{M}$.

Similarly, an almost Hermitian submanifold $(M, \bar{g}, I)$ is called totally complex submanifold if at each point $p \in M$, with respect to $\tilde{L} \in Q_p$ which anti-commute with $\tilde{I}_p$, $\tilde{L}T_p M \perp T_p M$ hold.

In quaternionic Kähler manifold, a submanifold is totally complex if and only if it is Kähler (Alekseevsky-Marchiafava, 2001).
Theorem (K., Diff. Geom. Appl. 2014) Let $M^{2n-1}$ be a real hypersurface in complex projective space $\mathbb{CP}^n$, and let $\gamma : M \rightarrow G_2(\mathbb{C}^{n+1})$ be the Gauss map.
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- If $M$ is not Hopf, then the Gauss map $\gamma$ is an immersion.
- If $M$ is a Hopf hypersurface, then the image $\gamma(M)$ is a half-dimensional totally complex submanifold of $G_2(\mathbb{C}^{n+1})$.
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- If $M$ is not Hopf, then the Gauss map $\gamma$ is an immersion.
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- And a Hopf hypersurface $M$ in $\mathbb{CP}^n$ is a total space of a circle bundle over a Kähler manifold such that the fibration is nothing but the Gauss map $\gamma : M \rightarrow \gamma(M)$. 
Let $\varphi : \Sigma^{n-1} \rightarrow G_2(\mathbb{C}^{n+1})$ be a totally complex immersion from a (half dimensional) Kähler manifold to complex 2-plane Grassmann manifold.
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Let \( \varphi : \Sigma^{n-1} \to G_2(\mathbb{C}^{n+1}) \) be a totally complex immersion from a (half dimensional) Kähler manifold to complex 2-plane Grassmann manifold.

Then, for each point \( p \) in \( \Sigma \), if we assign \( \tilde{I}_p \in Q_{\varphi(p)} \), then we have a submanifold \( \tilde{I}(\Sigma) \) of the twistor space \( Z = \{ \tilde{I} \in Q | \tilde{I}^2 = -1 \} \) of \( G_2(\mathbb{C}^{n+1}) \)(natural lift).
Let $\varphi : \Sigma^{n-1} \to G_2(\mathbb{C}^{n+1})$ be a totally complex immersion from a (half dimensional) Kähler manifold to complex 2-plane Grassmann manifold.

Then, for each point $p$ in $\Sigma$, if we assign $\tilde{I}_p \in Q_{\varphi(p)}$, then we have a submanifold $\tilde{I}(\Sigma)$ of the twistor space $\mathcal{Z} = \{ \tilde{I} \in Q | \tilde{I}^2 = -1 \}$ of $G_2(\mathbb{C}^{n+1})$(natural lift).

Since $\Sigma$ is a totally complex submanifold of $G_2(\mathbb{C}^{n+1})$, $\tilde{I}(\Sigma)$ is a **Legendrian submanifold** of the twistor space $\mathcal{Z}$ with respect to a complex contact structure (Alekseevsky-Marchiafava, 2004).
Twistor space $\mathcal{Z}$ of $G_2(C^{n+1})$ is naturally identified with the space $L(CP^n)$ of oriented geodesics in $CP^n$. 
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Let $E$ be the quotient space of complex Steifel manifold $V_2(\mathbb{C}^{n+1})$ under diagonal action of $S^1$. Then $E$ is $S^1$-bundle over $\mathcal{Z} \cong L(\mathbb{CP}^n)$ and each fiber is identified with oriented geodesic in $\mathbb{CP}^n$. 
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With respect to the following diagram:
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With respect to the following diagram:

\[ \tilde{I}^* E \xrightarrow{\eta} E \xrightarrow{\psi} \mathbb{CP}^n \]

\[ \downarrow \quad \downarrow \]

\[ \Sigma^{n-1} \xrightarrow{\tilde{I}} \mathcal{Z} \cong L(\mathbb{CP}^n) \]
The map $\Phi := \psi \circ \eta : \tilde{I}^*E \to \mathbb{CP}^n$ gives Hopf hypersurface with $A\xi = 0$ (on open subset of regular points of $M = \tilde{I}^*E$), and
The map $\Phi := \psi \circ \eta : \tilde{I}^*E \rightarrow \mathbb{CP}^n$ gives Hopf hypersurface with $A\xi = 0$ (on open subset of regular points of $M = \tilde{I}^*E$), and

its parallel hypersurface $\phi_r(\tilde{I}^*E)$ gives Hopf hypersurface with $A\xi = 2 \tan 2r\xi$ (on open subset of regular points of $M = \tilde{I}^*E$).
Recently K. Tsukada proved that conormal bundle of any complex submanifold in $\mathbb{CP}^n$ is realized as a half dimensional totally complex submanifold in $\mathbb{G}_2(\mathbb{C}^{n+1})$. 
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For real hypersurfaces in complex hyperbolic space $\mathbb{CH}^n$, we define Gauss map $\gamma : M \rightarrow \mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and we obtain similar results for Hopf hypersurfaces in $\mathbb{CH}^n$ by using para-quaternionic Kähler structure (J.T. Cho and M.K., Topol. Appl. 2015).
\[ \tilde{\mathbb{H}} = C(2, 0) = C(1, 1), \] 
Split-quaternions (or coquaternions, para-quaternions):

\[ q = q_0 + iq_1 + jq_2 + kq_3, \quad i^2 = -1, \quad j^2 = k^2 = 1, \]
\[ ij = -ji = -k, \quad jk = -kj = i, \quad ki = -ik = -j, \]
\[ |q|^2 = q_0^2 + q_1^2 - q_2^2 - q_3^2, \quad \exists \text{ zero divisors}, \]
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\[ \text{http://en.wikipedia.org/wiki/Split-quaternion} \]
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Introduced by James Cockle in 1849.
Para-quaternionic structure

\[ \{I_1, I_2, I_3\}, \quad I_1^2 = -1, \quad I_2^2 = I_3^2 = 1, \]
\[ I_1I_2 = -I_2I_1 = -I_3, \quad I_2I_3 = -I_3I_2 = I_1, \]
\[ I_3I_1 = -I_1I_3 = -I_2 \] gives para-quaternionic structure,
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\tilde{V} = \{aI_1 + bI_2 + cI_3 | \ a, b, c \in \mathbb{R} \} \cong \mathfrak{su}(1, 1) \cong \mathbb{R}^3_1,

and

Gauss map of real hypersurfaces
Para-quaternionic structure

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gives para-quaternionic structure,

\[ \tilde{V} = \{aI_1 + bI_2 + cI_3\mid a, b, c \in \mathbb{R}\} \cong \mathfrak{su}(1, 1) \cong \mathbb{R}^3, \]
and

\[ Q_+ = \{I \in \tilde{V}\mid I^2 = 1\} \cong S^2_1: \text{ de-Sitter space}, \]
\[ Q_- = \{I \in \tilde{V}\mid I^2 = -1\} \cong H^2: \text{ hyperbolic space}, \]
\[ Q_0 = \{I \in \tilde{V}\mid I^2 = 0, \ I \neq 0\} \cong \text{lightcone}. \]
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Theorem 2

Let $M^{2n-1}$ be a real hypersurface in $\mathbb{CH}^n$ and let $g : M \rightarrow G_{1,1}(\mathbb{C}^{n+1})$ be the Gauss map. Suppose $M$ is a Hopf hypersurface with $|\mu| > 2$ (resp. $0 \leq |\mu| < 2$). Then $g(M)$ is a real $(2n - 2)$-dimensional submanifold of $G_{1,1}(\mathbb{C}^{n+1})$, and
There exist sections $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_3$ of the bundle $\tilde{Q}|_{g(M)}$ of the para-quaternionic Kähler structure such that
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- $(\tilde{I}_1)^2 = -1$ (resp. $(\tilde{I}_1)^2 = 1$),
- $(\tilde{I}_2)^2 = 1$ (resp. $(\tilde{I}_2)^2 = -1$),
- $(\tilde{I}_3)^2 = 1$,

such that $dg_x(TxM)$ is invariant under $\tilde{I}_1$ and $\tilde{I}_2$, $\tilde{I}_3$ $dg_x(TxM)$ are orthogonal to $dg_x(TxM)$.
Theorem 2

There exist sections $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_3$ of the bundle $\tilde{Q}|_{g(M)}$ of the para-quaternionic Kähler structure such that they are orthonormal with respect to natural inner product on $\tilde{Q}_{g(p)}$ for $p \in \Sigma$ satisfying

\[ (\tilde{I}_1)^2 = -1 \quad (\text{resp.} (\tilde{I}_1)^2 = 1), \]
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\[(\tilde{I}_2)^2 = 1 \quad \text{(resp.} (\tilde{I}_2)^2 = -1)\] and \[(\tilde{I}_3)^2 = 1,\]

such that $dg_x(T_xM)$ is invariant under $\tilde{I}_1$ and $\tilde{I}_2 dg_x(T_xM)$, $\tilde{I}_3 dg_x(T_xM)$ are orthogonal to $dg_x(T_xM)$. 

Makoto Kimura (Ibaraki University, Japan)  Gauss map of real hypersurfaces
The induced metric on \( g(M) \) in \( G_{1,1}(\mathbb{C}_1^{n+1}) \) has signature \((p, q)\), where

- When \(|\mu| > 2\), \(p\) and \(q\) are both even.
- When \(0 \leq |\mu| < 2\), we have \(p = q\).
The induced metric on $g(M)$ in $\mathbb{G}_{1,1}(\mathbb{C}^{n+1}_1)$ has signature $(p, q)$, where

$$p = \sum_{|\lambda|>1} \dim \{ X \mid AX = \lambda X, \ X \perp \xi \},$$

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Theorem 2

The induced metric on $g(M)$ in $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$ has signature $(p, q)$, where

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The induced metric on $g(M)$ in $\mathbb{C}^{n+1}$ has signature $(p, q)$, where

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\]

\[
q = \sum_{|\lambda| < 1} \dim \{X \mid AX = \lambda X, \ X \perp \xi\}.
\]

- When $|\mu| > 2$, $p$ and $q$ are both even.
- When $0 \leq |\mu| < 2$, we have $p = q$. 
Furthermore if $p + q = 2n - 2$, the induced metric of $g(M)$ is non-degenerate and $g(M)$ is a pseudo-Kähler (resp. para-Kähler) submanifold of $G_{1}(C_{n}+1)$. 
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Let $M^{2n-1}$ be a real hypersurface in $\mathbb{CH}^n$ and let $g : M \to \mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$ be the Gauss map. Suppose $M$ is a Hopf hypersurface with $|\mu| = 2$. Then $g(M)$ is a real $(2n - 2)$-dimensional submanifold of $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and
There exist sections $\tilde{I}_1$ and $\tilde{I}_2$ of the bundle $\tilde{Q}|_{g(M)}$ of the para-quaternionic Kähler structure such that they are orthonormal with respect to natural inner product on $\tilde{Q}_x$ for $p \in M$ satisfying $(\tilde{I}_1)^2 = 1$, $(\tilde{I}_2)^2 = 0$, such that $\tilde{I}_1 \, dg_x(T_xM), \tilde{I}_2 \, dg_x(T_xM)$ are orthogonal to $dg_x(T_xM)$.
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