How many are affine connections with torsion

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Introduction

How big is a infinite well determined family of geometric objects? (pseudo-Riemannian metrics, affine connections,...)

To measure an infinite family of real analytic geometric objects we use

- a finite family of arbitrary functions of k variables,
- a family of arbitrary functions of less variables,
- modulo another family of arbitrary functions of less variables.

The last family of functions corresponds to automorphisms of any geometric object from the given family.

Introduction

In the real analytic case, the Cauchy-Kowalevski Theorem is the standard tool.

- Egorov, Yu.V., Shubin, M.A.: Foundations of the Classical Theory of Partial Differential Equations, Springer-Verlag, Berlin, 1998.
- Kowalevsky, S.: Zur theorie der partiellen differentialgleichungen, J. Reine Angew. Math. **80** (1875) 1–32.
- Petrovsky, I.G.: Lectures on Partial Differential Equations, Dover Publications, Inc., New York, 1991.

An example

How many real analytic Riemannian metrics in dimension 3?

- Every such metric can be put locally into a diagonal form

  Eisenhart, L.P.: Fields of parallel vectors in a Riemannian geometry, Trans. Amer. Math. Soc. **27** (4) (1925) 563–573.

  Kowalski, O., Sekizawa, M.: Diagonalization of three-dimensional pseudo-Riemannian metrics, J. Geom. Phys. **74** (2013), 251–255.

- All coordinate transformations preserving diagonal form of the given metric depend on 3 arbitrary functions of two variables.
- Hence all Riemannian metrics in dimension 3 can be locally described by 3 arbitrary functions of 3 variables modulo 3 arbitrary functions of 2 variables.

Overview of the results

An immediate question arise if we can determine the number of other basic geometric objects, namely the affine connections, in an arbitrary dimension n. We shall be occupied with real analytic connections in arbitrary dimension n.

- We give an alternative proof of the existence of a system of pre-semigeodesic coordinates.
- We describe the class of affine connections using
  $$\frac{n(n^2-1) \text{ functions of } n \text{ variables}}{\text{modulo } 2n \text{ functions of } n-1 \text{ variables.}}$$
- We describe the class of torsion-free affine connections using
  $$\frac{n(n-1)(n+2)/2 \text{ functions of } n \text{ variables}}{\text{modulo } 2n \text{ functions of } n-1 \text{ variables.}}$$

A well known fact from Riemannian geometry is that a Riemannian connection has symmetric Ricci form.

Overview of the results

- We prove that the class of all affine connections with *skew-symmetric* Ricci form depends on $\underline{n(2n^2 - n - 3)/2 \text{ functions of } n \text{ variables}}$ and $n(n+1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.
- Class of connections with *symmetric* Ricci form depends on $\underline{n(2n^2 - n - 1)/2 \text{ functions of } n \text{ variables}}$ and $n(n-1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.
- Class of all torsion-free affine connections with *skew-symmetric* Ricci form depends on $\underline{n(n^2 - 3)/2 \text{ functions of } n \text{ variables}}$ and $n(n+1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.
- Class of torsion-free connections with *symmetric* Ricci form depends on $\underline{(n^3 + n^2 - 4n + 2)/2 \text{ functions of } n \text{ variables}}$ modulo $2n$ functions of $n-1$ variables.

Overview of the results

- All equiaffine connections depends on $n^3 - 2n + 1$ functions of n variables modulo a constant and modulo $2n$ functions of $n-1$ variables.
- Equiaffine connections with *skew-symmetric* Ricci form depends on $(2n^3 - n^2 - 5n + 2)/2$ functions of n variables and $n(n+1)/2$ functions of $n-1$ variables, modulo a constant and modulo $2n$ functions of $n-1$ variables.
- Equiaffine connections with *symmetric* Ricci form depends on $(2n^3 - n^2 - 3n + 2)/2$ functions of n variables and $n(n-1)/2$ functions of $n-1$ variables, modulo a constant and modulo $2n$ functions of $n-1$ variables.

The Cauchy-Kowalevski Theorem of order 1

Consider a system of PDEs for unknown functions $U^1(x^1,\dots,x^n),\dots,U^N(x^1,\dots,x^n)$ on $\mathcal{U}\subset\mathbb{R}^n$ and of the form

$$\begin{array}{rcl}
\dfrac{\partial U^1}{\partial x^1} & = & H^1(x^1,\dots,x^n,U^1,\dots,U^N,\dfrac{\partial U^1}{\partial x^2},..,\dfrac{\partial U^1}{\partial x^n},\dots,\dfrac{\partial U^N}{\partial x^2},..,\dfrac{\partial U^N}{\partial x^n}) \\
\dfrac{\partial U^2}{\partial x^1} & = & H^2(x^1,\dots,x^n,U^1,\dots,U^N,\dfrac{\partial U^1}{\partial x^2},..,\dfrac{\partial U^1}{\partial x^n},\dots,\dfrac{\partial U^N}{\partial x^2},..,\dfrac{\partial U^N}{\partial x^n}) \\
 & \dots & \\
\dfrac{\partial U^N}{\partial x^1} & = & H^N(x^1,\dots,x^n,U^1,\dots,U^N,\dfrac{\partial U^1}{\partial x^2},..,\dfrac{\partial U^1}{\partial x^n},\dots,\dfrac{\partial U^N}{\partial x^2},..,\dfrac{\partial U^N}{\partial x^n}
\end{array}$$

where $H^i, i=1,\dots,N$, are real analytic functions of all variables in a neighborhood of

$$(x_0^1,\dots,x_0^n,a^1,\dots,a^N,a_2^1,\dots,a_n^1,\dots,a_2^N,\dots,a_n^N),$$

where x_0^i, a^i, a_j^i are arbitrary constants.

The Cauchy-Kowalevski Theorem of order 1

Further, let the functions $\varphi^1(x^2, \ldots, x^n), \ldots, \varphi^N(x^2, \ldots, x^n)$ be real analytic in a neighborhood of $(x_0^2, \ldots, x_0^n)$ and satisfy

$$\begin{aligned} \varphi^j(x_0^2, .., x_0^n) &= a^j, \qquad j = 1, \ldots, N, \\ \Big(\frac{\partial \varphi^1}{\partial x^2}, .., \frac{\partial \varphi^1}{\partial x^n}, .., \frac{\partial \varphi^N}{\partial x^2}, .., \frac{\partial \varphi^N}{\partial x^n}\Big)(x_0^2, .., x_0^n) &= (a_2^1, .., a_n^1, .., a_2^N, .., a_n^N). \end{aligned}$$

Then the system has a unique solution
$(U^1(x^1, \ldots, x^n), \ldots, U^N(x^1, \ldots, x^n))$
which is real analytic around $(x_0^1, \ldots, x_0^n)$, and satisfies

$$U^i(x_0^1, x^2, \ldots, x^n) = \varphi^i(x^2, \ldots, x^n), \qquad i = 1, \ldots, N.$$

The Cauchy-Kowalevski Theorem of order 2

The basic assumptions about the system of PDEs are analogous:
The left-hand sides are the second derivatives

$$\frac{\partial^2 U^1}{(\partial x^1)^2}, \dots, \frac{\partial^2 U^N}{(\partial x^1)^2}$$

and the right-hand sides $H^1, \dots, H^N$ involve, as arguments, the original coordinates, the unknown functions $U^1, \dots, U^N$, their first derivatives and their second derivatives except the derivatives written on the left-hand sides:

$$H^i(x^j, U^j, \frac{\partial U^j}{\partial x^k}, \frac{\partial^2 U^j}{\partial x^k \partial x^l}),$$

$$j = 1, \dots, N, \quad k = 1, \dots, n, \quad l = 2, \dots, n.$$

The Cauchy-Kowalevski Theorem of order 2

There exist locally a unique n-tuple $(U^1, \dots, U^N)$ of real analytic functions which is a solution of the new PDE system, and satisfies the initial conditions

$$\begin{aligned} U^i(x_0^1, x^2, \dots, x^n) &= \varphi_0^i(x^2, \dots, x^n), \\ \frac{\partial U^i}{\partial x^1}(x_0^1, x^2, \dots, x^n) &= \varphi_1^i(x^2, \dots, x^n). \end{aligned}$$

The general solution then depends
on $2N$ arbitrary functions φ_0^i, φ_1^i of $n-1$ variables.
See [1], [2] and [3] for the general case and more details.

Transformation of the connection

We work locally with the spaces $\mathbb{R}[u^1, \dots, u^n]$, or $\mathbb{R}[x^1, \dots, x^n]$.
We will use the notation $\mathbf{u} = (u^1, \dots, u^n)$ and $\mathbf{x} = (x^1, \dots, x^n)$.
For a diffeomorphism $f \colon \mathbb{R}[\mathbf{u}] \to \mathbb{R}[\mathbf{x}]$, we write
$x^k = f^k(u^l)$, or $\mathbf{x} = \mathbf{x}(\mathbf{u})$ for short.

We start with the standard formula for the transformation of the connection, which is

$$\bar{\Gamma}^h_{ij}(\mathbf{u}) = \big(\Gamma^\gamma_{\alpha\beta}(\mathbf{x}(\mathbf{u}))\frac{\partial f^\alpha}{\partial u^i}\frac{\partial f^\beta}{\partial u^j} + \frac{\partial^2 f^\gamma}{\partial u^i \partial u^j}\big)\frac{\partial f^h}{\partial u^\gamma}. \tag{1}$$

Transformation of the connection

Lemma

For any affine connection determined by $\Gamma^h_{ij}(\mathbf{x})$*, there exist a local transformation of coordinates determined by* $\mathbf{x} = f(\mathbf{u})$ *such that the connection in new coordinates satisfies* $\bar{\Gamma}^h_{11}(\mathbf{u}) = 0$*, for* $h = 1, \ldots, n$*. All such transformations depend on* $2n$ *arbitrary functions of* $n-1$ *variables.*

Proof. We consider the equations (1) with $\bar{\Gamma}^h_{11}(\mathbf{u}) = 0$, which are

$$0 = \Big(\Gamma^\gamma_{\alpha\beta}(\mathbf{x}(\mathbf{u}))\frac{\partial f^\alpha}{\partial u^1}\frac{\partial f^\beta}{\partial u^1} + \frac{\partial^2 f^\gamma}{(\partial u^1)^2}\Big)\frac{\partial f^h}{\partial u^\gamma}, \qquad h = 1, \ldots, n.$$

We multiply these equations by the inverse of the Jacobi matrix and we obtain the equivalent equations

$$\frac{\partial^2 f^\gamma}{(\partial u^1)^2} = -\Gamma^\gamma_{\alpha\beta}(\mathbf{x}(\mathbf{u}))\frac{\partial f^\alpha}{\partial u^1}\frac{\partial f^\beta}{\partial u^1}, \qquad \gamma = 1, \ldots, n.$$

On the right-hand sides, we have analytic functions depending on $f^1, \ldots, f^n$ and their first derivatives.

Transformation of the connection

We choose arbitrary analytic functions $\varphi^i_\lambda(u^2, \dots, u^n)$, for $i = 1, \dots, n$ and $\lambda = 0, 1$.
According to the Cauchy-Kowalevski Theorem (of pure order 2), there exist unique functions $f^i(u^1, \dots, u^n)$ such that

$$\begin{aligned} f^i(u^1_0, u^2, \dots, u^n) &= \varphi^i_0(u^2, \dots, u^n), \\ \frac{\partial f^i}{\partial u^1}(u^1_0, u^2, \dots, u^n) &= \varphi^i_1(u^2, \dots, u^n). \end{aligned}$$

Obviously, determinant of the Jacobi matrix for these functions will be nonzero for the generic choice of the functions $\varphi^i_\lambda(u^2, \dots, u^n)$.

□

Thus, the local existence of pre-semigeodesic coordinates is proved.

Transformation of the connection

Theorem

All affine connections with torsion in dimension n depend locally on $n(n^2-1)$ arbitrary functions of n variables modulo $2n$ arbitrary functions of $(n-1)$ variables.

Proof. After the transformation into pre-semigeodesic coordinates, we obtain n Christoffel symbols equal to zero.
We are left with $n^3 - n = n(n^2-1)$ functions.
The transformations into pre-semigeodesic coordinates is uniquely determined up to the choice of $2n$ functions $\varphi^i_0(u^2, \ldots, u^n), \varphi^i_1(u^2, \ldots, u^n)$ of $n-1$ variables. □

The Ricci tensor

We consider $\mathbb{R}^n[u^i]$ with the coordinate vector fields $E_i = \frac{\partial}{\partial u^i}$.
We will denote derivatives with respect to u^i by the bottom index.
Using the standard definition

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

we calculate the curvature operators

$$R(E_i, E_j)E_k = (\Gamma^\alpha_{jk})_i E_\alpha - (\Gamma^\beta_{ik})_j E_\beta + \Gamma^\alpha_{jk}\Gamma^\gamma_{i\alpha} E_\gamma - \Gamma^\beta_{ik}\Gamma^\delta_{j\beta} E_\delta.$$

For the Ricci form

$$Ric(X,Y) = \mathrm{trace}\big[W \mapsto R(W,X)Y\big],$$

we obtain

$$Ric(E_i, E_j) = \sum_{k,l=1}^{n} \Big[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}\Big].$$

Skew-symmetric Ricci tensor

We analyze the condition for the skew-symmetry of the Ricci form using previous formulas. Using the symmetry condition $\Gamma_{ij}^k = \Gamma_{ji}^k$, we consider just the Christoffel symbols Γ_{ij}^k such that $i < j$.
We split the skew-symmetry conditions into four cases:

$$\begin{array}{rcll} Ric(E_1, E_1) & = & 0, & \\ Ric(E_i, E_i) & = & 0, & i > 1, \\ Ric(E_1, E_i) + Ric(E_i, E_1) & = & 0, & i > 1, \\ Ric(E_i, E_j) + Ric(E_j, E_i) & = & 0, & 1 < i < j \leq n. \end{array}$$

In each formula which follows, we denote by Λ'_{ij} the terms which involve first derivatives with respect to $u^2, \ldots, u^n$ and by Λ_{ij} the terms which do not involve any differentiation (and which form a homogeneous polynomial of degree 2 in Γ_{ij}^k).

Skew-symmetric Ricci tensor

Corresponding to the four cases above, we obtain the equations

$$
\begin{aligned}
\sum_{k=2}^{n}(\Gamma^k_{k1})_1 &= \Lambda'_{11} + \Lambda_{11}, && \\
(\Gamma^1_{ii})_1 &= \Lambda'_{ii} + \Lambda_{ii}, && i > 1, \\
(\Gamma^1_{i1})_1 - \sum_{k=2}^{n}(\Gamma^k_{ki})_1 &= \Lambda'_{1i} + \Lambda_{1i}, && i > 1, \\
(\Gamma^1_{ij})_1 + (\Gamma^1_{ji})_1 &= \Lambda'_{ij} + \Lambda_{ij}, && 1 < i < j \leq n.
\end{aligned}
\tag{2}
$$

Skew-symmetric Ricci tensor

Theorem

The family of connections with torsion whose Ricci form is skew-symmetric depends locally, on $\frac{n(2n^2-n-3)}{2}$ functions of n variables and $\frac{n(n+1)}{2}$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.

Proof. After the transformation into pre-semigeodesic coordinates, the family of connections with torsion depends on $q(n) = n(n^2-1)$ functions (Christoffel symbols).

- We have $p(n) = n(n+1)/2$ conditions for the skew-symmetry of the Ricci form.
- These conditions involve first derivatives of the Christoffel symbols and they are written in a suitable way.
- Any Christoffel symbol appears on the left-hand side of the mentioned equations at most once.

Skew-symmetric Ricci tensor

We select one Christoffel symbol in each of the equations (to be determined later), for example the following:

- Γ^2_{21} (1 function),
- Γ^1_{ii}, for $i \geq 1$ ($n-1$ functions),
- Γ^1_{ij}, for $i \geq j$ ($n(n-1)$ functions).

We choose the other $q(n) - p(n) = n(2n^2 - n - 3)/2$ Christoffel symbols as arbitrary functions.

Skew-symmetric Ricci tensor

The $p(n)$ Christoffel symbols remain undetermined,
just one in each of the equations.
After transporting the arbitrarily chosen functions
to the right-hand side, we obtain a new system of equations
of the form

$$\begin{aligned}
(\Gamma^2_{12})_1 &= -\sum_{k=3}^{n}(\Gamma^k_{1k})_1 + \Lambda'_{11} + \Lambda_{11}, \\
(\Gamma^1_{ii})_1 &= \Lambda'_{ii} + \Lambda_{ii}, \qquad i > 1, \\
(\Gamma^1_{1i})_1 &= -\sum_{k=2}^{n}(\Gamma^k_{ik})_1 + \Lambda'_{1i} + \Lambda_{1i}, \qquad i > 1, \\
(\Gamma^1_{ij})_1 &= \Lambda'_{ij} + \Lambda_{ij}, \qquad 1 < i < j \leq n,
\end{aligned}$$

where the Christoffel symbols on the right-hand sides
are already fixed.

Skew-symmetric Ricci tensor

We have got a standard system of $p(n)$ equations
for the last $p(n)$ functions
at which the Cauchy-Kowalevski Theorem can be applied.

The general solution depends on $p(n)$ arbitrary functions of $n-1$ variables and, because we used pre-semigeodesic coordinates, this number is to be reduced by $2n$ functions. $\square$

Symmetric Ricci tensor

We recall the formula for the nondiagonal entries of the Ricci form

$$Ric(E_i, E_j) \;=\; \sum_{k,l=1}^{n} \Big[(\Gamma_{ij}^k)_k - (\Gamma_{kj}^k)_i + \Gamma_{ij}^l \Gamma_{kl}^k - \Gamma_{kj}^l \Gamma_{il}^k\Big]. \qquad (3)$$

The Ricci tensor is symmetric if it holds

$$Ric(E_i, E_j) - Ric(E_j, E_i) \;=\; 0, \qquad 1 \le i < j \le n. \qquad (4)$$

We will split the situation into the two cases, $i = 1$ and $i > 1$:

$$\begin{aligned} -\sum_{k=2}^{n} (\Gamma_{kj}^k)_1 - (\Gamma_{j1}^1)_1 &= \Lambda'_{1j} + \Lambda_{1j}, \qquad 1 < j \le n, \\ (\Gamma_{ij}^1)_1 - (\Gamma_{ji}^1)_1 &= \Lambda'_{ij} + \Lambda_{ij}, \qquad 1 < i < j \le n. \qquad (5) \end{aligned}$$

Symmetric Ricci tensor

Theorem

A family of connections with torsion whose Ricci form is symmetric depends locally on $\frac{n(2n^2-n-1)}{2}$ *functions of* n *variables and* $\frac{n(n-1)}{2}$ *functions of* $n-1$ *variables modulo* $2n$ *arbitrary functions of* $n-1$ *variables.*

Proof. Using now pre-semigeodesic coordinates, there are just $q(n) = n^3 - n = n(n^2 - 1)$ nontrivial Christoffel symbols.

In the system, there are $p(n) = n(n-1)/2$ conditions for the symmetry of the Ricci form.

We let the $p(n)$ Christoffel symbols Γ^1_{ij}, to be determined later and we fix arbitrarily the $q(n) - p(n) = n(2n^2 - n - 1)/2$ other Christoffel symbols.

If we transport the chosen Christoffel symbols to the right-hand sides of the equations, we obtain a standard system for which the Cauchy-Kowalevski Theorem can be applied.

Final remark

Let the symbol $\#$ denote the number of arbitrary functions of n variables on which a set of connections on an n-dimensional manifold depends (General connections with torsion, or those with symmetric Ricci tensor, or those with skew-symmetric Ricci tensor, respectively).

$\#Gen(n) = n^3 - n,$
$\#Sym(n) = n^3 - n(n+1)/2,$
$\#Skew(n) = n^3 - n(n+3)/2.$

- $\#Gen(n) > \#Sym(n) > \#Skew(n),$
- $\#Gen(n) - \#Sym(n) = O(n^2),$
  $\#Gen(n) - \#Skew(n) = O(n^2),$
- $\#Sym(n) - \#Skew(n) = n,$
- $lim_{n \to \infty}(\#Sym(n)/\#Gen(n)) =$
  $lim_{n \to \infty}(\#Skew(n)/\#Gen(n)) = 1.$

The last, limit rules seem to be like a paradox but this is connected with the fact that the operation of computing a Ricci tensor from the Christoffel symbols is a nonlinear operation.

References

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