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MERIDIAN SURFACES IN MINKOWSKI 4-SPACE

Velichka Milousheva

Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences, Sofia, Bulgaria

Plan of the talk:

- ***Meridian surfaces with constant mean curvature.***
- ***Marginally trapped meridian surfaces.***
- ***Chen meridian surfaces.***
- ***Meridian surfaces with parallel normalized mean curvature vector field.***
- ***Meridian surfaces with pointwise 1-type Gauss map.***

The talk is based on the following papers:

[1] Ganchev, G., Milousheva, V., *An invariant theory of marginally trapped surfaces in the four-dimensional Minkowski space*, J. Math. Phys. **53** (2012), Article ID: 033705, 15 pp.

[2] Ganchev, G., Milousheva, V., *Marginally trapped meridian surfaces of parabolic type in the four-dimensional Minkowski space*, Int. J. Geom. Methods Mod. Phys. **10**, no. 10 (2013), Article ID: 1350060, 17 pp.

[3] Ganchev, G., Milousheva, V., *Meridian surfaces of elliptic or hyperbolic type in the four-dimensional Minkowski space.* (submitted)

[4] Ganchev, G., Milousheva, V., *Meridian surfaces of parabolic type in the four-dimensional Minkowski space.* (submitted)

[5] Arslan, K., Milousheva, V., *Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-Space.* (submitted)

1. Meridian surfaces in $\mathbb{R}^4_1$

Meridian surfaces of elliptic type:

Let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in $\mathbb{R}^4_1$, i.e. $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, $\langle e_4, e_4\rangle = -1$.

Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $f'^2(u) - g'^2(u) > 0$, $u \in I$.

The standard rotational hypersurface $\mathcal{M}'$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the Oe_4-axis, is:

$$Z(u, w^1, w^2) = f(u) \cos w^1 \cos w^2 e_1 + f(u) \cos w^1 \sin w^2 e_2 + f(u) \sin w^1 e_3 + g(u) e_4.$$

$\mathcal{M}'$ is a **rotational hypersurface with timelike axis**.

Let $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$. We consider the two-dimensional surface $\mathcal{M}'_m$ lying on $\mathcal{M}'$, constructed in the following way:

$$\mathcal{M}'_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I, v \in J.$$

$\mathcal{M}'_m$ is a one-parameter system of meridians of $\mathcal{M}'$. That is why we call $\mathcal{M}'_m$ a ***meridian surface of elliptic type***.

If we denote $l(w^1, w^2) = \cos w^1 \cos w^2\, e_1 + \cos w^1 \sin w^2\, e_2 + \sin w^1\, e_3$, then

$$\mathcal{M}'_m : z(u,v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\, v \in J.$$

$l(w^1, w^2)$ is the unit position vector of the 2-dimensional sphere $S^2(1)$ lying in the Euclidean space $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$ and centered at the origin O; $c : l = l(v) = l(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ is a smooth curve on $S^2(1)$.

So, we can say that each meridian surface of elliptic type is determined by a meridian curve m of a rotational hypersurface with timelike axis and a smooth curve c lying on the unit 2-dimensional sphere $S^2(1)$.

All invariants of the meridian surface of elliptic type $\mathcal{M}'_m$ are expressed by the curvature $\kappa_m(u)$ of the meridian curve m and the spherical curvature $\kappa(v)$ of the curve c on $S^2(1)$.

The Gauss curvature and the curvature of the normal connection are:

$$K = -\frac{f''(u)}{f(u)}; \qquad \varkappa = 0.$$

The mean curvature vector field is:

$$H = \frac{\kappa}{2f} n_1 + \frac{f f'' + f'^2 - 1}{2f\sqrt{f'^2 - 1}} n_2$$

The length of the mean curvature vector field is:

$$||H|| = \sqrt{\frac{\varepsilon(\kappa^2(\dot{f}^2 - 1) - (f\ddot{f} + \dot{f}^2 - 1)^2)}{4f^2(\dot{f}^2 - 1)}}.$$

Meridian surfaces of hyperbolic type:

Let $f = f(u)$, $g = g(u)$ be smooth functions, such that $f'^2(u) + g'^2(u) > 0$. The rotational hypersurface $\mathcal{M}''$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the Oe_1-axis is:

$$Z(u, w^1, w^2) = g(u)e_1 + f(u)\cosh w^1 \cos w^2 e_2 + f(u)\cosh w^1 \sin w^2 e_3 + f(u)\sinh w^1 e_4.$$

$\mathcal{M}''$ is a **rotational hypersurface with spacelike axis**.

Let $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$. We consider the surface:

$$\mathcal{M}''_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I, v \in J.$$

$\mathcal{M}''_m$ is a one-parameter system of meridians of $\mathcal{M}''$. We call $\mathcal{M}''_m$ a ***meridian surfaces of hyperbolic type***.

If we denote $l(w^1, w^2) = \cosh w^1 \cos w^2\, e_2 + \cosh w^1 \sin w^2\, e_3 + \sinh w^1\, e_4$, then

$$\mathcal{M}''_m : z(u, v) = f(u)\, l(v) + g(u)\, e_1, \quad u \in I, v \in J,$$

$l(w^1, w^2)$ being the unit position vector of the timelike sphere $S^2_1(1)$ in the Minkowski space $\mathbb{R}^3_1 = \text{span}\{e_2, e_3, e_4\}$, i.e. $S^2_1(1) = \{V \in \mathbb{R}^3_1 : \langle V, V\rangle = 1\}$. $S^2_1(1)$ is a timelike surface in $\mathbb{R}^3_1$ known also as the ***de Sitter space***.

Meridian surfaces of parabolic type:

Instead of the standard orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we consider the pseudo-orthonormal base $\{e_1, e_2, \xi_1, \xi_2\}$, where $\xi_1 = \dfrac{e_3 + e_4}{\sqrt{2}}$, $\xi_2 = \dfrac{-e_3 + e_4}{\sqrt{2}}$.

A **rotational hypersurface with lightlike axis** in $\mathbb{R}^4_1$:

$$\mathcal{M}''' : Z(u, w^1, w^2) = f(u) w^1 \cos w^2 e_1 + f(u) w^1 \sin w^2 e_2 + \left(f(u) \frac{(w^1)^2}{2} + g(u) \right) \xi_1 + f(u)\, \xi_2,$$

where $f = f(u)$, $g = g(u)$ are smooth functions, such that $-f'(u)g'(u) > 0$, $f(u) > 0$.

Let $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$. We consider the surface:

$$\mathcal{M}'''_m : z(u, v) = Z(u, w^1(v), w^2(v)). \qquad (1)$$

$\mathcal{M}'''_m$ - a ***meridian surface of parabolic type***.

Each meridian surface of parabolic type is determined by a meridian curve of a rotational hypersurface with lightlike axis and a curve c lying on the paraboloid $\mathcal{P}^2$, defined by

$$\mathcal{P}^2 : l(w^1, w^2) = w^1 \cos w^2\, e_1 + w^1 \sin w^2\, e_2 + \frac{(w^1)^2}{2}\, \xi_1 + \xi_2.$$

Indeed, we can rewrite (1) as

$$\mathcal{M}'''_m : z(u, v) = f(u)\, l(v) + g(u)\, \xi_1, \quad u \in I,\ v \in J. \qquad (2)$$

2. Meridian surfaces with constant mean curvature

Constant mean curvature surfaces in arbitrary spacetime are important objects for their special role in the theory of general relativity. The study of constant mean curvature surfaces (CMC surfaces) involves not only geometric methods but also PDE and complex analysis, that is why the theory of CMC surfaces is of great interest not only for mathematicians but also for physicists and engineers. CMC surfaces in Minkowski space have been studied intensively in the last years, for example by R. López, J. Pastor, R. Souam, S. Montiel, N. Sasahara, H. Liu, G. Liu, D. Brander, R. Chaves, C. Cândido, J. Hano, K. Nomizu, etc.

For meridian surfaces of elliptic type:
$||H|| = a = const \ (a \neq 0)$ if and only if $(ff'' + f'^2 - 1)^2 = (f^2 - 1)(b^2 - 4a^2 f^2)$.

Theorem 2.1. *(i) $\mathcal{M}'_m$ has constant mean curvature $||H|| = a = const$, $a \neq 0$ if and only if the curve c on $S^2(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $f' = y(f)$ where*

$$y(t) = \sqrt{1 + \frac{1}{t^2}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a} \arcsin \frac{2at}{b}\right)^2}, \qquad C = const,$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2(u) - 1}$.

(ii) $\mathcal{M}''_m$ *has constant mean curvature* $||H|| = a = const$*,* $a \neq 0$ *if and only if the curve* c *on* $S^2_1(1)$ *has constant spherical curvature* $\kappa = const = b$*,* $b \neq 0$*, and the meridian* m *is determined by* $f' = y(f)$ *where*

$$y(t) = \sqrt{1 - \frac{1}{t^2}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right)^2}, \qquad C = const,$$

$g(u)$ *is defined by* $g'(u) = \sqrt{1 - f'^2(u)}$.

(iii) $\mathcal{M}'''_m$ *has constant mean curvature* $||H|| = a = const$*,* $a \neq 0$ *if and only if* $\kappa = const = b$*,* $b \neq 0$*, and the meridian* m *is determined by* $f' = y(f)$*, where*

$$y(t) = \frac{1}{t}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right), \quad \textit{if} \quad \langle H, H\rangle > 0,$$

$$y(t) = \frac{1}{t}\left(C \pm \frac{t}{2}\sqrt{b^2 + 4a^2t^2} \pm \frac{b^2}{4a}\ln|2at + \sqrt{b^2 + 4a^2t^2}|\right), \quad \textit{if} \quad \langle H, H\rangle < 0,$$

$g(u)$ *is defined by* $g'(u) = -\dfrac{1}{2f'(u)}$.

3. Marginally trapped meridian surfaces

The concept of trapped surfaces was introduced by Roger Penrose [*Phys. Rev. Lett.*, 1965] and plays an important role in general relativity. A surface in a 4-dimensional spacetime is called ***marginally trapped*** or ***quasi-minimal*** it its mean curvature vector H is lightlike at each point.

Marginally trapped surfaces satisfying some extra conditions on the mean curvature vector, the Gauss curvature or the second fundamental form have been intensively studied in the last few years. For example, marginally trapped surfaces with positive relative nullity were classified by B.-Y. Chen and J. Van der Veken in [*Class. Quantum Grav.*, 2007] and [*J. Math. Phys.*, 2007]. The classification of marginally trapped surfaces with parallel mean curvature vector in Lorenz space forms is given by B.-Y. Chen and J. Van der Veken in [*Houston J. Math.*, 2010].

Marginally trapped surfaces in Minkowski 4-space which are invariant under spacelike rotations, under boost transformations, and under screw rotations were classified by S. Haesen and M. Ortega in [*Gen. Relativ. Grav.*, 2009], [*Class. Quantum Grav.*, 2007], and [*J. Math. Anal. Appl.*, 2009], respectively.

Marginally trapped surfaces with pointwise 1-type Gauss map were classified by V. Milousheva in [*Int. J. Geom.*, 2013] and N. C. Turgay in [*Gen. Relativ. Grav.*, 2014].

In [*J. Math. Phys.*, 2012] and [*J. Geom. Methods Mod. Phys.*, 2013] we described all marginally trapped meridian surfaces.

Theorem 3.1. *(i)* $\mathcal{M}'_m$ *is marginally trapped if and only if* $\kappa = a,\; a \neq 0$, *and the meridian curve* m *is defined by*

$$\begin{aligned} f(u) = &\; u; \\ g(u) = &\; \frac{\pm a}{a^2+1}\sqrt{(\pm a u + c)^2 + u^2} \\ &+ \frac{c}{(a^2+1)^{\frac{3}{2}}} \ln\left(\sqrt{a^2+1}\,u \pm \frac{ac}{\sqrt{a^2+1}} + \sqrt{(\pm a u + c)^2 + u^2}\right) + b, \end{aligned}$$

where b *and* c *are constants,* $c \neq 0$.

(ii) $\mathcal{M}''_m$ *is marginally trapped if and only if* $\kappa = a,\; a \neq 0$, *and the meridian curve* m *is defined by*

$$\begin{aligned} f(u) = &\; u; \\ g(u) = &\; \frac{\pm a}{1-a^2}\sqrt{u^2 - (\pm a u + c)^2} \\ &+ \frac{c}{(1-a^2)^{\frac{3}{2}}} \ln\left(\sqrt{1-a^2}\,u \mp \frac{ac}{\sqrt{1-a^2}} + \sqrt{u^2 - (\pm a u + c)^2}\right) + b, \end{aligned}$$

where b *and* c *are constants,* $c \neq 0$.

(iii) $\mathcal{M}'''_m$ *is marginally trapped if and only if* $\kappa(v) = a = const,\ a \neq 0$, *and the meridian curve is defined by*

$$\begin{aligned} f(u) &= u; \\ g(u) &= \frac{\pm 1}{2a^3}\left(\frac{a^2u^2 \mp 2auc}{c \mp au} - 2c\ln|c \mp au| + b\right), \end{aligned}$$

where b *and* c *are constants,* $c \neq 0$.

Remark: As far as we know, all examples of marginally trapped surfaces known till now in the literature are surfaces with parallel mean curvature vector field. The first examples of marginally trapped surfaces with non-parallel mean curvature vector field are the meridian surfaces.

4. Chen meridian surfaces

The *allied vector field* of a normal vector field ξ of an n-dimensional submanifold M^n of $(n+m)$-dimensional Riemannian manifold $\widetilde{M}^{n+m}$ is defined by B.-Y. Chen by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^{m} \{\mathrm{tr}(A_1 \circ A_k)\} \xi_k,$$

where $\{\xi_1 = \dfrac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is a local orthonormal frame of the normal bundle of M^n, and $A_i = A_{\xi_i}$, $i = 1, \dots, m$ is the shape operator with respect to ξ_i. In particular, the allied vector field $a(H)$ of the mean curvature vector field H is called the *allied mean curvature vector field* of M^n.

B.-Y. Chen defined the $\mathcal{A}$-submanifolds to be those submanifolds of $\widetilde{M}^{n+m}$ for which $a(H)$ vanishes identically. The $\mathcal{A}$-submanifolds are also called *Chen submanifolds*. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be *trivial* $\mathcal{A}$*-submanifolds*.

The notion of allied mean curvature vector field is extended by S. Haesen and M. Ortega [*J. Math. Anal. Appl.*, 2009] to the case when the normal space is a two-dimensional Lorenz space and the mean curvature vector field is lightlike.

In the following theorem we classify all non-trivial Chen meridian surfaces.

Theorem 4.1. *(i)* $\mathcal{M}'_m$ *is a Chen surface if and only if* $\varkappa = const = b$, $b \neq 0$, *and the meridian* m *is determined by* $f' = y(f)$ *where*

$$y(t) = \frac{\pm 1}{2\,t^{\pm 1}} \sqrt{4\,t^{\pm 2} - a\left(t^{\pm 2} - \frac{b^2}{a}\right)^2}, \qquad a = const \neq 0,$$

$g(u)$ *is defined by* $g'(u) = \sqrt{f'^2(u) - 1}$.

(ii) $\mathcal{M}''_m$ *is a Chen surface if and only if* $\varkappa = const = b$, $b \neq 0$, *and the meridian* m *is determined by* $f' = y(f)$ *where*

$$y(t) = \frac{\pm 1}{2\,t^{\pm 1}} \sqrt{4\,t^{\pm 2} + a\left(t^{\pm 2} - \frac{b^2}{a}\right)^2}, \qquad a = const \neq 0,$$

$g(u)$ *is defined by* $g'(u) = \sqrt{1 - f'^2(u)}$.

(iii) $\mathcal{M}'''_m$ *is a Chen surface if and only if* $\varkappa = const = b$, $b \neq 0$, *and the meridian* m *is determined by* $f' = y(f)$ *where*

$$y(t) = \frac{1}{2c\,t^{\pm 1}} \left(c^2 t^{\pm 2} + b^2\right), \qquad c = const \neq 0,$$

$g(u)$ *is defined by* $g'(u) = -\dfrac{1}{2f'(u)}$.

5. Meridian surfaces with parallel normalized mean curvature vector field

A normal vector field ξ is said to be *parallel in the normal bundle* (or simply *parallel*), if $D_x \xi = 0$ holds identically for any tangent vector field x. A surface M is said to have *parallel normalized mean curvature vector field* if the mean curvature vector H is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [B.-Y. Chen, *Monatsh. Math.*, 1980]. Note that if M is a surface with non-zero parallel mean curvature vector field, then M is a surface with parallel normalized mean curvature vector field, but the converse is true only in the case $\|H\| = const.$

It is known that every surface in the Euclidean 3-space has parallel normalized mean curvature vector field but in the 4-dimensional Euclidean space, there exist abundant examples of surfaces which lie fully in $\mathbb{E}^4$ with parallel normalized mean curvature vector field, but not with parallel mean curvature vector field.

In pseudo-Euclidean spaces there are very few results on surfaces with parallel normalized mean curvature vector field. In [J. Math. Phys. Anal. Geom., 2011] S. Shu studied spacelike submanifolds with parallel normalized mean curvature vector field in the de Sitter space. He showed that compact spacelike submanifolds whose mean curvature does not vanish and whose corresponding normalized vector field is parallel, must be, under some suitable geometric assumptions, totally umbilical.

In the next theorem we describe the meridian surfaces of elliptic type with parallel normalized mean curvature vector field.

Theorem 5.1. *$\mathcal{M}'_m$ has parallel normalized mean curvature vector field if and only if one of the following cases holds:*

(a) the meridian m is defined by

$$f(u) = \pm\sqrt{u^2 + 2cu + d};$$
$$g(u) = \pm\sqrt{c^2 - d}\,\ln|u + c + \sqrt{u^2 + 2cu + d}| + a,$$

where a, c, and d are constants, $c^2 > d$;

(b) $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $f' = y(f)$ where

$$y(t) = \pm\frac{\sqrt{(a^2+1)\,t^2 + 2ac\,t + c^2}}{t}, \quad a = const \neq 0, \quad c = const,$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2(u) - 1}$.

Similar results hold for meridian surfaces of hyperbolic or parabolic type.

6. Meridian surfaces with pointwise 1-type Gauss map

A submanifold M of the Euclidean space $\mathbb{E}^m$ (or pseudo-Euclidean space $\mathbb{E}^m_s$) is said to have ***pointwise 1-type Gauss map*** if its Gauss map G satisfies

$$\Delta G = \lambda(G + C)$$

for some non-zero smooth function λ on M and some constant vector C.

Classification results on surfaces with pointwise 1-type Gauss map in Minkowski space have been obtained in the last few years. For example, the classification of ruled surfaces with pointwise 1-type Gauss map of first kind in Minkowski space $\mathbb{E}^3_1$ is given by Y. Kim and D. Yoon in [*J. Geom. Phys.*, 2000]. Ruled surfaces with pointwise 1-type Gauss map of second kind in Minkowski 3-space were classified by M. Choi, Y. H. Kim, and D. W. Yoon in [*Taiwanese J. Math.*, 2011]. In [*Rocky Mountain J. Math.*, 2005] Y. Kim and D. Yoon studied ruled surfaces with 1-type Gauss map in Minkowski space $\mathbb{E}^m_1$ and gave a complete classification of null scrolls with 1-type Gauss map.

The complete classification of flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ is given in [Y. Kim and D. Yoon, *J. Korean Math. Soc.*, 2004].

7.1. Meridian surfaces with harmonic Gauss map.

The classification of the meridian surfaces of elliptic or hyperbolic type with harmonic Gauss map is given in the next two theorems.

Theorem 6.1 (K. Arslan, V. M.). *Let $\mathcal{M}'_m$ be a meridian surface of elliptic type. The Gauss map of $\mathcal{M}'_m$ is harmonic if and only if $\mathcal{M}'_m$ is part of a plane.*

Theorem 6.2 (K. Arslan, V. M.). *Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type. The Gauss map of $\mathcal{M}''_m$ is harmonic if and only if one of the following cases holds:*

(i) $\mathcal{M}''_m$ is part of a plane;

(ii) the curve c has spherical curvature $\kappa = \pm 1$ and the meridian curve m is determined by $f(u) = a$; $g(u) = \pm u + b$, where $a = const$, $b = const$. In this case $\mathcal{M}''_m$ is a marginally trapped developable ruled surface in $\mathbb{E}^4_1$.

Remark: In the Euclidean space $\mathbb{E}^4$ planes are the only surfaces with harmonic Gauss map. However, in the Minkowski space $\mathbb{E}^4_1$ there are surfaces with harmonic Gauss map which are not planes. Theorem 6.1 and Theorem 6.2 show that in the class of the meridian surfaces of elliptic type there are no surfaces with harmonic Gauss map other than planes, while in the class of the meridian surfaces of hyperbolic type there exist surfaces with harmonic Gauss map, which are not planes.

7.2. Meridian surfaces with pointwise 1-type Gauss map of first kind

The classification of the meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map of first kind, i.e.

$$\Delta G = \lambda G$$

for some non-zero smooth function λ, is given in the following theorems:

Theorem 6.3 (K. Arslan, V. M.). *Let $\mathcal{M}'_m$ be a meridian surface of elliptic type. Then $\mathcal{M}'_m$ has pointwise 1-type Gauss map of first kind if and only if the curve c has zero spherical curvature and the meridian curve m is determined by a solution $f(u)$ of the following differential equation*

$$f\left(\frac{f f''}{\sqrt{f'^2-1}}\right)' - f'\sqrt{f'^2-1} = 0,$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2(u)-1}$.

Theorem 6.4 (K. Arslan, V. M.). *Let $\mathcal{M}''_m$ be a meridian surface of hyperbolic type. Then $\mathcal{M}''_m$ has pointwise 1-type Gauss map of first kind if and only if one of the following cases holds:*

(i) the curve c has zero spherical curvature and the meridian curve m is determined by a solution $f(u)$ of the following differential equation

$$f\left(\frac{f f''}{\sqrt{1-f'^2}}\right)' + f'\sqrt{1-f'^2} = 0,$$

$g(u)$ is defined by $g'(u) = \sqrt{1-f'^2(u)}$;

(ii) the curve c has non-zero constant spherical curvature k $(\kappa \neq \pm 1)$ and the meridian curve m is determined by $f(u) = a$; $g(u) = \pm u + b$, where $a = const$, $b = const$. Moreover, $\mathcal{M}''_m$ is a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3_1$ (if $\kappa^2 - 1 > 0$) or $\mathbb{E}^3$ (if $\kappa^2 - 1 < 0$) of $\mathbb{E}^4_1$.

7.3. Meridian surfaces with pointwise 1-type Gauss map of second kind

The classification of the meridian surfaces of elliptic type with pointwise 1-type Gauss map of second kind, i.e.

$$\Delta G = \lambda(G + C)$$

for some non-zero smooth function λ and a constant vector $C \neq 0$, is given in

Theorem 6.5 (K. Arslan, V. M.). *Let $\mathcal{M}'_m$ be a meridian surface of elliptic type. Then $\mathcal{M}'_m$ has pointwise 1-type Gauss map of second kind if and only if one of the following cases holds:*

(i) the curve c has non-zero constant spherical curvature κ and the meridian curve m is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = const$, $b = const$. In this case $\mathcal{M}'_m$ is a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3$ of $\mathbb{E}^4_1$.

(ii) the curve c has constant spherical curvature κ and the meridian curve m is determined by $f(u) = au + a_1$; $g(u) = bu + b_1$, where a, a_1, b and b_1 are constants, $a^2 \geq 1$, $a^2 - b^2 = 1$. In this case $\mathcal{M}'_m$ is either a marginally trapped developable ruled surface (if $\kappa^2 = b^2$) or a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3$ (if $\kappa^2 - b^2 > 0$) or $\mathbb{E}^3_1$ (if $\kappa^2 - b^2 < 0$) of $\mathbb{E}^4_1$.

(iii) the curve c has zero spherical curvature and the meridian curve m is determined by the solutions of the following differential equation

$$\left(\ln \frac{\sqrt{f'^2-1}\,\left(f(f'^2-1)(ff'')' - f^2f'f''^2 - f'(f'^2-1)^2\right)}{(f'^2-1)^2 + f^2f''^2 - ff'(f'^2-1)(ff'')'}\right)' = \frac{f'f''}{f'^2-1}.$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2(u)-1}$.

A similar result holds for meridian surfaces of hyperbolic type with pointwise 1-type Gauss map of second kind.

Similarly to the elliptic and hyperbolic type one can classify the meridian surfaces of parabolic type with pointwise 1-type Gauss map.

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THANK YOU FOR YOUR ATTENTION!