Lectures on Geometric Quantization

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- **Planck's relation**

$$E = h\nu,$$

where E is the energy, ν is the frequency of the wave, and h is a universal constant called Planck's constant.

- **Bohr's quantization condition**

$$\oint p_i dq^i = nh,$$

where $(q^1, ..., q^n)$ are position coordinates, $(p_1, ..., p_n)$ are the conjugate momenta, h is Planck's constant, and Einstein's convention of summation over repeated indices is adopted.

- Harmonic oscillator Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2).$$

Contour integral is taken over circles $H =$ constant.

- **Notation**. Symplectic form

$$\omega = d(p_i dq^i) = dp_i \wedge dq^i .$$

For each smooth function $f(p,q)$, the Hamiltonian vector field of f is the vector field

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}.$$

In general, X_f is defined by

$$X_f \lrcorner \omega = -df .$$

- **Hydrogen atom Hamiltonian**

$$H = \frac{1}{2m}p^2 - \frac{k}{|q|}.$$

Contour integration is taken over the integral curves of the Hamiltonian vector fields X_{p_φ}, X_{p_θ} and X_H, where p_φ and p_θ are canonical momenta conjugate to the spherical polar cocrdinates φ and θ, respectively.

- **Relativistic hydrogen atom Hamiltonian**

$$H = \sqrt{m^2 - p^2} - \frac{k}{|q|}.$$

  The motion is not periodic, but orbits lie on 3-dimensional tori. Sommerfeld interpreted the contour integral in Bohr's quantization condition as integration over generators of the tori.
- A Hamiltonian system with n-degrees of freedom is completely integrable if its orbits lie on n-dimensional tori.
- **Bohr-Sommerfeld quantization conditions** for completely integrable Hamiltonian systems

$$\oint p_i dq^i = nh,$$

  where integration is taken over generators of the tori.
- **Modified Bohr-Sommerfeld quantization conditions**

$$\oint pdq = (n + \frac{1}{2})h.$$

Heisenberg's advance

- Quotation from Dirac's lecture in 1975: "The great advance was made by Heisenberg in 1925. He made a very bold step. He had the idea that physical theory should concentrate on quantities which are very closely related to observed quantities. Now, the things you observe are only very remotely connected with the Bohr orbits. So Heisenberg said that the Bohr orbits are not very important. The things that are observed, or which are connected closely with the observed quantities, are all associated with two Bohr orbits and not with one Bohr orbit: *two* instead of *one*."
- In the following year, we had two competing theories: the matrix mechanics of Max Born and Pascuale Jordan and the wave mechanics of Ervin Schrödinger. A unification of both theories into the present day quantum mechanics came in the work of Paul Dirac.

Quantum Mechanics in Dirac's formulation

- Classical states are points of the phase space manifold P and dynamical variables are $f \in C^\infty(P)$.
- Quantum states of the system under consideration form a complex Hilbert space $\mathfrak{H}$.
- For $\Psi_1, \Psi_2 \in \mathfrak{H}$, the scalar product $\langle \Psi_1 \mid \Psi_2 \rangle$ is the relative probability amplitude; that is $|\langle \Psi_1 \mid \Psi_2 \rangle|^2$ is the probability that the measurement of state Ψ_1 gives state Ψ_2.
- Classical dynamical variables $f \in C^\infty(P)$ of the system are replaced by quantum observables that are self-adjoint operators on $\mathfrak{H}$ (possibly unbounded).
- Observables form an algebra $\mathfrak{A}$ over $\mathbb{C}$ with operations of multiplication and commutation of operators.
- A complete set of commuting observables is a set of observables which commute with one another and for which there is only one simultaneous eigenvector (possibly generalized) belonging to any set of eigenvalues.

- There exists a basis (Ψ_ξ) in $\mathfrak{H}$, consisting of joint eigenvectors of a complete set of commuting observables labelled by sets ξ of joint eigenvalues.
- Every vector $\Psi \in \mathfrak{H}$, can be expressed as follows:

$$\Psi = \sum_{\xi'} \left\langle \Psi \mid \Psi_{\xi'} \right\rangle \Psi_{\xi'} + \int \left\langle \Psi \mid \Psi_{\xi''} \right\rangle d\xi'' \Psi_{\xi''},$$

  where the sum is take over eigenvalues ξ' in the discrete joint spectrum, and integration is taken over the continuous spectrum.
- Different choice of of a complete set of commuting observables gives rise to a different decomposition the space $\mathfrak{H}$ of quantum states.
- If the spectrum of eigenvalues is discrete, we get matrix formalism.
- If the spectrum of eigenvalues is purely continuous, we get Schrödinger's wave mechanics.

Poisson algebra

Notation Symplectic structure ω of the phase space P of a classical system gives rise to the Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(P)$, defined by

$$\{f_1, f_2\} = X_{f_2}(f_1),$$

for all $f_1, f_2 \in C^\infty(P)$. Here, X_{f_2} is the Hamiltonian vector field of f_2. The Poisson bracket is bilinear, and antisymmetric. Moreover, it satisfies the Leibniz rule

$$\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + f_2\{f_1, f_3\}$$

and the Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3\{f_1, f_2\}\} = 0.$$

The space $C^\infty(P)$ of smooth functions on a symplectic manifold (P, ω), endowed with the Poisson bracket, is called the Poisson algebra of (P, ω).

Classical Analogy

- A classical analogy between a quantum observables $\mathfrak{A}$ and a classical dynamical variables $C^\infty(P)$ is given by a Poisson subalgebra A of $C^\infty(P)$ and a linear map

$$\mathbf{Q} : A \longrightarrow \mathfrak{A} : f \longmapsto \mathbf{Q}_f ,$$

  such that, for every $f_1, f_2 \in A$,

$$[\mathbf{Q}_{f_1}, \mathbf{Q}_{f_2}] = i\hbar \mathbf{Q}_{\{f_1, f_2\}},$$

  where $\hbar$ is Planck's constant divided by 2π.
- Note that Dirac's quantum conditions imply that that the map

$$f \longmapsto (1/i\hbar)\mathbf{Q}_f$$

  is a homomorphism of the Poisson algebra A into the Lie algebra of skew-adjoint operators on $\mathfrak{H}$.
- Dirac allows for existence of systems for which there is no non-trivial Poisson subalgebra A of $C^\infty(P)$, for which quantum conditions apply.

Kirillov-Kostant-Souriau forms

- The foundation of geometric quantization is the fact, discovered independently by Kirillov, Souriau and Kostant, that every co-adjoint orbit P of a Lie group G is endowed with a symplectic form ω.
- Coadjoint orbit of G through $\mu \in \mathfrak{g}^*$,

$$P = \{Ad_g^* \mu \mid g \in G\},$$

where $\langle Ad_g^* \mu \mid \xi \rangle = \langle \mu \mid Ad_{g^{-1}} \xi \rangle$ for every $\xi \in \mathfrak{g}$.
- Since the co-adjoint action of G is transitive on P, for each $\xi \in g$, there exists a unique vector field X^ξ on P generating the action of $\exp t\xi$ on P, and for every $\nu \in P$,

$$T_\nu P = \{X^\xi(\nu) \mid \xi \in g\}.$$

- The Kirillov-Kostant-Souriau form ω, also called the KKS-form, is given by

$$\omega(X^\xi(\nu), X^\zeta(\nu)) = -\langle \nu \mid [\xi, \zeta] \rangle .$$

Hamiltonian action of a Lie group

- An action
$$\Phi : G \times P \to P : (g, p) \mapsto \Phi_g(p)$$
of a connected Lie group G on a symplectic manifold (P, ω) is Hamiltonian if there exists an Ad^*-equivariant mmomentum map $J : P \to \mathfrak{g}^*$ such that, for every $\xi \in \mathfrak{g}$, the restriction of Φ to the 1-parameter subgroup $\exp t\xi$ is generated by the Hamiltonian vector field of
$$J_\xi = \langle J, \xi \rangle .$$
- If P is a co-adjoint orbit of a connected Lie group G and ω is the Kirillov-Kostant-Souriau form on P, then the co-adjoint action of G on P is Hamiltonian, with the momentum map $J : P \to \mathfrak{g}^*$ given by the inclusion of P in $\mathfrak{g}^*$.

Sources of geometric quantization

- In 1962, Aleksandr Kirillov constructed unitary representations of nilpotent Lie groups using the *orbit method*, which relied on the symplectic structure of co-adjoint orbits. Kirillov also conjectured that irreducible unitary representations of compact group were in one-to-one correspondence with integral co-adjoint orbits.
- Jean-Marie Souriau formulated, in his 1966 paper, a quantization scheme in terms of sections of a circle bundle over the phase space (P, ω) of the quantized system. Souriau's *quantification géométrique* did not provide for probability amplitudes in quantum mechanics.
- In 1965, Bertram Kostant outlined his geometric quantization theory at the US-Japan Seminar in Differential Geometry, Kyoto. A comprehensive presentation of the first step of geometric quantization, called *prequantization*, was given in his 1970 paper . Application of the complete theory to representations of solvable group appeared in a joint paper with L. Auslander published in 1971

Role of geometric quantization

- The theory of geometric quantization forms a bridge between quantum mechanics and the representation theory of Lie groups. I
- In representation theory, geometric quantization is a geometric technique of obtaining a unitary representation of a connected Lie group from its action on a symplectic manifold.
- In quantum mechanics, geometric quantization provides a differential-geometric method of constructing a quantum theory correponding to the classical system.
- This dual role of geometric quantization enables us to use representation theory to test hypotheses in quantum mechanics and vice versa.
- Reliance on differential geometry restricts the applicability of geometric quantization to smooth situations.
- Study of singularities requires further techniques like cohomology and singularity theory.

Elements of geometric quantization

- Prequantization gives a representation of the Poisson algebra of a symplectic manifolds by symmetric operators on a Hilbert space $\mathfrak{H}$.
- Polarization chooses an irreducible component of the prequantization representation to a sub-algebra of the Poisson algebra.
- If the space of the polarization representation consists of generalized vectors in $\mathfrak{H}$, the next step is to determine the scalar product for which the polarization representation is unitary.
- Determine a sub-algebra of the Poisson algebra which acts transitively on the space of the polarization representation.

Classical elastica

- An elastica is a curve in $\mathbb{R}^3$ that is stationary under variations of the integral of the square of its curvature.
- Consider a dynamical system $t \mapsto x(t)$ such that its evolution follows an elastica.
- The Lagrangian is

$$L(t, x, \dot{x}, \ddot{x}) = \kappa^2 |\dot{x}| = \frac{|\ddot{x}|^2}{|\dot{x}|^3} - \frac{\dot{x} \cdot \ddot{x}}{|\dot{x}|^5}.$$

- The action integral is invariant under the group of reparametrizations, that is the group $Diff_+\mathbb{R}$ of orientation preserving diffeomorphisms of $\mathbb{R}$.
- By the Second Noether Theorem the initial data for an elastica are not independent; they satisfy some identities.
- It is a toy model for general relativity.
- How should we quantize this system?

Constrained Hamiltonian system

- Ostrogradski's Legendre transformation $\mathcal{L} : J_0^3 \to T^* J_0^1$ is
$$p_{\dot{x}} = \frac{\partial L}{\partial \ddot{x}}, \text{ and } p_x = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}.$$
Here $J_0^1 = \{(t, x, \dot{x}) \in \mathbb{R}^7 \mid \dot{x} \neq 0\}$ and
$J_0^3 = \{(t, x, \dot{x}, \ddot{x}, \dddot{x}) \in \mathbb{R}^{13} \mid \dot{x} \neq 0\}$.
- $\omega = dp_t \wedge dt + dp_x \wedge dx + dp_{\dot{x}} \wedge d\dot{x}$ is the canonical symplectic form of $T^* J_0^1$.
- The function
$$h = 4\frac{\dot{x}}{|\dot{x}|} \cdot p_x + |\dot{x}|^2 \, p_{\dot{x}} \cdot p_{\dot{x}}$$
may be interpreted as a Hamiltonian of the motion.
- **Theorem**. The action of $Diff_+\mathbb{R}$ on $T^* J_0^1$ is Hamiltonian with the momentum map $\mathcal{J} : T^* J_0^1 \to (diff_+\mathbb{R})^*$ such that, for every $\tau\frac{\partial}{\partial t} \in diff_+\mathbb{R}$,
$$\mathcal{J}_\tau = \tau p_\tau - \dot{\tau}\dot{x} \cdot p_{\dot{x}}.$$
The range of $\mathcal{L}$ is $J^{-1}(0) \cap h^{-1}(0)$.

Reduction of reparametrization symmetries

- The projection map $\rho : \mathcal{J}^{-1}(0) \to \mathcal{J}^{-1}(0) / \mathit{Diff}_{+}\mathbb{R}$ is a submersion.
- $\mathcal{J}^{-1}(0) / \mathit{Diff}_{+}\mathbb{R}$ has a symplectic form ω_{red} such that $\rho^{*}\omega_{red}$ is the pull-back to $\mathcal{J}^{-1}(0)$ of the canonical symplectic form of $T^{*}J_{0}^{1}$.
- **Theorem** There is a diffeomorphism $\delta : J_{0}^{1} / \mathit{Diff}_{+}\mathbb{R} \to S$, where

$$S = \mathbb{R}^{3} \times \mathbb{P}^{3} = \{ (x, \dot{x} / |\dot{x}|) \mid x, \dot{x} \in \mathbb{R}^{3},\ \dot{x} \neq 0 \}$$

  is the product of $\mathbb{R}^{3}$ and the projective space $\mathbb{P}^{3}$.
- **Theorem** The orbit space $(\mathcal{J}^{-1}(0) / \mathit{Diff}_{+}\mathbb{R}, \omega_{red})$ is symplectomorphic to the cotangent bundle space $T^{*}S$ of endowed with its canonical symplectic form.
- The Hamiltonian function $h = 4\frac{\dot{x}}{|\dot{x}|} \cdot p_{x} + |\dot{x}|^{2}\, p_{\dot{x}} \cdot p_{\dot{x}}$ pushes forward to a function h_{red} on $T^{*}S$.

Quantization of the extended phase space

- Quantize $(T^* J_0^1, \omega)$ to obtain a quantization representation of $Diff_+\mathbb{R}$.
- Use Schrödinger quantization with wave functions Ψ on J_0^1 and the scalar product

$$(\Psi_1, \Psi_2) = \int_{J_0^1} \overline{\Psi}_1(t, x, \dot{x})\Psi_2(t, x, \dot{x}) dt d_3 x d_3 \dot{x}.$$

- For $\mathcal{J}_\tau = \tau p_\tau - \dot{\tau}\dot{x} \cdot p_{\dot{x}}$,

$$\mathbf{Q}_{\mathcal{J}_\tau} = \left(-i\hbar\tau \frac{\partial}{\partial t} + i\hbar\dot{\tau} \left\langle \dot{x} \cdot \frac{\partial}{\partial \dot{x}} \right\rangle \right).$$

- Moreover,

$$\mathbf{Q}_h = -\frac{\hbar^2}{4} \left(|\dot{x}|^2 \left\langle \frac{\partial}{\partial \dot{x}} \cdot \frac{\partial}{\partial \dot{x}} \right\rangle - \frac{1}{3} \right) - \frac{i\hbar}{|\dot{x}|} \left\langle \dot{x} \cdot \frac{\partial}{\partial \dot{x}} \right\rangle.$$

Implementation of the constraint conditions

- Physically admissible quantum states satisfy the quantum constraint conditions

$$\mathbf{Q}_{J_\tau}\Psi = 0 \text{ for all } \tau \in \mathit{diff}_+\mathbb{R}, \text{ and } \mathbf{Q}_h\Psi = 0.$$

- Thus physically admissible states states Ψ are $\mathit{Diff}_+\mathbb{R}$-invariant and satisfy the equation

$$\mathbf{Q}_h\Psi \equiv -\frac{\hbar^2}{4}\left(|\dot{x}|^2\left\langle\frac{\partial}{\partial\dot{x}}\cdot\frac{\partial}{\partial\dot{x}}\right\rangle - \frac{1}{3}\right)\Psi - \frac{i\hbar}{|\dot{x}|}\left\langle\dot{x}\cdot\frac{\partial}{\partial\dot{x}}\right\rangle\Psi = 0.$$

- $\mathit{Diff}_+\mathbb{R}$-invariant functions Ψ on J_0^1 are pull-backs by δ of functions ψ on $S = \mathbb{R}^3 \times \mathbb{P}^3$.
- $\mathit{Diff}_+\mathbb{R}$-invariant wave functions Ψ on J_0^1 that satisfy the equation $\mathbf{Q}_h\Psi = 0$ are pull-backs of wave functions ψ on $S = \mathbb{R}^3 \times \mathbb{P}^3$ satisying the equation

$$\mathbf{Q}_{h_{red}}\psi = 0,$$

where $\mathbf{Q}_{h_{red}}$ is the quantization of the reduced Hamiltonian on T^*S.

Commutation of quantization and reduction

- Quantizing a system with constraints, we have to decide on the order of quantization and reduction.
- Gupta and Bleuler first quantized the extended phase space of electrodynamics in the space of quantum states with indefinite metric. They followed with the quantum implementation of the (linear) constraint conditions.
- In general relativity, the constraint equations are not linear. Therefore, Dirac's approach was to reduce the constraint first and to quantize the reduced system.
- In the context of the theory of representations of compact groups, Guillemin and Sternberg showed that the quantization and reduction commute.
- In this example quantization commutes with reduction even though $\mathit{Diff}_+\mathbb{R}$ is not locally compact.

Complex line bundle

- Let $\lambda : L \to P$ be a complex line bundle with connection ∇.
- For each section $\sigma : P \to L$, and each vector field X on P, the covariant derivative $\nabla_X \sigma$ is a section of L such that, for every $f \in C^\infty(P)$,

$$\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma \quad \text{and} \quad \nabla_{fX}\sigma = f\nabla_X\sigma.$$

- The curvature of the connection ∇ is a 2-form α on P such that

$$(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})\sigma = 2\pi i\alpha(X_1, X_2)\sigma.$$

- A form α on P is the curvature form of a complex line bundle with connection if and only if the de Rham cohomology class $[\alpha]$ is in $H^2(\mathbb{Z})$.
- For every line bundle L with connection, there exists a connection invariant Hermitian form $\langle \cdot \mid \cdot \rangle$ defined up to a constant factor. For every pair of sections σ_1, σ_2 of L and every vector field X on P,

$$X(\langle \sigma_1 \mid \sigma_2 \rangle) = \langle \nabla_X\sigma_1 \mid \sigma_2 \rangle + \langle \sigma_1 \mid \nabla_X\sigma_2 \rangle.$$

Prequantization line bundle

- A symplectic manifold (P, ω) is prequantizable if, for every closed surface S in P,
$$\int_S \omega = nh,$$
where n is an integer.
In quantum mechanics, $\omega = \sum dp_i \wedge dq^i$, where (q^i) are position coordinates, p_i is the momentum dual to q^i, and h denotes Planck's constant.
In the theory of representations of Lie groups, h is usually taken to be $-i$.
- If (P, ω) is prequantizable, there exists a complex line bundle $\lambda : L \to P$ with a connection ∇ and a connection invariant Hermitian form such that the curvature of ∇ is $\alpha = -\frac{1}{h}\omega$.
- Equivalence classes of complex line bundles with connection with the curvature form $\alpha = -\frac{1}{h}\omega$ are parametrized by $H^1(\mathbb{Z})$.

Quantomorphisms

- A function $f \in \mathcal{C}^\infty(P)$ generates a local one-parameter group $\exp tX_f$ of local symplectomorphisms of (P, ω).
- The Hamiltonian vector field X_f corresponding to f can be lifted to a unique vector field $\widehat{X}_f$ on L such that $\exp t\widehat{X}_f$ is the lift of $\exp tX_f$ that preserves the connection ∇.
- For each $\sigma \in S^\infty(\mathcal{L})$ we set

$$\mathbf{P}_f\sigma = i\hbar \frac{d}{dt}(\exp t\widehat{X}_f \circ \sigma \circ \exp(-tX_f))_{|t=0}.$$

- Direct computation yields

$$\mathbf{P}_f\sigma = (-i\hbar\nabla_{X_f} + f)\sigma.$$

  For each $f_1, f_2 \in \mathcal{C}^\infty(P)$ and $\sigma \in S^\infty(\mathcal{L})$

$$[\mathbf{P}_{f_1}, \mathbf{P}_{f_2}] = i\hbar\mathbf{P}_{\{f_1, f_2\}}.$$

- Prequantization map

$$\mathbf{P} : \mathcal{C}^\infty(P) \times S^\infty(L) \to S^\infty(L) : (f, \sigma) \mapsto \mathbf{P}_f\ \sigma.$$

Prequantization representation

- The map $\mathcal{C}^\infty(P) \times S^\infty(L) \to S^\infty(L) : (f, \sigma) \mapsto \frac{i}{\hbar}\mathbf{P}_f\ \sigma$ is a representation of the Lie algebra structure of $\mathcal{C}^\infty(P)$ on $S^\infty(L)$.
- The space $S_0^\infty(L)$ of compactly supported smooth sections of $\mathcal{L}$ has a Hermitian scalar product
$$((\sigma_1 \,||\, \sigma_2)) = \int_P \langle \sigma_1 \mid \sigma_2 \rangle \omega^n,$$
where $n = \frac{1}{2} \dim P$.
- For each $f \in \mathcal{C}^\infty(P)$, the prequantization operator $\mathbf{P}_f$ is symmetric with respect to this scalar product.
- If the Hamiltonian vector field X_f of f is complete, then $\mathbf{P}_f$ is self-adjoint on the Hilbert space $\mathfrak{H}$ obtained by the completion of $S_0^\infty(L)$ with respect to the norm above.
- Prequantization does not correspond to the quantum theory, because the interpretation of $((\sigma \,||\, \sigma))\,(p)$ as the probability density of localizing the state σ at a point $p \in P$ fails to satisfy Heisenberg's Uncertainty Principle.

Hamiltonian action of a Lie group

- An action

$$\Phi : G \times P \to P : (g, p) \mapsto \Phi_g(p)$$

  of a connected Lie group G on a symplectic manifold (P, ω) is Hamiltonian if there exists an Ad^*-equivariant momentum map $J : P \to \mathfrak{g}^*$ such that, for every $\xi \in \mathfrak{g}$, the restriction of Φ to the 1-parameter subgroup $\exp t\xi$ is generated by the Hamiltonian vector field of

$$J_\xi = \langle J, \xi \rangle .$$

- If P is a co-adjoint orbit of a connected Lie group G and ω is the Kirillov-Kostant-Souriau form on P, then the co-adjoint action of G on P is Hamiltonian, with the momentum map $J : P \to \mathfrak{g}^*$ given by the inclusion of P in $\mathfrak{g}^*$.

Prequantization representation of a Lie group

- For a Hamiltonian action of G on (P, ω) with the momentum map $J : P \longrightarrow \mathfrak{g}^*$, the map $\xi \mapsto J_\xi$ is a homomorphism of $\mathfrak{g}$ to the Poisson algebra $\mathcal{C}^\infty(P)$.
- The map $\xi \mapsto (i/\hbar)\mathbf{P}_{J_\xi}$ is the prequantization representation of the Lie algebra $\mathfrak{g}$ by skew-adjoint operators on the Hilbert space $\mathfrak{H}$.
- If the prequantization representation of $\mathfrak{g}$ on $\mathfrak{H}$ integrates to a unitary representation

$$\mathbf{U} : G \times \mathfrak{H} \longrightarrow \mathfrak{H}$$

  of G on $\mathfrak{H}$, we call $\mathbf{U}$ the prequantization representation of G.
- The prequantization representation $\mathbf{U}$ of a Lie group G need not be irreducible.

Polarization

- A polarization of a symplectic manifold (P, ω) is an involutive complex Lagrangian distribution F such that
$$D = F \cap \overline{F} \cap TP \text{ and } E = (F + \overline{F}) \cap TP.,$$
where D and E are involutive distributions on P.
  1. Real polarization $F = D \otimes \mathbb{C}$, where D is an integrable Lagranian distribution on P such that the space P/D of integral manifolds of D is a quotient manifold of P.
  2. Complex polarization F such that $F \cap \overline{F} = 0$. Involutivity of F ensures that P has a complex structure and F is spanned by anti-holomorphic directions.
- The representation space corresponding to F is
$$\mathcal{S}_F^\infty(L) = \{\sigma \in \mathcal{S}^\infty(L) \mid \nabla_u \sigma = 0 \text{ for all } u \in F\}.$$
- A function $f \in C^\infty(Q)$ is directly quantizable in terms of F if X_f preserves F. Such functions form a Poisson subalgebra $\mathcal{C}_F^\infty(P)$ of $C^\infty(P)$.

Quantization

- The quantization map $\mathbf{Q}$ relative to a polarization F is the restriction of the prequantization map $\mathbf{P}$ to domain $\mathcal{C}^\infty_F(P) \times \mathcal{S}^\infty_F(L) \subset \mathcal{C}^\infty(P) \times \mathcal{S}^\infty(L)$ and codomain $\mathcal{S}^\infty_F(L) \subset \mathcal{S}^\infty(L)$:

$$\mathbf{Q} : \mathcal{C}^\infty_F(P) \times \mathcal{S}^\infty_F(L) \to \mathcal{S}^\infty_F(L) : (f, \sigma) \mapsto \mathbf{Q}_f \sigma = (i\hbar \nabla_{X_f} + f)\sigma.$$

- Note that sections $\mathcal{S}^\infty_F(L)$ may have zero intersection with $\mathfrak{H}$. In this case we need to introduce a new scalar product in $\mathcal{S}^\infty_F(L)$ to construct the Hilbert space $\mathfrak{H}_F$ of normalizable quantum states.
- If $P = T^*Q$, ω is the canonical symplectic form of the cotangent bundle, and D is the kernel of the projection map, then sections in $S^\infty_F(L)$ are given by pull-backs to T^*P of complex valued functions on Q. The space $\mathcal{C}^\infty_F(P)$ consists of functions on T^*Q that are linear in momenta. In order to get the usual Schrödinger quantum mechanics, we need an additional structure on P, called the metaplectic structure.

Blattner-Kostant-Sternberg kernels

- This approach to quantization allows to compare quantizations corresponding to different polarizations.
- Let F_1 and F_2 be two transverse polarizations, so that $F_1 \oplus F_2 = TP \otimes \mathbb{C}$. For $\sigma_i \in \mathcal{S}^\infty_{F_i}(L)$, the integral

$$\mathcal{K}_{21}(\sigma_1, \sigma_2) = \int_P \langle \sigma_1, \sigma_2 \rangle \, \omega^n$$

  defines a sesquilinear pairing of $\mathcal{S}^\infty_{F_1}(L)$ and $\mathcal{S}^\infty_{F_2}(L)$, which can be used to construct a linear transformation between the corresponding Hilbert spaces of quantum states.
- This transformation is a generalization of the Fourier transform between the position and the momentum representations in the Schrödinger formulation.

III Prequantization of spin

- Co-adjoint orbits of $SO(3)$ are spheres
$$S_r^2 = \{x \in \mathbb{R}^3 \mid x^2 = r^2\}.$$
- For a fixed $r > 0$, let $\mathbf{s} = x_{|S_r}$ denote the restriction of x to S_r^2. The Kirillov-Kostant-Souriau ω form on S_r^2 can be written as
$$\omega = -\frac{1}{2} r^{-2} \sum_{i,j,k} \varepsilon_{ijk} s^i ds^j \wedge ds^k = \frac{1}{r} \mathrm{vol}_{S_r^2},$$
where s^1, s^2, s^3 are components of the spin vector $\mathbf{s}$, ε_{ijk} is the completely antisymmetric tensor with $\varepsilon_{123} = 1$, and $\mathrm{vol}_{S_r^2}$ is is the standard area form on S_r^2 with $\int_{S_r^2} \mathrm{vol}_{S_r^2} = 4\pi r^2$.
- The orbit (S_r^2, ω) is prequantizable if
$$\int_{S_r^2} \omega = \int_{S_r^2} \frac{1}{r} \mathrm{vol}_{S_r^2} = 4\pi r = nh$$
or $r = n\hbar/2$.

Complex structure

- For even $n = 2s$, prequantization of (S_r^2, ω) gives a representation of $SO(3)$.
- Let V_+ and V_- be complements of the south pole and the north pole in S_r^2, respectively,

$$V_+ = \{\mathbf{s} \in S_r^2 \mid s^3 + r > 0\} \text{ and } V_- = \{\mathbf{s} \in S_r^2 \mid s^3 - r < 0\}.$$

- On V_+ and V_- define complex functions

$$z_+ = \frac{s^1 - is^2}{r + s^3} \text{ and } z_- = \frac{s^1 + is^2}{r - s^3}.$$

- In $V_+ \cap V_-$,

$$z_+ z_- = 1.$$

- The functions z_+ and z_- define a complex structure on S_r^2.

Computations

- Solving for the spin vector **s** we obtain

$$
\begin{aligned}
s^1 &= r(z_\pm + \bar{z}_\pm)(1 + z_\pm \bar{z}_\pm)^{-1}, \\
s^2 &= \pm ir(z_\pm - \bar{z}_\pm)(1 + z_\pm \bar{z}_\pm)^{-1}, \\
s^3 &= \pm r(1 - z_\pm \bar{z}_\pm)(1 + z_\pm \bar{z}_\pm)^{-1}.
\end{aligned}
$$

- Hence,

$$
\begin{aligned}
\omega_{|V_\pm} &= d\theta_\pm = -2ir(1 + z_\pm \bar{z}_\pm)^{-2} d\bar{z}_\pm \wedge dz_\pm, \\
\theta_\pm &= -2ir(1 + z_\pm \bar{z}_\pm)^{-1} \bar{z}_\pm dz_\pm,
\end{aligned}
$$

and

$$
\theta_+ - \theta_- = id(\log z_-^{2r}) = i\hbar d(\log z_-^{2r/\hbar}) = i\hbar d(\log z_-^n).
$$

- Since n is an integer, the transition function z_-^n is globally defined and single-valued

Prequantization line bundle

- Consider an equivalence relation $\sim$ on $(\mathbb{C} \times V_+ \times \{+\}) \cup (\mathbb{C} \times V_- \times \{-\})$ defined by
$$(c, x, \alpha) \sim (c', x', \alpha')$$
if (i) $(c, x, \alpha) = (c', x', \alpha')$ or (ii) $\alpha = +$, $\alpha' = -$, $x = x' \in V_+ \cap V_-$, and $c = z_-(x)^n c'$, or
(iii) $\alpha = -$, $\alpha' = +$, $x = x' \in V_+ \cap V_-$, and $c' = z_-(x)^n c$.
- The space L of $\sim$-equivalence classes is a complex line bundle over S^2_r with projection map
$$\lambda : L \to S^2_r : [(c, x, \alpha)] \mapsto x.$$
- The restrictions of L to $V_\pm$ are trivial, with trivializing sections
$$\sigma_\pm : V_\pm \to L : x \mapsto [(1, x, \pm)].$$
- For $x \in V_+ \cap V_-$,
$$\sigma_+(x) = z_-(x)^n \sigma_-(x).$$

Connection

- In $V_\pm$ set
$$\nabla_\pm \sigma_\pm = -i\hbar^{-1}\theta_\pm \otimes \sigma_\pm .$$

- In $V_+ \cap V_-$,

$$\begin{aligned}
\nabla_- \sigma_+ &= \nabla_-(z_-^n \lambda_-) = dz_-^n \otimes \sigma_- + z_-^n \nabla_- \sigma_- \\
&= dz_-^n \otimes \sigma_- - i\hbar^{-1} z_-^n \theta_- \otimes \sigma_- \\
&= dz_-^n \otimes \sigma_- + i\hbar^{-1} z_-^n (\theta_+ - \theta_- - \theta_+) \otimes \sigma_- \\
&= dz_-^n \otimes \sigma_- + i\hbar^{-1} z_-^n (\theta_+ - \theta_-) \otimes \sigma_- - i\hbar^{-1} z_-^n \theta_+ \otimes \sigma_- \\
&= dz_-^n \otimes \sigma_- + i\hbar^{-1} z_-^n (i\hbar d(\log z_-^n)) \otimes \sigma_- - i\hbar^{-1} \theta_+ \otimes z_-^n \sigma_- \\
&= \nabla_+ \sigma_+ .
\end{aligned}$$

- Hence, there exists a unique connection ∇ on L that restricts to $\nabla_\pm$ on $L_{|V_\pm}$. By construction, the curvature of ∇ is $\frac{-1}{h}\omega$, as required.

Hermitian form

- If $\langle \cdot, \cdot \rangle$ is a connection invariant Hermitian form on L, then

$$\begin{aligned} d\langle \sigma_\pm, \sigma_\pm \rangle &= \langle \nabla_\pm \sigma_\pm, \sigma_\pm \rangle + \langle \sigma_\pm, \nabla_\pm \sigma_\pm \rangle \\ &= i\hbar^{-1}(\bar{\theta}_\pm - \theta_\pm) \langle \sigma_\pm, \sigma_\pm \rangle \\ &= -2\hbar^{-1} r d \log(1 + z_\pm \bar{z}_\pm) \langle \sigma_\pm, \sigma_\pm \rangle . \end{aligned}$$

- Since $r = n\frac{\hbar}{2} = s\hbar$, it follows that

$$d\langle \sigma_\pm, \sigma_\pm \rangle = -2sd\log(1 + z_\pm \bar{z}_\pm) \langle \sigma_\pm, \sigma_\pm \rangle$$

or

$$d\log\langle \sigma_\pm, \sigma_\pm \rangle = -2sd\log(1 + z_\pm \bar{z}_\pm) = d\log(1 + z_\pm \bar{z}_\pm)^{-2s}.$$

- Therefore, we may choose

$$\langle \sigma_\pm, \sigma_\pm \rangle = (1 + z_\pm \bar{z}_\pm)^{-2s}.$$

Prequantization representation of so(3)

- For every section section $\sigma : S_r^2 \longrightarrow L$,

$$\sigma_{|V_\pm} = \psi_\pm \sigma_\pm,$$

where $\psi_\pm$ are functions on $V_\pm$.

- For each $f \in C^\infty(S^2)$, the prequantization operator is

$$\mathbf{P}_f \sigma_{|V_\pm} = \mathbf{P}_f(\psi_\pm \sigma_\pm) = (-i\hbar X_f \psi_\pm + f - \langle \theta_\pm \mid X_f \rangle)\sigma_\pm.$$

- In particular,

$$\begin{aligned}
\mathbf{P}_{s^1}\sigma_{|V_\pm} &= -\frac{\hbar}{2}\left[(z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm} - (\bar{z}_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\right]\sigma_\pm + s\hbar z_\pm \psi_\pm \sigma_\pm \\
\mathbf{P}_{s^2}\sigma_{|V_\pm} &= \mp\frac{i\hbar}{2}\left[(z_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial z_\pm} + (\bar{z}_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial \bar{z}_\pm}\right]\sigma_\pm \pm is\hbar z_\pm \psi_\pm \sigma_\pm, \\
\mathbf{P}_{s^3}\sigma_{|V_\pm} &= \pm\hbar\left[\bar{z}_\pm \frac{\partial \psi_\pm}{\partial \bar{z}_\pm} - z_\pm \frac{\partial \psi_\pm}{\partial z_\pm}\right]\sigma_\pm \pm s\hbar \psi_\pm \sigma_\pm .
\end{aligned}$$

Polarization

- The sphere S_r^2 does not admit a real 1-dimensional distribution.
- However, S_r^2 has a complex structure defined by functions $z_\pm$ on $V_\pm$.
- Choose the polarization

$$F = \operatorname{span}\left\{ \frac{\partial}{\partial \bar{z}_+}, \frac{\partial}{\partial \bar{z}_-} \right\}$$

  corresponding to anti-holomorphic directions.
- A section $\sigma = \psi_\pm \sigma_\pm$ is covariantly constant along F if $\nabla_{\frac{\partial}{\partial \bar{z}_\pm}} (\psi_\pm \sigma_\pm) = 0$.
- But

$$\nabla_\pm \sigma_\pm = -i\hbar^{-1}\theta_\pm \otimes \sigma_\pm = -2\hbar^{-1} r(1 + z_\pm \bar{z}_\pm)^{-1} \bar{z}_\pm dz_\pm \otimes \sigma_\pm$$

  implies that $\sigma = \psi_\pm \sigma_\pm$ is covariantly constant along F if $\psi_\pm$ are holomorphic functions of $z_{\pm.}$.

Quantization

- Thus σ is covariantly constant along $F \Leftrightarrow \sigma$ is holomorphic.
- The subspace $\mathfrak{H}_F$ of $\mathfrak{H}$ consisting of holomorphic sections in $\mathfrak{H}$ is non-zero.
- The Hamiltonian vector fields of s^1, s^2 and s^3 preserve F.
- The the quantum operators $\mathbf{Q}_{s^i}$ are restrictions of $\mathbf{P}_{s^i}$ to holomorphic sections.

$$\begin{aligned}
\mathbf{Q}_{s^1}\sigma_{|V_\pm} &= -\frac{\hbar}{2}\left[(z_\pm^2 - 1)\frac{\partial \psi_\pm}{\partial z_\pm}\right]\sigma_\pm + s\hbar z_\pm \psi_\pm \sigma_\pm \\
\mathbf{Q}_{s^2}\sigma_{|V_\pm} &= \mp\frac{i\hbar}{2}\left[(z_\pm^2 + 1)\frac{\partial \psi_\pm}{\partial z_\pm}\right]\sigma_\pm \pm is\hbar z_\pm \psi_\pm \sigma_\pm, \\
\mathbf{Q}_{s^3}\sigma_{|V_\pm} &= \mp\hbar\left[z_\pm\frac{\partial \psi_\pm}{\partial z_\pm}\right]\sigma_\pm \pm s\hbar \psi_\pm \sigma_\pm .
\end{aligned}$$

- The operators $(i/\hbar)\mathbf{Q}_{s^1}$, $(i/\hbar)\mathbf{Q}$ and $(i/\hbar)\mathbf{Q}_{s^3}$ give the usual spin s representation of $\mathfrak{so}(3)$. It integrates to the spin s representation of $SO(3)$.

IV Polarizations

- A complex distribution $F \subset T^{\mathbb{C}}P = \mathbb{C} \otimes TP$ on a symplectic manifold (P, ω) is Lagrangian if, for each $p \in P$, the restriction of the symplectic form ω to the subspace $F_p \subset T_p^{\mathbb{C}}P$ vanishes identically, and $\mathrm{rank}_{\mathbb{C}} F = \frac{1}{2} \dim P$.
- A polarization of (P, ω) is an involutive complex Lagrangian distribution F such that

$$D = F \cap \overline{F} \cap TP \text{ and } E = (F + \overline{F}) \cap TP.,$$

  where D and E are involutive distributions on P.
- F is strongly admissible if the spaces P/D and P/E of integral manifolds of D and P, respectively, are quotients manifolds of P and the natural projection $P/D \to P/E$ is a submersion.
- F is positive if $i\omega(w, \bar{w}) \geq 0$ for every $w \in F$.
- A positive polarization F is semi-definite if $\omega(w, \bar{w}) = 0$ for $w \in F$ implies that $w \in D^{\mathbb{C}}$.
- F is real if $F = \bar{F}$. In this case $F = D \otimes \mathbb{C}$.
- F is complex if $F \oplus \bar{F} = TP$.

Quantization

- $\mathcal{C}^\infty(P)^{\mathbb{C}}_F$ complex-valued functions on P constant along F.
- If F is strongly admissible, it is spanned by the Hamiltonian vector fields of functions in $\mathcal{C}^\infty(P)^{\mathbb{C}}_F$.
- $\mathcal{C}^\infty_F(P)$ denote the space of functions on P whose Hamiltonian vector fields preserve F.
- $\mathcal{C}^\infty_F(P)$ is a Poisson subalgebra of $\mathcal{C}^\infty(P)$.
- The space of polarized sections of L is

$$\mathcal{S}^\infty_F(L) = \{\sigma \in \mathcal{S}^\infty(L) \mid \nabla_u \sigma = 0 \text{ for all } u \in F\}.$$

- The quantization map $\mathbf{Q}$ relative to a polarization F is the restriction of the prequantization map $\mathbf{P}$ to domain $\mathcal{C}^\infty_F(P) \times \mathcal{S}^\infty_F(L) \subset \mathcal{C}^\infty(P) \times \mathcal{S}^\infty(L)$ and codomain $\mathcal{S}^\infty_F(L) \subset \mathcal{S}^\infty(L)$. Thus,

$$\mathbf{Q} : \mathcal{C}^\infty_F(P) \times \mathcal{S}^\infty_F(L) \to \mathcal{S}^\infty_F(L) : (f, \sigma) \mapsto \mathbf{Q}_f \sigma = \mathbf{P}_f \sigma = (i\hbar \nabla_{X_f} + f)\sigma.$$

Quantization of momentum map

- Assume that a Hamiltonian action $\Phi : G \times P \to P$ preserves F.
- For each $\xi \in \mathfrak{g}$, the momentum J_ξ is in $\mathcal{C}_F^\infty(P)$.
- The map $\xi \mapsto (i\hbar)^{-1}\mathbf{Q}_{J_\xi}$ is a representation of $\mathfrak{g}$ on $\mathcal{S}_F^\infty(\mathcal{L})$.
- If the action Φ of G on P lifts to an action of G on L by connection preserving automorphisms, then this representation of $\mathfrak{g}$ integrates to a linear representation

  $$\mathbf{R} : G \times \mathcal{S}_F^\infty(\mathcal{L}) \to \mathcal{S}_F^\infty(\mathcal{L}) : (g, \sigma) \mapsto \mathbf{R}_g \sigma$$

  of G on $\mathcal{S}_F^\infty(\mathcal{L})$, which we call the quantization representation of G.
- If F is Kähler, $\mathbf{R}$ restricts to a unitary representation

  $$\mathbf{U} : G \times \mathfrak{H}_F \to \mathfrak{H}_F : (g, \sigma) \mapsto \mathbf{R}_g \sigma$$

  of G on $\mathfrak{H}_F$.

Kähler polarizations

- Recall that a polarization F of (P, ω) is Kähler if $F \oplus \bar{F} = T^{\mathbb{C}}P$ and $i\omega(w, \bar{w}) > 0$ for all non-zero $w \in F$. These assumptions imply that there is a complex structure $\mathbf{J}$ on P such that F is the space of antiholomorphic directions. Moreover, P is a Kähler manifold such that $-\omega$ is the Kähler form on P.
- If F is Kähler, L is a holomorphic line bundle over P and the space $\mathcal{S}_F^\infty(L)$ of polarized sections coincides with the space of holomorphic sections. and the linear representation $\mathbf{R}$ of G on $\mathcal{S}_F^\infty(\mathcal{L})$ gives rise to a unitary representation $\mathbf{U}$ of G on

$$\mathcal{H}_F = \mathcal{H} \cap S_F^\infty(\mathcal{L}).$$

- Co-adjoint orbits of a compact connected Lie group admit Kähler polarizations.
- For a quantizable co-adjoint orbit (P, ω) of G, the unitary quantization representation $\mathbf{U}$ of G on $\mathcal{H}_F$ is irreducible, and the map $P \mapsto \mathbf{U}$ is a bijection.

Cotangent polarization

- Assume that $P = T^*Q$ is the cotangent bundle of a manifold Q, with the cotangent bundle projection $\pi : T^*Q \to Q$.
- The Liouville form θ of the cotangent bundle T^*Q is defined as follows. For each $p \in T^*Q$ and every $u \in T_pT^*Q$,

$$\theta(u) = \langle p \mid T\vartheta(u)\rangle.$$

- $\omega = d\theta$ is the canonical symplectic form of T^*Q.
- The cotangent polarization is the complexification $F = (\ker T\pi) \otimes \mathbb{C}$ of the kernel of the cotangent bundle projection.
- Since ω is exact, the prequantization line bundle is trivial; that is $L = \mathbb{C} \times P$.
- We choose a trivializing section $\sigma_0 : P \to L : p \mapsto (1, p)$ of L, $\langle \sigma_0, \sigma_0 \rangle = 1$ and the covariant derivative operator ∇ such that

$$\nabla \sigma_0 = i\hbar^{-1}\theta \otimes \sigma_0.$$

Non-normalizability of polarized sections

- $\mathcal{C}^\infty(P)^{\mathbb{C}}_F$ consists of complex-valued functions on $P = T^*Q$ that are constant along the fibres of $\pi : T^*Q \to Q$; that is
$$\mathcal{C}^\infty(P)^{\mathbb{C}}_F = \{\pi^* k \mid k \in (\mathbb{C} \otimes \mathcal{C}^\infty(Q))\}.$$
- The space $\mathcal{S}^\infty_F(L)$ of polarized sections of $\mathcal{L}$ is given by
$$\mathcal{S}^\infty_F(L) = \{\pi^*\Psi\sigma_0 \mid \Psi \in \mathbb{C} \otimes \mathcal{C}^\infty(Q)\}.$$
- If $\sigma_i = \pi^*\Psi\sigma_0$, for $i = 1, 2$,
$$\langle \pi^*\Psi_1\sigma_0, \pi^*\Psi_2\sigma_0 \rangle = \pi^*(\overline{\Psi}_1\Psi_2) = (\overline{\Psi}_1\Psi_2) \circ \pi.$$
- Since the fibres of the cotangent bundle projection π are not compact,
$$\int_{T^*Q} \langle \sigma, \sigma \rangle \, \omega^n = \int_{T^*Q} \vartheta^*(\overline{\Psi}\Psi) \, \omega^n = \infty$$
unless $\Psi = 0$.
- Hence, by passing to polarized sections we have lost the scalar product.

Quantization

- $\pi^* C^\infty(Q)$ is the space of functions in $C^\infty(T^*Q)$ that are constant along $\ker T\pi$,

$$\pi^* C^\infty(Q) == \{\pi^* k \mid k \in C^\infty(Q)\}.$$

- $C^\infty_F(T^*Q)$ is a module over $\pi^* C^\infty(Q)$ generated by functions in $\pi^* C^\infty(Q)$ and of functions k in and evaluations $\langle \theta, X \rangle$ of the Liouville form on smooth vector fields on Q.
- For each $f_1 = \pi^* k$, $f_2 = \langle \theta, X \rangle$ and $\sigma = \pi^* \Psi \sigma_0$

$$\begin{aligned} \mathbf{Q}_{f_1}\sigma &= \mathbf{Q}_{\pi^* k}(\pi^* \Psi \sigma_0) = \pi^*(k\Psi)\sigma_0, \\ \mathbf{Q}_{f_2}\sigma &= \mathbf{Q}_{\langle \theta, X \rangle}(\pi^* \Psi \sigma_0) = -i\hbar \pi^* X(\Psi)\sigma_0. \end{aligned}$$

- Functions that are not linear on the fibres of the cotangent bundle projection are not quantizable in this scheme, because their Hamiltonian vector fields do not preserve F.

Alternative scalar product

- We can introduce the scalar product in $\mathcal{S}_F^\infty(L) = \{\pi^*\Psi\sigma_0 \mid \Psi \in \mathbb{C} \otimes \mathcal{C}^\infty(Q)\}$ by choosing a volume density κ on and set

$$((\pi^*\Psi_1\sigma_0 \,||\, \pi^*\Psi_2\sigma_0))_\kappa = \int_Q \overline{\Psi}_1\Psi_2\kappa \ .$$

- Let $\mathfrak{H}_\kappa$ be the Hilbert space of square-integrable sections with respect to $((.\,||\,.))_\kappa$.
- The operator $\mathbf{Q}_{\pi^*k}$ is self adjoint on $\mathfrak{H}_\kappa$.The operator $\mathbf{Q}_{\langle\theta,X\rangle}$ is self-adjoint on $\mathfrak{H}_\kappa$ only if X preserves κ.
  In order to achieve self-adjointness, we can modify the representation space by multiplying $\pi^*\Psi\sigma_0$ by $\sqrt{\kappa}$. This will modify the operator $\mathbf{Q}_{\langle\theta,X\rangle}$ by the inclusion of the action of X on κ, and the modified operator will be self-adjoint.
- We have to define what we mean by a $\sqrt{\kappa}$. First, we discuss $\sqrt{|\kappa|}$ and then lift our considerations to the double covering.

Half-densities

- Consider a strongly admissible real polarization $F = D \otimes \mathbb{C}$.
- $\pi_D : P \to P/D$ associates to each $p \in P$ the integral manifold of D through p.
- The bundle $\mathcal{B}F$ of all linear frames of F is a principal bundle with structure group $Gl(n, \mathbb{C})$ and the projection map $\beta : \mathcal{B}F \to T^*Q$.
- $\mathcal{B}F$ restricted to integral manifolds of D has a flat affine connection. We say that the bundle $\mathcal{B}F$ has a flat partial affine connection covering D.
- The bundle $\sqrt{|\wedge^n|\, F}$ of half-densities associated to F is the associated bundle of $\mathcal{B}F$ corresponding to the homomorphism

$$|\det|^{1/2} : Gl(n, \mathbb{C}) \to \mathbb{C}^\times : A \mapsto |\det A|^{1/2}.$$

- A section $\mu : P \to \sqrt{|\wedge^n|\, F}$ corresponds to an equivariant function $\mu^\sharp : \mathcal{B}F \to \mathbb{C}$ such that

$$\mu^\sharp(bA) = |\det A|^{-1/2}\, \mu^\sharp(b).$$

Scalar product given by half-densities

- Let $(u_1, ..., u_n, v_1, ..., v_n)$ be a symplectic frame in T_pP such that $v_1, ..., v_n$ are in D_p. Then $(T\pi_D(u_1), ..., T\pi_D(u_n))$ is a basis in $T_{\pi_P(p)}(P/D)$. Define
$$\langle \mu_1, \mu_2 \rangle \left((T\pi(u_1), ..., T\pi(u_n) \right) = \overline{\mu_1(v_1, ..., v_2)}\mu_2(v_1, ..., v_n).$$
- For half-densities μ_1 and μ_2 that are covariantly constant along D, the pairing $\langle \mu_1, \mu_2 \rangle$ given is a well defined density on P/D.
- As an alternative to choosing a density κ on Q, we can take the representation space to consist of sections of $L \otimes \sqrt{|\wedge^n| F}$ that are covariantly constant along F, and define the scalar product of $\sigma_1 \otimes \mu_1$ and $\sigma_2 \otimes \mu_2$ by
$$((\sigma_1 \otimes \mu_1 \,||\, \sigma_2 \otimes \mu_2))_{|F|} = \int_Q \pi_* \langle \sigma_1, \sigma_2 \rangle \langle \mu_1, \mu_2 \rangle$$
- Denote by $\mathfrak{H}_{|F|}$ the space of sections of $L \otimes \sqrt{|\wedge^n| F}$ that are covariantly constant along F and square integrable with respect to this scalar product.

Pairing of half-densities

- $F_1 = D_1 \otimes \mathbb{C}$ and $F_2 = D_2 \otimes \mathbb{C}$ such that $D_1 \oplus D_2 = TP$.
- Sections $\sigma_i \otimes \mu_i \in \mathfrak{H}_{|F_i|}$ pair to a function $\langle \sigma_1 \otimes \mu_1, \sigma_2 \otimes \mu_2 \rangle$ on P such that, for each $p \in P$, and a symplectic basis $(\mathbf{u}_1, \mathbf{u}_2)$ in $T_p P$ such that $\mathbf{u}_1$ is a basis in D_1 and $\mathbf{u}_2$ is a basis in D_2,

$$\langle \sigma_1 \otimes \mu_1, \sigma_2 \otimes \mu_2 \rangle (p) = \langle \sigma_1, \sigma_2 \rangle (p) \overline{\mu_1(\mathbf{u}_1)}\, \mu_2(\mathbf{u}_2).$$

- The integral

$$\mathcal{K}_{21}(\sigma_1 \otimes \mu_1, \sigma_2 \otimes \mu_2) = \int_P \langle \sigma_1 \otimes \mu_1, \sigma_2 \otimes \mu_2 \rangle \, |\omega^n|$$

gives a sesquilinear pairing

$$\mathcal{K}_{21} : \mathfrak{H}_{|F_1|} \times \mathfrak{H}_{|F_2|} \to \mathbb{C},$$

- It defines a $\mathbb{C}$-linear map $\mathcal{L}_{21} : \mathfrak{H}_{|F_1|} \to \mathfrak{H}_{|F_2|}$ such that $\mathcal{K}_{21}(\sigma_1 \otimes \mu_1, \sigma_2 \otimes \mu_2)$ is the scalar product in $\mathfrak{H}_{|F_2|}$ of $\mathcal{L}_{21}(\sigma_1 \otimes \mu_1)$ and $\sigma_2 \otimes \mu_2$; that is

$$((\mathcal{L}_{21}(\sigma_1 \otimes \mu_1) \,||\, \sigma_2 \otimes \mu_2))_{|F_2|} = \mathcal{K}_{21}(\sigma_1 \otimes \mu_1, \sigma_2 \otimes \mu_2).$$

Half-forms

- $Ml(n, \mathbb{C})$ is the double cover of $Gl(n, \mathbb{C})$.
- The bundle of metalinear frames of F is a principal $Ml(n, \mathbb{C})$ bundle $\widetilde{\mathcal{B}}F$ that double covers $\mathcal{B}F$
- The bundle $\sqrt{\wedge^n F}$of half-forms relative to F is the associated fibre bundle of $\mathcal{B}F$ corresponding to the holomorphic character $\sqrt{\det} : Ml(n, \mathbb{C}) \to \mathbb{C}^{\times}$.
- To each pair ν_1, ν_2 of sections of $\sqrt{\wedge^n F}$, there corresponds a density $\langle \nu_1, \nu_2 \rangle$ on P/D.
- The space $\mathfrak{H}_F$ of sections of $L \otimes \sqrt{\wedge^n F}$ that are covariantly constant along F, has scalar product

$$(\sigma_1 \otimes \nu_1 \mid \sigma_2 \otimes \nu_2)_F = \int_{P/D} \pi_* \langle \sigma_1, \sigma_2 \rangle \langle \nu_1, \nu_2 \rangle .$$

Blattner-Kostant-Sternberg kernel

- The metaplectic group $Mp(n)$ is the double covering group of the symplectic group $Sp(n)$.
- The the metaplectic frame bundle $\widetilde{\mathcal{B}}_\omega TP$ is the principal $Mp(n)$-bundle over P that double covers the bundle $\mathcal{B}_\omega TP$ of symplectic frames of (P, ω).
- The choice of the metaplectic frame bundle $\widetilde{\mathcal{B}}_\omega TP$ determines uniquely the metalinear frame bundle $\widetilde{B}F$ corresponding to F.
- The action on P of each 1-parameter group φ_t of symplectomorphisms of (P, ω) lifts to to the metaplectic frame bundle $\widetilde{B}_\omega TP$ and it induces a principal fibre bundle isomorphism from $\widetilde{B}F$ to $\widetilde{B}(T\varphi_t(F))$.
- If F_1 and F_2 are real polarizations such that $F_1 \oplus F_2 = TP \otimes \mathbb{C}$, there is a sesquilinear map $\widetilde{\mathcal{K}}_{21} : \mathfrak{H}_{F_1} \times \mathfrak{H}_{F_2} \to \mathbb{C}$, called the BKS kernel.
- $\widetilde{\mathcal{K}}_{21}$ defines a $\mathbb{C}$-linear map $\widetilde{\mathcal{L}}_{21} : \mathfrak{H}_{F_1} \to \mathfrak{H}_{F_2}$ such that

$$\Big((\widetilde{\mathcal{L}}_{21}(\sigma_1 \otimes \nu_1) \,||\, \sigma_2 \otimes \nu_2) \Big)_{F_2} = \widetilde{\mathcal{K}}_{21}(\sigma_1 \otimes \nu_1, \sigma_2 \otimes \nu_2).$$

Example

- $(T^*\mathbb{R}^3, d\mathbf{p} \wedge d\mathbf{q})$ the phase space of a free particle with position variables $\mathbf{q} \in \mathbb{R}^3$ and conjugate momenta $\mathbf{p}$.
- The one parameter group $\exp tX_H$ generated by the Hamiltonian vector field X_H of the Hamiltonian $H = \frac{1}{2}\mathbf{p}^2$ maps the vertical polarization F to $F_t = T(\exp tX_H)F$ such that

$$F_t \oplus F = TP \otimes \mathbb{C} \text{ for } t \neq 0,$$

  and maps sections $\sigma \otimes \nu$ of $\mathfrak{H}_F$ to sections $\sigma_t \otimes \nu_t$ of $\mathfrak{H}_{F_t}$.
- Let $\widetilde{\mathcal{L}}_t : \mathfrak{H}_{F_t} \longrightarrow \mathfrak{H}_F$ be the map defined by the *BKS* kernel.
- Set $\mathbf{Q}_H(\sigma \otimes \nu) = i\hbar \frac{d}{dt}\widetilde{\mathcal{L}}_t(\sigma_t \otimes \nu_t)_{|t=0}$.
- If $\sigma \otimes \nu = \Psi \otimes \sigma_0 \otimes \nu_0$, for an appropriate section ν_0 of $\sqrt{\wedge^3 F}$, then

$$\mathbf{Q}_H(\Psi \otimes \sigma_0 \otimes \nu_0) = -\frac{\hbar^2}{2}\Delta\Psi \otimes \sigma_0 \otimes \nu_0$$

  gives the usual expression for the quantum Hamiltonian of a free particle.

MpC quantization

- In the presentation above, we defined the representation space $\mathfrak{H}_F$ as the space of square integrable sections of $L \otimes \sqrt{\wedge^n F}$ that are covariantly constant along F. This presumes the existence of the prequantization line bundle L and the bundle $\sqrt{\wedge^n F}$ of $\frac{1}{2}$-forms relative to F, which are equivalent to the vanishing of the corresponding characteristic classes.
- The tensor product $L \otimes \sqrt{\wedge^n F}$ may exist, even though individual factors do not exist.
- The $\mathrm{Mp}^{\mathbb{C}}$-quantization was developed by Rawnsley and Robinson in order to circumvent this difficulty. Their theory is more general than the geometric quantization presented here, but it is less intuitive.

V Completely integrable systems

- Consider a completely integrable system on a symplectic manifold (P, ω).
- Let (A_i, φ_i) be action angle coordinates defined on an open dense subset P_0 of P.
- The restriction to P_0 of the symplectic form ω is an exact symplectic form $\omega_0 = d\theta$ on P_0, where $\theta = \sum_{i=1}^{n} A_i d\varphi_i$.
- We have a symplectic action on (P_0, ω_0) of the the torus group $\mathbb{T}^n$ with the momentum map
  $J : P_0 \to \mathbb{R}^n : p \mapsto J(p) = (A_1(p), ..., A_n(p))$.
- For each $i = 1, ..., n$, the Hamiltonian vector field X_{A_i} generates the action on P_0 of the i^{th} component $\mathbb{T}_i$ of the torus group $\mathbb{T}^n = \mathbb{T} \times \mathbb{T} \times ... \times \mathbb{T}$.
- $O_{i,p}$ the orbit of $\mathbb{T}_i$ through $p \in P$. Clearly, A_i is constant on each orbit $O_{i,p}$.
- The Hamiltonian vector fields $X_{A_i} = \frac{\partial}{\partial \varphi_i}$ span a Lagrangian distribution D on P_0.

Bohr-Sommerfeld conditions

- We consider a polarization $F = D \otimes \mathbb{C}$ of (P_0, ω_0).
- Since ω_0 is exact, the prequantization line bundle is trivial, $L = \mathbb{C}_0 \oplus P_0$.
- Trivializing section $\sigma_0 : P \to \mathcal{L} : p \mapsto (1, p)$.
- A section $\sigma = \Psi\sigma_0$ is covariantly constant along F if, or all $j = 1, ..., n$,

$$\nabla_{\frac{\partial}{\partial\varphi_j}}(\Psi\sigma_0) = \left(\frac{\partial\Psi}{\partial\varphi_j} + i\hbar^{-1}A_j\Psi\right)\sigma_0 = 0.$$

- This equation has single-valued solution Ψ only if the Bohr-Sommerfeld conditions

$$\int_0^{2\pi} A_j d\varphi_j = m_j h$$

are satisfied for $j = 1, ..., n$, and integers $m_1, ..., m_n$.
- Hence, Ψ cannot be a smooth function on P. It is a distribution supported on the Bohr-Sommerfeld tori.

Representation space

- To each Bohr-Sommerfeld torus

$$Q_{m_1,\dots,m_n} = \{p \in P \mid A_j(p) = m_j \hbar \ \text{ for } \ j = 1, \dots, n\},$$

we may associate a non-zero distribution section $\sigma_{m_1,\dots,m_n}$ given by

$$\begin{aligned} &\sigma_{m_1,\dots,m_n}(p) = \\ = \ & \exp\left(-i(\sum_{j=1}^{n} m_j \varphi_j\right) \delta(A_1(p) - m_1 \hbar) \dots \delta(A_n(p) - m_n \hbar) \sigma_0(p). \end{aligned}$$

- The collection $\{\sigma_{m_1,\dots,m_n}\}$ of sections of L forms a basis of an infinite dimensional vector space $\mathfrak{E}$, in which we may define a scalar product $(. \mid .)$ such that the basis $\{\sigma_{m_1,\dots,m_n}\}$ is orthonormal.
- The Hilbert space $\mathfrak{H}$ of distribution sections in $\mathfrak{E}$ with of finite norm is the space of quantum states of Bohr-Sommerfeld quantization.

Inclusion of half-forms

- In the preceding lectures we saw that, for the cotangent bundle polarization, we have to consider sections of $L \otimes \sqrt{\wedge^n F}$ in order to get the Schrödinger quantum mechanics of a free particle.
- For a completely integrable Hamiltonian system, replacing sections of L by sections of $L \otimes \sqrt{\wedge^n F}$ gives modified Bohr-Sommerfeld conditions

$$\int_0^{2\pi} A_j d\varphi_j = (m_j + \epsilon_j)\, h,$$

where $\epsilon_j \in \{0, \frac{1}{2}\}$ generates the holonomy group of the flat partial connection in $\sqrt{\wedge^n F}$ restricted to the orbit O_j.

Bohr-Sommerfeld quantization

- The space $C^\infty_F(P_0)$ of functions $f \in C^\infty(P_0)$ such that X_f preserves the polarization $F = D \otimes \mathbb{C}$ coincides with the space $C^\infty(P_0)_F$ of functions that are constant along D.
- Thus, quantizable functions in the Bohr-Sommerfeld theory are smooth functions of the action variables $A_1, ..., A_n$.
- For each $j = 1, ..., n$, the quantum operator $\mathbf{Q}_{A_j}$ corresponding to A_j is diagonal in the basis $\{\sigma_{m_1,...,m_n}\}$ and

$$\mathbf{Q}_{A_j}\sigma_{m_1,...,m_n} = m_j \hbar \sigma_{m_1,...,m_n}.$$

- A single particle in $\mathbb{R}^3$ with spherically symmetric Hamiltonian H is completely integrable with the commuting variables J_3, $|J|$ and H, where $|J|$ is the length of the angular momentum vector. The Bohr-Sommerfeld quantization gives the spectra of quantum operators $\mathbf{Q}_{J_3}$, $\mathbf{Q}_{|J|}$ and $\mathbf{Q}_H$ but it does not allow for quantization of the position variables $\mathbf{x}$ and the momentum variables $\mathbf{p}$.

Problems with Bohr-Sommerfeld quantization

- In the formulation given above, only functions of the momenta A_j are quantizable in the Bohr-Sommerfeld.
- Recall the quotation from Dirac's lecture:
  "So Heisenberg said that the Bohr orbits are not very important. The things that are observed, or which are connected closely with the observed quantities, are all associated with two Bohr orbits and not with one Bohr orbit: *two* instead of *one*."
- In other words, the problem with Bohr-Sommerfeld quantization is that the operators, which we obtain by quantization of dynamical variables do not act transitively on the space of states.
- On the other hand, the Bohr-Sommerfeld conditions, as well as their modifications due to half-forms imply that the orthonormal basis $\{\sigma_{m_1,...,m_n}\}$ in $\mathfrak{H}$ is locally a lattice.

Completely integrable systems

- If $\{\sigma_{m_1,\dots,m_n}\}$ is a lattice, we have well defined operators corresponding to shifting along the generators of the lattice.
- For each $i = 1, \dots, n$, let

$$\begin{aligned}\mathbf{m} &= \{m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_n\} \\ \mathbf{m}_i &= \{m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_n\} \\ \mathbf{m}^i &= \{m_1, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots, m_n\}.\end{aligned}$$

- We can define shifting operators $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ by

$$\mathbf{a}_i \sigma_{\mathbf{m}} = \sigma_{\mathbf{m}_i} \text{ and } \mathbf{a}_i^\dagger \sigma_{\mathbf{m}} = \sigma_{\mathbf{m}^i}.$$

- The Poisson bracket relations between actions and angles are $\{e^{-\imath\varphi_k}, A_j\} = -\imath\delta_{kj}\, e^{-\imath\varphi_k}$. Hence, Dirac's quantization conditions $[\mathbf{Q}_{f_1}, \mathbf{Q}_{f_2}] = \imath\hbar\, \mathbf{Q}_{\{f_1, f_2\}}$ suggest the identification

$$\mathbf{a}_k = \mathbf{Q}_{e^{-\imath\varphi_k}} \text{ and } \mathbf{a}_k^\dagger = \mathbf{Q}_{e^{\imath\varphi_k}}.$$

Shifting operators

- The basic vectors $\sigma_{\mathbf{m}}$ introduced above have supports in an open subset P_0 of (P, ω) with independent action-angle variables.
- In order to extend our discussion to the whole (P, ω) we consider functions $\chi_k = r_k e^{-\imath\varphi_k}$, where the coefficient r_k depends only on the actions and vanishes at the points at which $e^{\imath\varphi_k}$ is undefined.
- Using the Poisson bracket relations $\{\chi_k, A_j\} = -\imath\delta_{kj}\chi_k$ and Dirac's quantization conditions $[\mathbf{Q}_{\chi_k}, \mathbf{Q}_{A_j}] = \delta_{kj}\hbar\mathbf{Q}_{\chi_k}$, we get for every basic vector $\sigma_{\mathbf{m}}$ of $\mathfrak{H}$,

$$\mathbf{Q}_{A_j}(\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}) = \hbar(m_j - 1)\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}.$$

  Thus, $\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}$ is proportional to $\sigma_{\mathbf{m}_j}$. Similarly $\mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}}$ is proportional to $\sigma_{\mathbf{m}^j}$.
- Hence, $\mathbf{Q}_{\chi_j}$ and $\mathbf{Q}_{\bar{\chi}_j}$ act as shifting operators,

$$\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} = b_{\mathbf{m},j}\sigma_{\mathbf{m}_j} \text{ and } \mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}} = c_{\mathbf{m},j}\sigma_{\mathbf{m}^j}$$

  for some coefficients $b_{\mathbf{m},j}$ and $c_{\mathbf{m},j}$.

Determination of shifting operators

- $$\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} = b_{\mathbf{m},j}\sigma_{\mathbf{m}_j} \text{ and } \mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}} = c_{\mathbf{m},j}\sigma_{\mathbf{m}^j}$$
- We can use Dirac's quantization conditions
$$[\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\chi_k}] = \imath\hbar\,\mathbf{Q}_{\{\chi_j,\chi_k\}} \quad \text{and} \quad [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\bar{\chi}_k}] = \imath\hbar\,\mathbf{Q}_{\{\chi_j,\bar{\chi}_k\}}$$
the identification
$$\mathbf{Q}^{\dagger}_{\chi_j} = \mathbf{Q}_{\bar{\chi}_j}$$
to determine the coefficients $b_{\mathbf{m},j}$ and $c_{\mathbf{m},j}$, which must satisfy the boundary conditions:
$b_{\mathbf{m},j} = 0$ if $Q_{\mathbf{m}_j} = \emptyset$ and $c_{\mathbf{m},j} = 0$ if $Q_{\mathbf{m}^j} = \emptyset$.
- We refer to the Bohr-Sommerfeld quantization enriched by the introduction of shifting operators as the Bohr-Sommerfeld-Heisenberg quantization.

Examples

- The phase space of the 1-dimensional harmonic oscillator is $P = \mathbb{R}^2$ with coordinates (p, q) and the symplectic form $\omega = p \wedge q$. The Hamiltonian is $H = \frac{1}{2}(p^2 + q^2)$, In polar coordinates $(p, q) = (r \cos \varphi, r \sin \varphi)$, where $r = \sqrt{p^2 + q^2}$ and $\varphi = \tan \frac{q}{p}$, we have $\omega = H \wedge d\varphi$. Here $H = \frac{1}{2} r^2$ is the action variable, while φ is the corresponding angle. Choosing $\chi = re^{-\imath\varphi}$ and $\bar{\chi} = re^{i\varphi}$ leads to the Bargmann-Fock quantization. It should be noted that $r = \sqrt{2H}$ is not a smooth function of H, but χ is in $C^\infty(P)$.
- The Bohr-Sommerfeld-Heisenberg quantization applied to the 2-dimensional harmonic oscillator gives its full quantum theory including a quantization representation of $SU(2)$.
- In the same way we can quantize co-adjoint orbits of $SO(3)$ and obtain the corresponding irreducible unitary representations.
- At present, Richard Cushman and I are working on the BSH-quantization of the mathematical pendulum.

Remarks

- Note that completely integrable Hamiltonian systems lead to real polarizations with singularities.
- The main problem for the BSH-quantization is to find a way of taking these singularities into account.
- The aim is to obtain a consistent quantum theory with quantum operators acting transitively of the Hilbert space of states.
- Each type of singularity requires individual attention.
- Additional problem: the presence of quantum monodromy.
- Only when we have solved problems posed by many individual systems, we can claim that we have a sufficiently general theory.

Further topics

- Cohomological quantization of completely integrable systems
- Commutation of quantization and reduction.
- Index approach to quantization.
- Independence of polarization.
- Observability of half-form corrections.
- Toeplitz quantization and semi-classical approximations.
- Connections with "hard" symplectic topology.

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