On the Geometry of Pseudo-Euclidean Spaces

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Gyrodecomposition of Groups

$$G = BH$$

Gyrodecomposition of Groups: $G = BH$.

1. $B \subseteq G$ (B is a subset of the group G)
2. $H < G$ (H is a subgroup of G)
3. Unique ($g \in G \Rightarrow g = bh, \quad b \in B, \quad h \in H$)
4. $I_G \in B$
5. $B = B^{-1}$
6. B is normalized by H ($hBh^{-1} \subseteq B$)

Any Group Gyrodecomposition

$G = BH$

Induces a

(1) Binary Operation, $\oplus$, in B called Gyroaddition; and

(2) Gyroautomorphisms of the Gyrogroupoid $(B, \oplus)$, called gyrations.

$$b_1, b_2 \in B \Rightarrow b_1 b_2 \in G \Rightarrow b_1 b_2 = b_{12} h(b_1, b_2)$$

Definition

$b_1 \oplus b_2 = b_{12}$

$\mathrm{gyr}[b_1, b_2] b_3 = h(b_1, b_2) b_3 (h(b_1, b_2))^{-1}$

for all $b_1, b_2, b_3 \in B$.

Here

$b_1 \oplus b_2$ is the gyroaddition of b_1 and b_2; and

$\mathrm{gyr}[b_1, b_2] b_3$ is the application to b_3 of the gyration

$\mathrm{gyr}[b_1, b_2]$ generated by b_1 and b_2.

The gyrogroupoid $(B, \oplus)$ is a gyrogroup, the definition of which follows.

Definition

A groupoid $(B, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms for all $a, b, c \in B$:

1. $0 \oplus a = a$ (Left Identity)
2. $\ominus a \oplus a = 0$ (Left Inverse)
3. $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$ (Left Gyroassociative Law)
4. $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ (Gyrations are Automorphisms)
5. $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ (Gyration Left Reduction Law)

Definition

A gyrogroup $(B, \oplus)$ is gyrocommutative if its binary operation satisfies the Gyrocommutative Law
$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

The famous concrete example of a

group gyrodecomposition

is the decomposition of the Lorentz group $SO(1, n)$, $n \in \mathbb{N}$, into *boosts* and space rotations of time-space coordinates.
Remarkably, the binary operation in the ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}$$

that the gyrodecomposition of the Lorentz group $SO(1, n)$ induces turns out to be the Einstein addition of relativistically admissible velocities.
Accordingly the gyrodecomposition of the Lorentz group $SO(1, n)$ enables us to

1. recover Einstein addition, $\oplus$, in the ball $\mathbb{R}^n_c$; and to
2. discover the gyrogroup structure of the resulting Einstein groupoid $(\mathbb{R}^n_c, \oplus)$.

Einstein addition, $\oplus$, is a binary operation in the c-ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}$$

$n = 1, 2, 3, \ldots$, of the Euclidean n-space $\mathbb{R}^n$. It is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}$$

where

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}} \geq 1$$

Gyrations in Einstein gyrogroups capture abstractly the special relativistic effect known as *Thomas precession*, which we extend to *Thomas gyration*. Our use of the prefix "gyro" thus stems from Thomas gyration.

Gyrations in Einstein gyrogroups $(\mathbb{R}^n_c, \oplus)$ are automorphisms of $(\mathbb{R}^n_c, \oplus)$ given in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$.

$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ measures the extent of deviation of Einstein addition from associativity.

The Rich Mathematical Life of Einstein Addition

$\mathbf{u}\oplus\mathbf{v} = \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u})$	Gyrocommutative Law
$\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) = (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}$	Left Gyroassociative Law
$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} = \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w})$	Right Gyroassociative Law
$\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}]$	Gyration Left Reduction Law
$\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}]$	Gyration Right Reduction Law
$\mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}]$	Gyration Even Property
$(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1} = \mathrm{gyr}[\mathbf{v},\mathbf{u}]$	Gyration Inversion Law

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^n_c$.

Einstein addition admits scalar multiplication, giving rise to Einstein gyrovector spaces.

$k \otimes \mathbf{v} = \mathbf{v} \oplus \mathbf{v} \ldots \oplus \mathbf{v}$ (k terms, $k = 1, 2, 3, \ldots$)
is the Einstein addition of k copies of $\mathbf{v} \in \mathbb{R}^n_c$
Then

$$k \otimes \mathbf{v} = \frac{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^k - \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^k}{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^k + \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^k} \frac{c\mathbf{v}}{\|\mathbf{v}\|}$$

Suggestively, Einstein scalar multiplication is defined by this equation with $k \in \mathbb{N}$ replaced by $r \in \mathbb{R}$.

Definition

Einstein scalar multiplication is given by the equation

$$r \otimes \mathbf{v} = \frac{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^r} \frac{c\mathbf{v}}{\|\mathbf{v}\|}$$

where r is any scalar, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_c^n$, $\mathbf{v} \neq \mathbf{0}$, and $r \otimes \mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v} \otimes r = r \otimes \mathbf{v}$.

Example: Einstein half

$$\tfrac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v}$$

Classical and Relativistic Kinetic Energy of a moving particle with velocity $\mathbf{v}$ relative to a rest frame

Classical Kinetic Energy:

$$K_{cls} = \frac{1}{2} m\mathbf{v}^2 = (\tfrac{1}{2}\mathbf{v})\cdot(m\mathbf{v})$$

Relativistic Kinetic Energy:

$$K_{rel} = c^2 m(\gamma_{\mathbf{v}} - 1) = (\tfrac{1}{2}\otimes\mathbf{v})\cdot(m\gamma_{\mathbf{v}}\mathbf{v})$$

The remarkable analogy that Einstein Scalar Multiplication, $\otimes$, captures here is clear.

From Einstein Addition $\oplus_{\mathrm{E}}$

to Möbius Addition $\oplus_{\mathrm{M}}$ in the Ball $\mathbb{R}_c^n$

Einstein half is involved in the gyroisomorphism between Einstein addition and Möbius addition in the ball:

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v})$$

$$\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} = 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes\mathbf{v})$$

The Einstein gyrogroup $(\mathbb{R}_c^n, \oplus_{\mathrm{E}})$ and the Möbius gyrogroup $(\mathbb{R}_c^n, \oplus_{\mathrm{M}})$ are thus gyroisomorphic (that is, they are isomorphic in the sense of gyrogroups and gyrovector spaces).

Einstein gyrovector spaces $(\mathbb{R}^n_c, \oplus_{\mathrm{E}}, \otimes)$ form the algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry,
and
Möbius gyrovector spaces $(\mathbb{R}^n_c, \oplus_{\mathrm{M}}, \otimes)$ form the algebraic setting for the Poincaré ball model of hyperbolic geometry,
just as
vector spaces form the algebraic setting for the standard model of Euclidean geometry.
As a result, analytic hyperbolic geometry can now be studied in full analogy with the study of analytic Euclidean geometry, as evidenced from 7 books on analytic hyperbolic geometry published during 2001 – 2015.
We thus see that gyrogroups and gyrovector spaces play a universal computational role, which extends far beyond the domain of Einstein's special relativity theory.

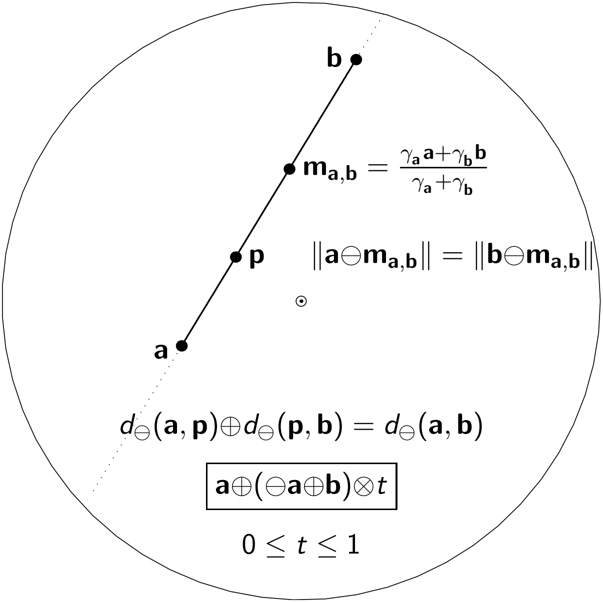


Figure: $\oplus = \oplus_E$. A gyroline in an Einstein gyrovector plane

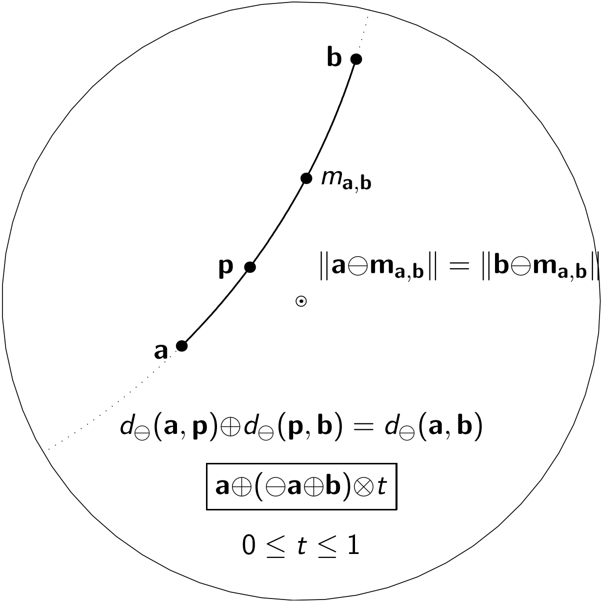
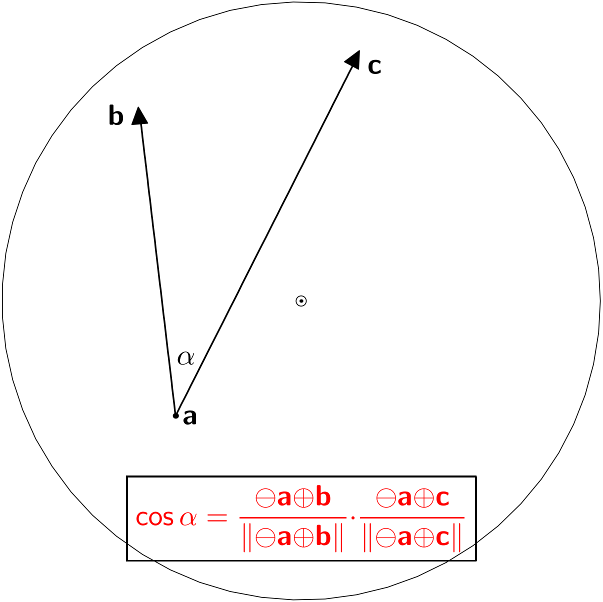
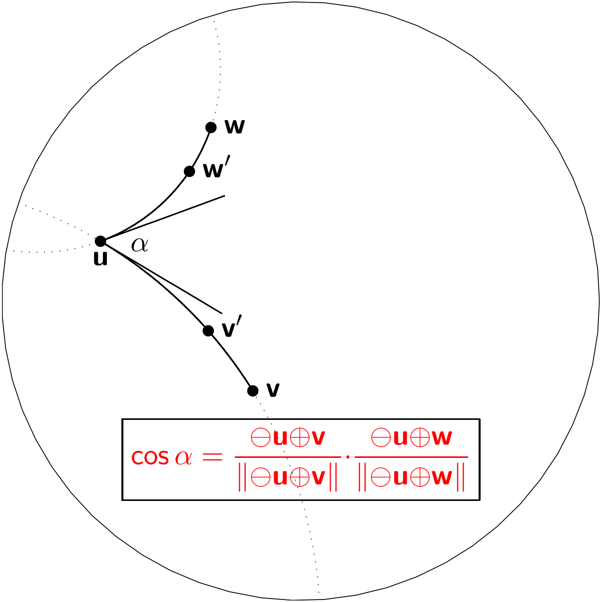


Figure: $\oplus = \oplus_M$. A gyroline in a Möbius gyrovector plane.



$$\boxed{\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}}$$

Figure: $\oplus = \oplus$ The Hyperbolic Angle in the Einstein gyrovector plane



$$\boxed{\cos\alpha = \frac{\ominus\mathbf{u}\oplus\mathbf{v}}{\|\ominus\mathbf{u}\oplus\mathbf{v}\|}\cdot\frac{\ominus\mathbf{u}\oplus\mathbf{w}}{\|\ominus\mathbf{u}\oplus\mathbf{w}\|}}$$

Figure: $\oplus = \oplus_M$. A Möbius angle α generated by the two intersecting

Covariance of Barycentric Coordinate Representations:

Let

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{1}$$

be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ in a Euclidean n-space $\mathbb{R}^n$ with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$. The barycentric coordinate representation (1) is covariant, that is,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k} \tag{2}$$

for all $X \in \mathbb{R}^n$, and

$$RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k} \tag{3}$$

for all $R \in SO(n)$.

Covariance of Gyrobarycentric Coordinate Representations:

Let

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{4}$$

be a gyrobarycentric coordinate representation of a point $P \in \mathbb{R}_c^n$ in an Einstein gyrovector space $(\mathbb{R}_c^n, \oplus, \otimes)$ with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_c^n$.
Then

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} \ (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{5}$$

and

$$RP = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} \ RA_k}{\sum_{k=1}^{N} m_k \gamma_{RA_k}} \tag{6}$$

From

gyrodecomposition of Groups

to

bi-gyrodecomposition of Groups

Past (1988 – 2015) and future (2015 –)

Each Lorentz transformation group $SO(1, n)$, $n > 1$, in a pseudo-Euclidean space of signature $(1, n)$ possesses a gyrodecomposition

$$SO(1, n) = BH$$

This gyrodecomposition along with the 1988 parametric realization of Lorentz transformations in pseudo-Euclidean spaces of signature $(1, n)$, $n > 1$, opened the door to the exploration of group gyrodecomposition.

The 1988 – 2015 exploration of the group gyrodecomposition, in turn, resulted in the discovery of the algebraic gyrostructures, gyrogroup and gyrovector space.

These gyrostructures play a universal computational role that extends far beyond the domain of Einstein's special relativity theory.

Of particular interest are applications in the hyperbolic geometry of Lobachevsky and Bolyai, resulting in the equation

$$\{\text{Hyperbolic Geometry}\} \quad = \quad \{\text{gyroeuclidean Geometry}\}$$

Each Lorentz transformation group $SO(1, n)$, $n > 1$, in a pseudo-Euclidean space of signature $(1, n)$ possesses a gyrodecomposition

$$SO(1, n) = BH$$

The exploration of the gyrodecomposition is far reaching.

Similarly:

Each Lorentz transformation group $SO(m, n)$, $m, n > 1$, in a pseudo-Euclidean space of signature (m, n) possesses a bi-gyrodecomposition

$$SO(m, n) = H_L B H_R$$

The exploration of the bi-gyrodecomposition is far reaching.

Guided by analogies with the 1988 – 2015 exploration of the group gyrodecomposition

$$G = BH$$

that was suggested by the Lorentz group gyrodecomposition

$$SO(1, n) = BH$$

our first step in the exploration of group bi-gyrodecomposition

$$G = H_L B H_R$$

is to study the special case of the bi-gyrodecomposition

$$SO(m, n) = H_L B H_R$$

$m, n > 1$.

The study of the bi-gyrodecomposition

$$SO(m, n) = H_L B H_R$$

$m, n > 1$, is based on the novel
Parametric Realization of the Lorentz Transformation Group in Pseudo-Euclidean Spaces of signature (m, n), $m, n > 1$.

This is in full analogy with:

The 1988 study of the gyrodecomposition

$$SO(1, n) = BH$$

$n > 1$, which was based on the 1988 – novel
Parametric Realization of the Lorentz Transformation Group in Pseudo-Euclidean Spaces of signature $(1, n)$, $n > 1$.

Parametric Realization of the boost in $SO(1, n)$ (STR)

$$B(\mathbf{v}_1)B(\mathbf{v}_2) = B(\mathbf{v}_1 \oplus \mathbf{v}_2)H(\mathbf{v}_1, \mathbf{v}_2)$$

$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$.
The application of two successive boosts is equivalent to the application of a single boost and a Thomas gyration of space coordinates.
Einstein velocity addition, $\oplus$, is involved.
The space of the parameter $\mathbf{v}$ is the ball $\mathbb{R}^n_c$ of all relativistically admissible velocities,

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}$$

The resulting Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ is a gyrocommutative gyrogroup.

Hence,

1. the parametric realization of the boost in $SO(1,n)$ (STR)

$$B(\mathbf{v}), \quad \mathbf{v} \in \mathbb{R}_c^n$$

   and

2. the gyrodecomposition of the Lorentz group

$$SO(1,n) = B(\mathbf{v})H$$

are rewarding for the following two reasons:
They enable us to

1. recover Einstein addition, $\oplus$, in the ball $\mathbb{R}_c^n$; and to
2. discover the gyrogroup structure of the resulting Einstein groupoid $(\mathbb{R}_c^n, \oplus)$.

The parametric realization of the bi-boost $B(P)$ in $SO(m,n)$

$$B(P) = \begin{pmatrix} \sqrt{I_m + P^tP} & P^t \\ P & \sqrt{I_n + PP^t} \end{pmatrix} \quad \in \quad \mathbb{R}^{(m+n)\times(m+n)}$$

$P \in \mathbb{R}^{n\times m}$, $m, n \in \mathbb{N}$.

$$B(P)\begin{pmatrix} \mathbf{t} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{t}' \\ \mathbf{x}' \end{pmatrix}$$

$\mathbf{t}, \mathbf{t}' \in \mathbb{R}^m$, $\quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$.

$$\mathbf{t}^2 - \mathbf{x}^2 = (\mathbf{t}')^2 - (\mathbf{x}')^2$$

$$B(P)\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot B(P)\begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \mathbf{t}_1\cdot\mathbf{t}_2 - \mathbf{x}_1\cdot\mathbf{x}_2$$

The parametric realization of the Lorentz group $SO(m,n)$

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + PP^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix}$$

$$\begin{aligned} O_n : \quad & P \rightarrow O_n P \\ O_m : \quad & P \rightarrow P O_m \\ (O_n, O_m) : \quad & P \rightarrow O_n P O_m \end{aligned}$$

The three parameters of the (m,n)-Lorentz transformation $\Lambda \in SO(m,n)$ are:

1. $O_m \in SO(m)$
2. $O_n \in SO(n)$
3. $P \in \mathbb{R}^{n \times m}$

Parametric Realization of the bi-boost in $SO(m, n)$

$$B(P_1)B(P_2) = H_L(P_1, P_2)B(P_1 \oplus P_2)H_R(P_1, P_2)$$

$P_1, P_2 \in \mathbb{R}^{n\times m}$.
The application of two successive bi-boosts is equivalent to the application of a single bi-boost and

1. a Thomas gyration of space-like coordinates (coming from H_R); and
2. a Thomas gyration of time-like coordinates (coming from H_L).

A novel binary operation, $\oplus$, between real $n \times m$ matrices is involved.
The space of the parameter P is the space $\mathbb{R}^{n\times m}$ of all real $n \times m$ matrices.
The resulting groupoid $(\mathbb{R}^{n\times m}, \oplus)$ is a bi-gyrocommutative bi-gyrogroup.

Hence,

1. the parametric realization of the bi-boost in $SO(m, n)$

$$B(P), \quad P \in \mathbb{R}^{n \times m}$$

   and

2. the bi-gyrodecomposition of the Lorentz group

$$SO(m, n) = H_L B(P) H_R$$

are rewarding for the following two reasons:
They enable us to

1. discover a binary operation, $\oplus$, in the space of all real $n \times m$ matrices and to
2. discover the bi-gyrogroup structure of the resulting groupoid $(\mathbb{R}^{n \times m}, \oplus)$.

The study of the bi-gyrodecomposition

$$SO(m,n) = H_L B H_R$$

$m, n > 1$, leads to our discovery of the two novel algebraic structures

bi-gyrogroup and bi-gyrovector space,

which play a universal computational role that extends far beyond the domain of Lorentz groups,

including Generalized Analytic Hyperbolic Geometry,

just as:

The 1988 – 2015 study of the gyrodecomposition

$$SO(1,n) = BH$$

$n > 1$, led us to the discovery of the algebraic structures

gyrogroup and gyrovector space.

Our study of the
Parametric Realization of the Lorentz Transformation Group in Pseudo-Euclidean Spaces of signature (m, n), $m, n > 1$,
leads us to the discovery of the novel bi-gyrogroup and bi-gyrovector space structures.

Naturally, a bi-gyrogroup involves two families of gyrations, left gyrations and right gyrations ,

as opposed to

a gyrogroup, which involves a single family of gyrations.

Moreover, in full analogy with the equation

$$\{\text{Hyperbolic Geometry}\} = \{\text{gyroeuclidean Geometry}\}$$

we will have the equation

$$\{\text{Generalized Hyperbolic Geometry}\} = \{\text{bi-gyroeuclidean Geometry}\}$$

The definition of the resulting bi-gyrogroup follows

Definition

A groupoid $(B, \oplus)$ is a bi-gyrogroup if its binary operation satisfies the following axioms for all $a, b, c \in B$:

1. $0 \oplus a = a$ (Left Identity)
2. $\ominus a \oplus a = 0$ (Left Inverse)
3. $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{lgyr}[a, b] c \mathrm{rgyr}[b, a]$
   (Left bi-gyroassociative Law)
4. $\mathrm{lgyr}[a, b],\ \mathrm{rgyr}[a, b] \in \mathrm{Aut}(G, \oplus)$
   (bi-gyrations are Automorphisms)
5. $\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus b, b]$ and $\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a \oplus b, b]$
   (Bi-gyration Left Reduction Law)

Definition

A bi-gyrogroup $(B, \oplus)$ is bi-gyrocommutative if its binary operation satisfies the Bi-gyrocommutative Law

$$a \oplus b = \mathrm{lgyr}[a, b](b \oplus a)\mathrm{rgyr}[b, a]$$

A presentation of the concrete example of a bi-gyrocommutative bi-gyrogroup that results from the bi-gyrodecomposition of Lorentz groups in Pseudo-Euclidean Spaces of signature (m, n), $m, n > 1$, appears in

A.A. Ungar, *Parametric Realization of the Lorentz Transformation Group in Pseudo-Euclidean Spaces.*

A presentation that captures abstractly the notion of the bi-gyrogroup that results from a group bi-gyrodecomposition appears in

T. Suksumran and A.A. Ungar, *Bi-gyrogroup: The Group-like Structure Induced by Bi-decomposition of Groups.*

Thank You