Space-like Surfaces of Pseudo-Hyperbolic Space $\mathbb{H}_1^4(-1)$

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Introduction

- In late 1970’s B.Y. Chen introduced the notion of **finite type submanifold** of a Euclidean space.
- Since then the **finite type submanifolds** of **Euclidean spaces** or **pseudo-Euclidean spaces** have been studied extensively, and many important results have been obtained ([2],[3],[6], etc.).
In [5], Chen and Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds.

A smooth map $\phi$ from a compact Riemannian manifold $M$ into a Euclidean space $\mathbb{E}^m$ is said to be of finite type if $\phi$ can be expressed as a finite sum of $\mathbb{E}^m$-valued eigenfunctions of the Laplacian $\Delta$ of $M$, that is,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_k, \quad (1)$$

where $\phi_0$ is a constant map, $\phi_1, \ldots, \phi_k$ non-constant maps such that $\Delta \phi_i = \lambda_{p_i} \phi_i$, $\lambda_{p_i} \in \mathbb{R}$, $i = 1, \ldots, k$. 


• If \( \lambda_{p_1}, \ldots, \lambda_{p_k} \) are mutually distinct, then the map \( \phi \) is said to be of \textbf{k-type}.

• If \( \phi \) is an isometric immersion, then \( M \) is called a \textbf{submanifold of finite type} (or of \( k \)-type).

• In the spectral decomposition of an immersion \( \phi \) on a compact manifold, the constant vector \( \phi_0 \) is the center of mass.
In [6], Chen introduced the notion of a map of finite type on a non-compact manifold.

When \( M \) is non-compact the vector \( \phi_0 \) in the spectral decomposition in (1) is not necessary a constant vector.
**Classical Gauss map**

- Let $\mathbf{x} : M^n \rightarrow \mathbb{E}^m$ be an isometric immersion from a Riemannian $n$-manifold $M^n$ into a Euclidean $m$-space $\mathbb{E}^m$.
- Let $G(n, m)$ denote the Grassmannian manifold consisting of linear $n$-subspaces of $\mathbb{E}^m$.
- The **classical Gauss map**

$$\nu^c : M^n \rightarrow G(n, m)$$

associated with $\mathbf{x}$ is the map which carries each point $p \in M$ to the linear subspace of $\mathbb{E}^m$ obtained by parallel displacement of the tangent space $T_p M$ to the origin of $\mathbb{E}^m$. 
Since $G(n, m)$ can be canonically imbedded in the vector space $\bigwedge^n \mathbb{E}^m = \mathbb{E}^N$, $N = \binom{m}{n}$, obtained by the exterior products of $n$-vectors in $\mathbb{E}^m$, the classical Gauss map gives rise to a well-defined map from $M^n$ into the Euclidean $N$-space $\mathbb{E}^N$ where $N = \binom{m}{n}$. 
Obata’s sense generalized Gauss map

- Let $\mathbf{x} : M^n \rightarrow \tilde{M}^m$ be an isometric immersion from a Riemannian $n$-manifold $M^n$ into a simply-connected complete $m$-space $\tilde{M}^m$ of constant curvature.
- In [11], Obata studied the generalized Gauss map which assigns to each $p \in M$ the totally geodesic $n$-space tangent to $\mathbf{x}(M)$ at $\mathbf{x}(p)$.
- In the case $\tilde{M}^m = \mathbb{S}^m$, the generalized Gauss map is also called the spherical Gauss map.
- If $\tilde{M}^m = \mathbb{H}^m$, the generalized Gauss map is called the hyperbolic Gauss map.
In [8], Chen and Lue studied spherical submanifolds with finite type spherical Gauss map. They obtained several results in this respect.
In [10], we investigated submanifolds of hyperbolic spaces with \textbf{finite type hyperbolic Gauss map}.

We characterized and classified submanifolds of the hyperbolic $m$-space $\mathbb{H}^m(-1)$ with finite type hyperbolic Gauss map.
Basic notations and formulas

- Let $\mathbb{E}_t^m$ denote the pseudo-Euclidean $m$-space with the canonical pseudo-Euclidean metric of index $t$ given by

$$g_0 = \sum_{i=1}^{t} dx_i^2 - \sum_{j=t+1}^{m} dx_j^2, \quad (2)$$

- where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}_t^m$. 
\[ \mathbb{S}^{m-1}_t(x_0, c) = \{ x \in \mathbb{E}^{m}_t | \langle x - x_0, x - x_0 \rangle = c^{-1} > 0, \ c > 0 \} \]
\[ \mathbb{H}^{m-1}_t(x_0, -c) = \{ x \in \mathbb{E}^{m+1}_t | \langle x - x_0, x - x_0 \rangle = -c^{-1} < 0, \ c > 0 \} , \]

- \( \mathbb{S}^{m-1}_t(x_0, c) \) and \( \mathbb{H}^{m-1}_t(x_0, -c) \) are complete pseudo-Riemannian manifolds with index \( t \) of constant curvature \( c \) and \( -c \).
An $n$-dimensional submanifold $M$ of $\mathbb{H}^m_t(-1) \subset \mathbb{E}^{m+1}_{t+1}$ is said to be **space-like** if the metric induced on $M$ from the ambient space $\mathbb{H}^m_t(-1)$ is positive definite.

The mean curvature vector $H$ of $M$ in $\mathbb{E}^m_t$ is defined by

$$H = \frac{1}{n} \sum_{r=n+1}^{m} \varepsilon_r \text{tr}A_r e_r.$$ 

(3)
• If $H = 0$ holds identically, we call $M$ is maximal.

• The **scalar curvature** $S$ of $M$ in $\mathbb{H}^{m-1}_{t-1}(-c)$, $c > 0$ is given

$$ S = -cn(n - 1) + n^2|\hat{H}|^2 - \|\hat{h}\|^2. $$

(4)
A submanifold $M$ is said to be **totally geodesic** if the second fundamental form $h$ of $M$ vanishes identically.

$M$ is called **totally umbilical** if its second fundamental form satisfies

$$h(X, Y) = \langle X, Y \rangle H$$

for vectors $X$ and $Y$ tangent to $M$.  

Pseudo-hyperbolic Gauss map

Let $\mathbf{x} : M^n \rightarrow \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ be an isometric immersion from a space-like oriented Riemannian $n$-manifold $M^n$ into a pseudo-hyperbolic $m-1$-space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$. The Obata’s map can be written as

$$\hat{\nu} : M^n \rightarrow G(n + 1, m)$$

$$\hat{\nu}(p) = (\mathbf{x} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p).$$

Considering the natural inclusion of $G(n + 1, m)$ into $\mathbb{E}_q^N$, the pseudo-hyperbolic Gauss map $\tilde{\nu}$ associated with $\mathbf{x}$ is thus given by

$$\tilde{\nu} = \mathbf{x} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n : M^n \rightarrow G(n + 1, m) \subset \mathbb{E}_q^N, \quad (5)$$

where $N = \binom{m}{n+1}$. 

The Laplacian formula is given by

\[ \triangle \tilde{\nu} = \sum_{i=1}^{n} (\nabla_{e_i} e_i - e_i e_i) \tilde{\nu}. \]  

(6)
Then we have

\[ \Delta \tilde{\nu} = \| \hat{h} \|^2 \tilde{\nu} + n \hat{H} \wedge e_1 \wedge \cdots \wedge e_n \]

\[- n \sum_{k=1}^{n} x \wedge e_1 \wedge \cdots \wedge D_{e_k} \hat{H} \wedge \cdots \wedge e_n \]

\[ + \sum_{j,k=1}^{n} \sum_{r,s=n+1 \atop s<r}^{m-1} \varepsilon_r \varepsilon_s R^r_{sjk} x \wedge e_1 \wedge \cdots \wedge e_s \wedge \cdots \wedge e_r \wedge \cdots \wedge e_n, \]

where \( R^r_{sjk} = R^D(e_j, e_k; e_r, e_s) \).
In [4], Chen investigated non-compact finite type pseudo-Riemannian submanifolds of a pseudo-Euclidean spaces.

He gave the definition for the finite type submanifolds of the pseudo-Riemannian sphere $S_t^{m-1}$ or the pseudo-hyperbolic space $\mathbb{H}_t^{m-1}$.

In [9], Dursun constructed the definition for a smooth map as the following:
A smooth map

\[ \phi : M_q \longrightarrow \mathbb{H}_{t-1}^{m-1}(-1) \subset \mathbb{R}_t^m \]

from a pseudo-Riemannian manifold \( M_q \) into a pseudo-hyperbolic space \( \mathbb{H}_{t-1}^{m-1}(-1) \) is called of \( k \)-type in \( \mathbb{H}_{t-1}^{m-1}(-1) \) if the map \( \phi \) has the following form:

\[ \phi = \phi_1 + \phi_2 + \cdots + \phi_k, \quad \Delta \phi_i = \lambda_i \phi_i, \quad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, k, \]

(8)
such that \( \lambda_1, \ldots, \lambda_k \) are all distinct.

Moreover, according to definition one of the component in the spectral decomposition may be constant.
First, we state the following Proposition.

**Proposition**

Let \( \mathbf{x} : (M^n, g) \rightarrow \mathbb{H}_1^{m-1}(-1) \subset \mathbb{E}_2^m \) be an isometric immersion from a space-like manifold \( M^n \) in an anti-de Sitter space \( \mathbb{H}_1^{m-1}(-1) \subset \mathbb{E}_2^m \). Then, one of the following cases occurs:

(a) the Obata’s map \( \hat{\nu} : (M^n, g) \rightarrow G(n+1, m) \) is a harmonic map if and only if \( \mathbf{x} : (M^n, g) \rightarrow \mathbb{H}_1^{m-1}(-1) \) is a maximal immersion;

(b) the pseudo-hyperbolic Gauss map \( \tilde{\nu} : (M^n, g) \rightarrow \mathbb{E}_S^N \) with \( N = \binom{m}{n+1} \) and \( S = 2\binom{m-2}{n} \) is a harmonic map if and only if

(b.1) \( \mathbf{x} : (M^n, g) \rightarrow \mathbb{H}_1^{m-1}(-1) \subset \mathbb{E}_2^m \) is a totally geodesic immersion, or

(b.2) \( M^n \) is maximal, has flat normal bundle and scalar curvature should satisfy the following equality \( S = -n(n-1) \).
In this section, we classify space-like surfaces in $\mathbb{H}_1^4(-1) \subset \mathbb{E}_2^5$ with 1-type pseudo-hyperbolic Gauss map.

**Theorem 1**
A space-like surface $M$ of $\mathbb{H}_1^4(-1) \subset \mathbb{E}_2^5$ has 1-type pseudo-hyperbolic Gauss map if and only if $M$ is a maximal surface of $\mathbb{H}_1^4(-1)$ with $M$ has constant scalar curvature and flat normal bundle.

**Proof** [here](#)
We obtain the following corollaries as an immediate consequence of Theorem 1.

**Corollary 1**
Let \( M \) be a space-like surface in an anti-de Sitter space \( \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2 \). Then, \( M \) has 1-type pseudo-hyperbolic Gauss map if and only if it is maximal surface of \( \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2 \) with constant scalar curvature.

**Corollary 2**
A totally geodesic hyperboloid \( \mathbb{H}^2(-1) \) in \( \mathbb{H}^4_1(-1) \) has biharmonic pseudo-hyperbolic Gauss map which is of 1-type.
**Example 1**

**Maximal space-like surface in** $\mathbb{H}^3_1(-1)$

Let $\mathbf{x} : M = \mathbb{H}^1(-a^{-2}) \times \mathbb{H}^1(-b^{-2}) \rightarrow \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ be an isometric immersion from $M$ into $\mathbb{H}^3_1(-1)$ defined by

$$
\mathbf{x}(u, v) = (a \sinh u, b \sinh v, a \cosh u, b \cosh v)
$$

with $a^2 + b^2 = 1$. If we put $\mathbf{e}_1 = \frac{1}{a} \frac{\partial}{\partial u}, \quad \mathbf{e}_2 = \frac{1}{b} \frac{\partial}{\partial v}$,

$$
\mathbf{e}_3 = (b \sinh u, -a \sinh v, b \cosh u, -a \cosh v), \quad \mathbf{e}_4 = \mathbf{x}
$$

then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ form an orthonormal frame field on $M$ in $\mathbb{E}^4_2$. 
A straightforward computation gives

\[ h_{11}^3 = -\frac{b}{a}, \quad h_{12}^3 = h_{12}^4 = 0, \quad h_{22}^3 = \frac{a}{b}, \]

\[ h_{11}^4 = h_{22}^4 = -1, \quad \omega_{12} = \omega_{34} = 0, \quad \omega_{13} = -\frac{b}{a} \omega^1, \]

\[ \omega_{23} = \frac{a}{b} \omega^2, \quad \omega_{14} = -\omega^1, \quad \omega_{24} = -\omega^2. \]  

- The equation (9) yields \( \hat{H} = \frac{a^2 - b^2}{ab} e_3 \) which gives \( M \) is a maximal surface if and only if \( a = b = \frac{1}{\sqrt{2}} \).
- Therefore, \( \mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \subset \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2 \) is a maximal and flat surface. It is obvious that \( \mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \) has 1-type pseudo-hyperbolic Gauss map by Theorem 1.
Theorem 2

Let $M$ be a space-like surface in an anti-de Sitter space $\mathbb{H}^{m-1}_1(-1) \subset \mathbb{E}^m$. Then, $M$ has 1-type pseudo-hyperbolic Gauss map if and only if $M$ is congruent to an open part of $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2)$ lying in $\mathbb{H}^3_1(-1) \subset \mathbb{H}^{m-1}_1(-1) \subset \mathbb{E}^m$ for some $m \geq 5$ or the totally geodesic hyperbolic space $\mathbb{H}^2(-1)$ lying in $\mathbb{H}^{m-1}_1(-1) \subset \mathbb{E}^m$ for some $m \geq 5$. 
Example 2

- **Space-like surface with flat normal bundle and null mean curvature vector in \( \mathbb{H}^4_1(−1) \)**

  Let \( \mathbf{x}: M \rightarrow \mathbb{H}^4_1(−1) \subset \mathbb{E}^5_2 \) be a space-like isometric immersion from a surface \( M \) into an anti-de Sitter space \( \mathbb{H}^4_1(−1) \). We consider a surface \( M \) in \( \mathbb{H}^4_1(−1) \subset \mathbb{E}^5_2 \) as follows

  \[
  \mathbf{x}(u, v) = (1, \cosh u \sinh v, \sinh u, \cosh u \cosh v, 1),
  \]

  [7].

  If we put

  \[
  e_1 = \frac{\partial}{\partial u}, \quad e_2 = \frac{1}{\cosh u} \frac{\partial}{\partial v},
  \]

  \[
  e_3 = \left( \frac{3}{2}, \cosh u \sinh v, \sinh u, \cosh u \cosh v, \frac{1}{2} \right)
  \]

  and

  \[
  e_4 = \left( \frac{1}{2}, \cosh u \sinh v, \sinh u, \cosh u \cosh v, -\frac{1}{2} \right), \quad e_5 = \mathbf{x}
  \]

  then \( \{e_i\} \) for \( i = 1, 2, 3, 4, 5 \) form an orthonormal frame field on \( M \).
A straightforward computation gives

\[ h_{11}^3 = h_{22}^3 = h_{11}^4 = h_{22}^4 = -1, \quad h_{12}^3 = h_{12}^4 = 0, \]
\[ \omega_{12}(e_1) = 0, \quad \omega_{12}(e_2) = \tanh u, \quad \omega_{34} = 0, \quad (10) \]
\[ \|\hat{h}\|^2 = 0, \quad \hat{H} = e_4 - e_3 = (-1, 0, 0, 0, -1). \]

If we use (10), then equation (7) reduces to

\[ \triangle \tilde{\nu} = 2\hat{H} \wedge e_1 \wedge e_2 = -2e_3 \wedge e_1 \wedge e_2 + 2e_4 \wedge e_1 \wedge e_2. \quad (11) \]
If we put

\[ \tilde{c} = \tilde{\nu} - e_3 \wedge e_1 \wedge e_2 + e_4 \wedge e_1 \wedge e_2 \quad (12) \]

and

\[ \tilde{\nu}_p = e_3 \wedge e_1 \wedge e_2 - e_4 \wedge e_1 \wedge e_2 \quad (13) \]

then we have \( \tilde{\nu} = \tilde{c} + \tilde{\nu}_p \).

It can be shown that \( e_i(\tilde{c}) = 0 \) for \( i = 1, 2 \), i.e., \( \tilde{c} \) is a constant vector. Using (11), (12) and (13), we get \( \Delta \tilde{\nu}_p = -2\tilde{\nu}_p \).

Thus, \( M \) has 1-type pseudo-hyperbolic Gauss map with a nonzero constant component in its spectral decomposition.
Now, we determine space-like surfaces of $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ with 1-type pseudo-hyperbolic Gauss map containing a nonzero constant component in its spectral decomposition.
Theorem 3 A space-like surface $M$ in the anti-de Sitter space $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ has 1-type pseudo-hyperbolic Gauss map with a nonzero constant component in its spectral decomposition if and only if $M$ is an open part of the following surfaces:

(1) A non-totally geodesic, totally umbilical space-like surface in a totally geodesic hyperbolic space $\mathbb{H}^3(-1) \subset \mathbb{H}^4_1(-1)$ with mean curvature $|\hat{\alpha}| \neq 1$, that is, $M$ is an open part of a hyperbolic 2-space $\mathbb{H}^2(-c)$ of curvature $-c$ for $0 < c < 1$ in $\mathbb{H}^3(-1) \subset \mathbb{H}^4_1(-1)$ or a 2-sphere $\mathbb{S}(c)$ of curvature $c$ for $c > 0$ in $\mathbb{H}^3(-1) \subset \mathbb{H}^4_1(-1)$ or

(2) A hyperbolic 2-space $\mathbb{H}^2(-c)$ of curvature $-c$ for $c > 1$ in $\mathbb{H}^3_1(-1) \subset \mathbb{H}^4_1(-1)$ or

(3) The surface defined by

$$\mathbf{x}(u, v) = (1, \cosh u \sinh v, \sinh u, \cosh u \cosh v, 1)$$

which is of curvature $-1$ and totally umbilical with constant lightlike mean curvature vector.
In this section, we give a characterization of space-like hypersurfaces in an anti-de Sitter space $\mathbb{H}^{7+1}_1(-1)$ with constant mean curvature vector and 2-type pseudo-hyperbolic Gauss map.
**Theorem 4** Let $M$ be a space-like, non-totally umbilical hypersurface with nonzero constant mean curvature $\hat{\alpha}$ in an anti-de Sitter space $\mathbb{H}^{n+1}_{1}(-1) \subset \mathbb{E}^{n+2}_{2}$. Then, $M$ has 2-type pseudo-hyperbolic Gauss map $\tilde{\nu}$ if and only if it has constant scalar curvature.
In [1], Zhen-qi and Xian-hua determined space-like, isoparametric hypersurface $M^n$ in $\mathbb{H}^{n+1}_1(-1) \subset \mathbb{E}^{n+2}$. They showed that $M$ is congruent to an open subset of a umbilical hypersurface $\mathbb{H}^n(-c)$ of curvature $-c$ with $c > 0$ or the product of two hyperbolic spaces,

$$\mathbb{H}^k(-c_1) \times \mathbb{H}^{n-k}(-c_2) = \{ (x, y) \in \mathbb{R}^{k+1}_1 \times \mathbb{R}^{n-k+1}_1 : <x, x> = -c_1^{-1}, <y, y> = -c_2^{-1} \}, \quad (14)$$

where $c_1, c_2 > 0$.

The product hypersurface $\mathbb{H}^k(-c_1) \times \mathbb{H}^{n-k}(-c_2)$ with $c_1 \neq c_2 > 0$ is a non-umbilical isoparametric hypersurface in an anti-de Sitter space $\mathbb{H}^{n+1}_1(-1)$ having non-zero mean curvature and constant scalar curvature. Therefore, we obtain the following Corollary.
**Corollary 3** A product two hyperbolic spaces 
\[ \mathbb{H}^k(-c_1) \times \mathbb{H}^{n-k}(-c_2) \] with \( c_1 \neq c_2 > 0 \) in \( \mathbb{H}_1^{n+1}(-1) \) is the only isoparametric hypersurface with 2-type pseudo-hyperbolic Gauss map.
We also classify space-like surfaces with constant mean curvature in an anti-de Sitter space $\mathbb{H}^3_1(−1) \subset \mathbb{E}^4_2$ having 2-type pseudo-hyperbolic Gauss map.

**Theorem 4** A space-like surface $M$ in an anti-de Sitter space $\mathbb{H}^3_1(−1) \subset \mathbb{E}^4_2$ with a non-totally umbilical and nonzero constant mean curvature in $\mathbb{H}^3_1(−1)$ has 2-type pseudo-hyperbolic Gauss map it is congruent to open portion of the $\mathbb{H}^1(-a^{-2}) \times \mathbb{H}^1(-b^{-2})$ in $\mathbb{H}^3_1(−1)$ with $a^2 + b^2 = 1$, $a \neq b$. 


THANK YOU ...
Assume that $M$ is a space-like surface in an anti-de Sitter space $H_4^4(−1) \subset E_2^5$ with 1-type pseudo-hyperbolic Gauss map.

Then, $\Delta \tilde{v} = \lambda_p \tilde{v}$ for some $\lambda_p \in \mathbb{R}$.

From equation (7) the pseudo-hyperbolic Gauss map $\tilde{v}$ is 1-type if and only if $\hat{H} = R^D = 0$ and $||\hat{h}||^2$ is constant.

Moreover, by (4) it seen that $M$ has constant scalar curvature.
First, we will calculate $\Delta(e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n)$. We take $\bar{\nu} = e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n$. If we differentiate $\bar{\nu}$, we obtain

$$e_i \bar{\nu} = e_{n+1} \wedge e_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge e_n. \quad (15)$$

Considering (6) and $\hat{\alpha} = \sum_{i=1}^{n} h_{ii}^{n+1}$ we get

$$\Delta \bar{\nu} = -n\hat{\alpha} \bar{\nu} - n\bar{\nu} + \sum_{i,j=1}^{n} (\omega_{ij}(e_i) + \omega_{ji}(e_i)) e_{n+1} \wedge e_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge e_n$$

$$= -n(\hat{\alpha} \bar{\nu} + \bar{\nu}). \quad (16)$$

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