f-Biminimal Immersions

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Introduction and Preliminaries

Let (M, g) and (N, h) be Riemannian manifolds. A map $\varphi : (M, g) \to (N, h)$ is called a harmonic map if it is a critical point of the energy functional

$$E(\varphi) = \frac{1}{2} \int_M \|d\varphi\|^2 \, d\nu_g.$$

Introduction and Preliminaries

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$$E(\varphi) = \frac{1}{2} \int_M \|d\varphi\|^2 \, d\nu_g.$$

The map φ is said to be biharmonic if it is a critical point of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M \|\tau(\varphi)\|^2 \, d\nu_g,$$

where $\tau(\varphi) = tr(\nabla d\varphi)$ is the tension field. If $\tau(\varphi) = 0$ then φ is called harmonic [Eells-Sampson].

The Euler-Lagrange equation for the bienergy functional were obtained by Jiang in [Jiang-86] by $\tau_2(\varphi) = 0$, where

$$\tau_2(\varphi) = tr(\nabla^N \nabla^N - \nabla^N_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi), \qquad (1)$$

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Using the Euler-Lagrange equation for the *f-energy functional*, in [OND] and [Course] the *f-tension field* $\tau_f(\varphi)$ was obtained by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(gradf). \qquad (2)$$

If $\tau_f(\varphi) = 0$ then the map is called f-harmonic [Course]. The map φ is said to be *f-biharmonic* (see [Lu]) if and only if it is a critical point of the f-bienergy functional

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The Euler-Lagrange equation for the f-bienergy functional is given by $\tau_{2,f}(\varphi) = 0$, where $\tau_{2,f}(\varphi)$ is the f-bitension field and is defined by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla^N_{grad f}\tau(\varphi), \tag{3}$$

(see [Lu]). It can be easily seen that any f-harmonic map is f-biharmonic. If the map is non-f-harmonic f-biharmonic then we call it by proper f-biharmonic [Lu].

In [Loubeau-Montaldo], Loubeau and Montaldo considered biminimal immersions. They studied biminimal curves in a Riemannian manifold, curves in a space form, and isometric immersions of codimension 1 in a Riemannian manifold. They investigated biminimal surfaces using Riemannian and horizontally homothetic submersions.

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An immersion φ, is called *biminimal* (see [Loubeau-Montaldo]) if it is a critical point of the bienergy functional $E_2(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi) \tag{4}$$

for any smooth variation of the map $\varphi_t :]-\epsilon, +\epsilon[$, $\varphi_0 = \varphi$, such that $V = \frac{d\varphi_t}{dt} \mid_{t=0} = 0$ is normal to $\varphi(M)$.

The Euler-Lagrange equation for λ-biminimal immersion is,

$$[\tau_{2,\lambda}(\varphi)]^{\perp} = [\tau_2(\varphi)]^{\perp} - \lambda[\tau(\varphi)]^{\perp} = 0. \tag{5}$$

for some value of $\lambda \in \mathbb{R}$, where $[\cdot]^{\perp}$ denotes the normal component of $[\cdot]$. An immersion is called free biminimal if it is biminimal for $\lambda = 0$ [Loubeau-Montaldo].

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In this study, we define f-biminimal immersions. We consider f-biminimal curves in a Riemannian manifold. We also consider f-biminimal submanifolds of codimension 1 in a Riemannian manifold. We give a non-trivial example for an f-biminimal Legendre curve in a Sasakian space form and we investigate the Riemannian and horizontally homothetic submersions for proper f-biminimal surface in a three dimension Riemannian manifold.

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Definition 1

An immersion φ, is called *f-biminimal* if it is a critical point of the f-bienergy functional $E_{2,f}(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-f-bienergy

$$E_{2,\lambda,f}(\varphi) = E_{2,f}(\varphi) + \lambda E_f(\varphi)$$

for any smooth variation of the map φ_t which is defined above.

Using the Euler-Lagrange equations for f-harmonic and f-biharmonic maps, an immersion is f-biminimal if

$$[\tau_{2,\lambda,f}(\varphi)]^{\perp} = [\tau_{2,f}(\varphi)]^{\perp} - \lambda[\tau_f(\varphi)]^{\perp} = 0 \tag{6}$$

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for some value of $\lambda \in \mathbb{R}$.

We call an immersion free f-biminimal if it is f-biminimal for $\lambda = 0$. If φ is a f-biminimal but not biminimal immersion then it is called as proper f-biminimal.

f-Biminimal Curves

Let $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ be a curve parametrized by arc length in a Riemannian manifold (M^m, g). We recall the definition of Frenet frames:

Definition 2 (Laugwitz)

The Frenet frame $\{E_i\}_{i=1,2,\ldots m}$ associated with a curve $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ is the orthonormalization of the $(m+1)-$tuple

$$\left\{ \nabla^{(k)}_{\frac{\partial}{\partial t}} d\gamma(\frac{\partial}{\partial t}) \right\}_{k=0,1,\ldots,m}$$

described by

$$E_1 = d\gamma(\frac{\partial}{\partial t}),$$

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} E_1 = k_1 E_2,$$

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad 2 \leq i \leq m-1,$$

$$\nabla^{\gamma}_{\frac{\partial}{\partial t}} E_m = -k_{m-1} E_{m-1},$$

where the function $\{k_1 = k > 0, k_2 = \tau, k_3, ..., k_{m-1}\}$ are called the curvatures of γ. In addition $E_1 = T = \gamma'$ is the unit tangent vector field to the curve.

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Firstly we have the following proposition for f-biminimal curve in Riemannian manifold:

Proposition 3

Let M^m be a Riemannian manifold and $\gamma : I \subset \mathbb{R} \longrightarrow (M^m, g)$ be an isometric curve. Then γ is f-biminimal if and only if there exists a real number λ such that

$$f\left\{\left(k_1'' - k_1^3 - k_1 k_2^2\right) - k_1 g(R(E_1, E_2)E_1, E_2)\right\} + \left(f'' - \lambda f\right) k_1 + 2f'k' = 0, \quad (7)$$

$$f\left\{\left(k_1' k_2 + (k_1 k_2)'\right) - k_1 g(R(E_1, E_2)E_1, E_3)\right\} + 2f' k_1 k_2 = 0, \quad (8)$$

$$f\left\{k_1 k_2 k_3 - k_1 g(R(E_1, E_2)E_1, E_4)\right\} = 0, \quad (9)$$

$$f k_1 g(R(E_1, E_2)E_1, E_j) = 0, \quad 5 \leq j \leq m, \quad (10)$$

where R is the curvature tensor of (M^m, g).

Now we investigate f-biminimality conditions for a surface or a three dimensional Riemannian manifold with a constant sectional curvature. Then we have the following corollary:

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Corollary 4

1) *A curve γ on a surface of Gaussian curvature G is f-biminimal if and only if its signed curvature k satisfies the ordinary differential equation*

$$f\left(k'' - k^3 - kG\right) + \left(f'' - \lambda f\right)k + 2f'k' = 0 \tag{11}$$

for some $\lambda \in \mathbb{R}$.

2) A curve γ on Riemannian 3-manifold of constant sectional curvature c is f-biminimal if and only if its curvature k and torsion τ satisfy the system

$$f\left(k'' - k^3 - k\tau^2 - kc\right) + \left(f'' - \lambda f\right)k + 2f'k' = 0$$

$$f\left(k'\tau + (k\tau)'\right) + 2f'k\tau = 0. \tag{12}$$

for some $\lambda \in \mathbb{R}$.

Codimension-1 f-Biminimal Submanifolds

Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion. We shall denote by B, η, A, Δ and $H_1 = H\eta$ the second fundamental form, the unit normal vector field, the shape operator, the Laplacian and the mean curvature vector field of φ (H the mean curvature function), respectively. Then we have the following proposition:

Proposition 5

Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension 1 and $H_1 = H\eta$ its mean curvature vector. Then φ is f-biminimal if and only if

$$\Delta H - H\|B\|^2 + HRicci(N) + \left(\frac{\Delta f}{f} + 2grad\ln f - \lambda\right)H = 0.$$

Corollary 6

Let $\varphi : M^m \longrightarrow N^{m+1}(c)$ be an isometric immersion of a Riemannian manifold $N^{m+1}(c)$ of constant curvature c. Then φ is f-biminimal if and only if there exists a real number λ such that

$$\Delta H - \left(m^2 H^2 - s + m(m-2)c - \frac{\Delta f}{f} - 2 grad \ln f + \lambda \right) H = 0 \tag{13}$$

where H is the mean curvature and s the scalar curvature of M^m. In addition, let $\varphi : M^2 \longrightarrow N^3(c)$ be an isometric immersion from a surface to a three-dimension space form. Then φ is f-biminimal if and only if

$$\Delta H - 2H \left(2H^2 - G - \frac{1}{2}\frac{\Delta f}{f} - grad \ln f + \frac{1}{2}\lambda \right) = 0 \tag{14}$$

Examples of f-Biminimal Surfaces on 3-Dimensional Riemannian Manifolds

Now,we find some examples of f-biminimal immersions similar to the methods given in [Loubeau-Montaldo]. A submersion $\varphi : (M, g) \longrightarrow (N, h)$ between two Riemannian manifolds in horizontally homothetic if there exists a function $\wedge : M \longrightarrow \mathbb{R}$, the dilation, such that

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i) at each point $p \in M$ the differential $d\varphi_p : H_p \longrightarrow T_{\varphi(p)}N$ is a conformal map with factor $\wedge(p)$, i.e.,
$\wedge^2(p)g(X, Y)(p) = h(d\varphi_p(X), d\varphi_p(Y))(\varphi(p))$ for all
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ii) $X(\wedge^2) = 0$, for all horizontal vector fields [Loubeau-Montaldo].

Lemma 7 (Loubeau-Montaldo)

Let $\varphi : (M^n, g) \longrightarrow (N^2, h)$ be a horizontally homothetic submersion with $\wedge$ and minimal fibres and let $\gamma : I \subset \mathbb{R} \longrightarrow N^2$ be a curve parametrized by arc length, of signed curvature k_γ. Then the codimension-1 submanifold $S = \varphi^{-1}(\gamma(I)) \subset M$ has mean curvature $H_s = \frac{\wedge k_\gamma}{n-1}$.

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Using the above lemma, we have the following theorem:

Theorem 8

Let $\varphi : M^3(c) \longrightarrow (N^2, h)$ be horizontally homothetic submersion with dilation $\wedge$, from a space form of constant sectional curvature c to a surface. Let $\gamma : I \subset \mathbb{R} \longrightarrow N^2$ be a curve parametrized by arc length such that the surface $S = \varphi^{-1}(\gamma(I)) \subset M^3$ has constant Gaussian curvature c. The $S = \varphi^{-1}(\gamma(I)) \subset M^3$ is a f-biminimal surface (with respect to $2c$) if and only if γ is a free f-biminimal curve with $k_\gamma = c_1 e^t$ where c_1 is a real constant.

Theorem 9

Let $\varphi : M^3(c) \longrightarrow N^2(\overline{c})$ be a Riemannian submersion with minimal fibres from a space of constant sectional curvature c to surface of constant Gaussian curvature $\overline{c}$. Let $\gamma : I \subset \mathbb{R} \longrightarrow N^2$ be a curve parametrized by arc length. Then $S = \varphi^{-1}(\gamma(I)) \subset M^3$ is a f-biminimal surface if and only if γ is a f-biminimal curve with $k_\gamma = c_1 e^t$ where c_1 is a real constant.

We consider the Riemannian submersion with totally geodesic fibres, given by the projection onto the first factor $\pi : N^2 \times \mathbb{R} \longrightarrow N^2$ and $\gamma : I \subset \mathbb{R} \longrightarrow N^2$ be a curve parametrized by arc length. Then we can state the following proposition:

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Proposition 10

The cylinder $S = \pi^{-1}(\gamma(I))$ is a proper f-biminimal surface in $N^2 \times \mathbb{R}$ if and only if γ is a proper f-biminimal curve on N^2 (S^2or H^2) with curvature $k = c_1 e^t$, where c_1 is a real constant.

The three-dimensional Heisenberg space $\hat{H}_3$ is the two-step nilpotent Lie group standardly represented in $GL_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \text{ with } x, y, z \in \mathbb{R}.$$

It is endowed with the left-invariant metric

$$g = dx^2 + dy^2 + (dz - xdy)^2. \tag{15}$$

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Let $\pi : \hat{H}_3 \longrightarrow \mathbb{R}^2$ be the projection $(x, y, z) \longrightarrow (x, y)$. It is easy to see that π is a Riemannian submersion (for more details see [Loubeau-Montaldo]). Take a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2$, parametrized by arc length, with signed curvature k.

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Proposition 11

The flat cylinder $S = \pi^{-1}(\gamma(I)) \subset \hat{H}_3$ *is a proper* f*-biminimal surface (with respect to* λ*) of* $\hat{H}_3$ *if and only if* γ *is a proper* f*-biminimal curve (with respect to* $\lambda + 1$*) of* $\mathbb{R}^2$ *with curvature* $k = c_1 e^t$*, where* c_1 *is a real constant.*

f-Biminimal Legendre Curves in Sasakian Space Forms

Let $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ be a contact metric manifold. If the Nijenhuis tensor of φ equals $-2d\eta \otimes \xi$, then $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is called a Sasakian manifold [Blair]. If a Sasakian manifolds has constant $\varphi-$sectional curvature c, then it is called a Sasakian space form. The curvature tensor of a Sasakian space form is given by

$$
\begin{aligned}
R(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\
+\frac{c-1}{4}\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\
+2g(X, \varphi Y)\varphi Z + \eta(X)\eta(Z)Y \\
-\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \qquad (16)
\end{aligned}
$$

A submanifold of a Sasakian manifold is called an integral submanifold if $\eta(X) = 0$, for every tangent vector X. A 1-dimension integral submanifold of a Sasakian manifold is called a Legendre curve of M. Hence a curve $\gamma : I \longrightarrow M = \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is called a Legendre curve if $\eta(T) = 0$, where T is the tangent vector field of γ [Blair 2002].

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Theorem 12

Let $\gamma : (a, b) \longrightarrow M$ be a non-geodesic Legendre Frenet curve of osculating order r in a Sasakian space form $M = \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$. Then γ is f-biminimal if and only if the following three equations hold

$$k_1'' - k_1^3 - k_1 k_2^2 + \frac{(c+3)}{4} k_1 + 2k_1' \frac{f'}{f} + k_1 \frac{f''}{f}$$
$$-\lambda k_1 + \frac{3(c-1)}{4} \left[k_1 g(\varphi T, E_2)^2 \right]^{\perp} = 0,$$
$$k_1' k_2 + (k_1 k_2)' + 2k_1 k_2 \frac{f'}{f} + \frac{3(c-1)}{4} [k_1 g(\varphi T, E_2) g(\varphi T, E_3)]^{\perp} = 0$$

and

$$k_1 k_2 k_3 + \frac{3(c-1)}{4} [k_1 g(\varphi T, E_2) g(\varphi T, E_4)]^{\perp} = 0.$$

Let's recall some notions about the Sasakian space form $\mathbb{R}^{2m+1}(-3)$ [Blair 2002]:
Let us take $M=\mathbb{R}^{2m+1}$ with the standard coordinate functions $(x_1,...,x_m,y_1,...,y_m,z)$, the contact structure $\eta=\frac{1}{2}(dz-\sum_{i=1}^{m}y_i dx_i)$, the characteristic vector field $\xi=2\frac{\partial}{\partial z}$ and the tensor field φ given by

$$\varphi=\begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

The Riemannian metric is $g = \eta \otimes \eta + \frac{1}{4}{}_{i=1}^{m} \left((dx_i)^2 + (dy_i)^2 \right)$.
Then $\left(M^{2m+1}, \varphi, \xi, \eta, g \right)$ is a Sasakian space form with constant $\varphi-$sectional curvature $c = -3$ and it is denoted by $\mathbb{R}^{2m+1}(-3)$.
The vector fields

$$X_i = 2\frac{\partial}{\partial y_i}, \ X_{i+m} = \varphi X_i = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}), \ 1 \leq i \leq m, \ \xi = 2\frac{\partial}{\partial z}, \tag{17}$$

form a g-orthonormal basis and Levi-Civita connection is calculated

$$\nabla_{X_i} X_j = \nabla_{X_{i+m}} X_{j+m} = 0, \ \nabla_{X_i} X_{j+m} = \delta_{ij} \xi, \ \nabla_{X_{i+m}} X_j = -\delta_{ij} \xi,$$

$$\nabla_{X_i} \xi = \nabla_{\xi} X_i = -X_{m+i}, \ \nabla_{X_{i+m}} \xi = \nabla_{\xi} X_{i+m} = X_i$$

(see [Blair]).

Now, let us produce example of proper f-biminimal Legendre curves in $\mathbb{R}^5(-3)$:

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Example. Let $\gamma = (\gamma_1, ..., \gamma_5)$ be a unit speed Legendre curve in $\mathbb{R}^5(-3)$. The tangent vector field of γ is

$$T = \frac{1}{2}\left\{\gamma_3' X_1 + \gamma_4' X_2 + \gamma_1' X_3 + \gamma_2' X_4 + \left(\gamma_5' - \gamma_1'\gamma_3 - \gamma_2'\gamma_4\right)\xi\right\}.$$

Using the above equation, since γ is a unit speed Legendre curve we have $\eta(T) = 0$ and $g(T, T) = 1$, that is,

$$\gamma_5' = \gamma_1'\gamma_3 - \gamma_2'\gamma_4$$

and

$$(\gamma_1')^2 + ... + (\gamma_5')^2 = 4.$$

For a Legendre curve, we can use the Levi-Civita connection and equation (17) to write

$$\nabla_T T = \frac{1}{2}\left(\gamma_3'' X_1 + \gamma_4'' X_2 + \gamma_1'' X_3 + \gamma_2'' X_4\right), \tag{18}$$

$$\varphi T = \frac{1}{2}\left(-\gamma_1' X_1 - \gamma_2' X_2 + \gamma_3' X_3 + \gamma_4' X_4\right). \tag{19}$$

From equations (18), (19) and $\varphi T \perp E_2$ if and only if

$$\gamma_1'\gamma_3'' + \gamma_2'\gamma_4'' = \gamma_3'\gamma_1'' + \gamma_4'\gamma_2''.$$

Finally, we can give the following explicit example:

Let us take $\gamma(t) = (\sin 2t, -\cos 2t, 0, 0, 1)$ in $\mathbb{R}^5(-3)$. Using the above equations and Theorem 12, γ is a proper f-biminimal Legendre curve with osculating order $r = 2$, $k_1 = 2$, $f = e^t$, $\varphi T \perp E_2$. We can easily check that the conditions of Theorem 12 are verified.

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Thank you...