Alternative Description of Rigid Body Kinematics and Quantum Mechanical Angular Momenta

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Two-Axes Decompositions

Given two non-parallel unit vectors $\hat{\mathbf{c}}_{1,2} \in \mathbb{S}^2$ the NSC for decomposing

$$\mathcal{R}(\mathbf{n}, \phi) = \mathcal{R}(\hat{\mathbf{c}}_2, \phi_2)\mathcal{R}(\hat{\mathbf{c}}_1, \phi_1)$$

is the coincidence of the matrix entries

$$r_{21} = g_{21}$$

and the solution has the form

$$\phi_1 = 2\arctan\frac{(\mathbf{n},\, \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2)}{(\mathbf{n},\, \hat{\mathbf{c}}_{[1})g_{2]1}}, \qquad \phi_2 = 2\arctan\frac{(\mathbf{n},\, \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2)}{(\mathbf{n},\, \hat{\mathbf{c}}_{[2})g_{1]2}}$$

where we denote

$$r_{ij} = (\hat{\mathbf{c}}_i,\, \mathcal{R}(\mathbf{n}, \phi)\,\hat{\mathbf{c}}_j), \qquad g_{ij} = (\hat{\mathbf{c}}_i,\, \hat{\mathbf{c}}_j), \qquad a_{[i}b_{j]} = a_i b_j - a_j b_i.$$

Euler Angles and Conjugation

Consider the Euler decomposition of a rotation $\mathcal{R} \in \mathrm{SO}(3)$

$$\mathcal{R}(\mathbf{n}, \phi) = \mathcal{R}(\hat{\mathbf{c}}_1, \psi)\mathcal{R}(\hat{\mathbf{c}}_2, \vartheta)\mathcal{R}(\hat{\mathbf{c}}_1, \varphi)$$

where ϕ, φ, ϑ and ψ denote the angles and $\mathbf{n}$, $\hat{\mathbf{c}}_{1,2}$ - the invariant axes (with the Davenport condition $\hat{\mathbf{c}}_1 \perp \hat{\mathbf{c}}_2$). A simple conjugation yields

$$\mathcal{R}_1(\psi)\mathcal{R}_2(\vartheta)\mathcal{R}_1(\varphi) = \mathcal{R}_1(\psi)\mathcal{R}_2(\vartheta)\mathcal{R}_1^{-1}(\psi)\mathcal{R}_1(\varphi + \psi)$$

so we have an equivalent two-factor decomposition with respect to a (shifted) pair of orthogonal axes

$$\mathcal{R}(\mathbf{n}, \phi) = \mathcal{R}\left(\mathcal{R}(\hat{\mathbf{c}}_1, \psi)\, \hat{\mathbf{c}}_2,\ \vartheta\right) \mathcal{R}(\hat{\mathbf{c}}_1,\ \varphi + \psi).$$

One Particular Solution

It is not difficult to satisfy the condition $r_{21} = g_{21}$ choosing arbitrary $\hat{\mathbf{c}}_1$

$$\hat{\mathbf{c}}_2 = \lambda\, \hat{\mathbf{c}}_1 \times \mathcal{R}(\mathbf{c})\, \hat{\mathbf{c}}_1, \qquad \lambda = (1 - r_{11}^2)^{-1/2}$$

and as for the solutions, we have (note that $\phi_1 = \vartheta$ and $\phi_2 = \varphi + \psi$)

$$\phi_1 = 2 \arctan \rho_1, \qquad \phi_2 = \arccos r_{11}$$

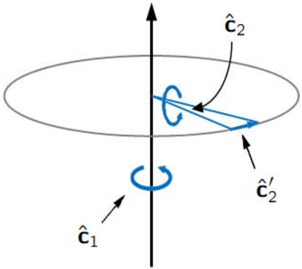
with the notation $\rho_i = \tan \frac{\phi}{2}(\mathbf{n}, \hat{\mathbf{c}}_i)$. One exception is the setting

$$\mathcal{R}(\mathbf{n}, \pi) = 2\, \mathbf{n} \otimes \mathbf{n}^t - \mathcal{I}, \qquad \mathbf{n} \perp \hat{\mathbf{c}}_1$$

that yields a one-parameter set of solutions. Choosing $\hat{\mathbf{c}}_2 \perp \hat{\mathbf{c}}_1$ one has

$$\mathcal{R}(\mathbf{n}, \pi) = \mathcal{R}(\hat{\mathbf{c}}_2, \pi)\mathcal{R}(\hat{\mathbf{c}}_1, \phi_1), \qquad \phi_1 = 2\measuredangle(\mathbf{c}_2, \mathbf{n}).$$

The Orthonormal Frame



One easily constructs a basis with a third vector

$$\hat{\mathbf{c}}_3 = \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2 = \lambda \left[r_{11} \mathcal{I} - \mathcal{R} \right] \hat{\mathbf{c}}_1 .$$

In order to parameterize the SO(3) we choose the third coordinate κ as the normal component of the rate, at which $\hat{\mathbf{c}}_2$ varies with $\mathcal{R}$, i.e.,

$$\hat{\mathbf{c}}_2' = \kappa' \, \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2 .$$

Kinematics

The $\{\phi_1, \phi_2, \kappa\}$ coordinates provide the kinematic equations in the form

$$\begin{aligned} \dot{\phi}_1 &= \Omega_1 - \Omega_3 \tan \frac{\phi_2}{2} \\ \dot{\phi}_2 &= \Omega_2 \\ \dot{\kappa} &= \Omega_1 + \Omega_3 \cot \phi_2 \end{aligned}$$

where Ω_k denote the components of the angular velocity in the so chosen basis. Inverting the matrix of the above system, one easily obtains

$$\begin{aligned} \Omega_1 &= \dot{w} - \cos v \, \dot{u} \\ \Omega_2 &= \dot{v} \\ \Omega_3 &= \sin v \, \dot{u} \end{aligned}$$

where we make use of the notation $u = \kappa - \phi_1$, $v = \phi_2$ and $w = \kappa$.

Dynamics

Consider the free Euler equations for a rotational inertial ellipsoid

$$\begin{aligned}\ddot{u} &= -\cot v\, \dot{u}\, \dot{v}\\ \ddot{v} &= \mu(\cos v\, \dot{u} - \dot{w}) \sin v\, \dot{u}\\ \ddot{w} &= (\mu \sin v - \csc v)\, \dot{u}\, \dot{v}\end{aligned}$$

with $I_1 = I_2 = I$ and $\mu = 1 - I_3/I$. One has $\Omega_3 = \mathrm{const.}$ and

$$v = a\cos(\omega t + \varphi_\circ) + b, \qquad \omega = \mu\,\Omega_3.$$

The kinematic equations then yield directly

$$\Omega_1(t) = a\omega\cos(\omega t + \varphi_\circ), \qquad \Omega_2(t) = -a\omega\sin(\omega t + \varphi_\circ)$$

while for the u and v variables one ends up with

$$u = \mp\frac{1}{\mu}\int\frac{\csc v\,\mathrm{d}v}{\sqrt{a^2-(v-b)^2}}\,, \qquad w = \mp\frac{1}{\mu}\int\frac{\cot v + \mu(v-b)}{\sqrt{a^2-(v-b)^2}}\,\mathrm{d}v.$$

Infinitesimal Variations

Infinitesimal left and right deck transformations yield the differential

$$\mathrm{d}\phi = \sin\phi \left(\csc\phi_1 \,\mathrm{d}\phi_1 + \frac{1-\cos\phi_2}{\cos\phi_1 - \cos\phi} \csc\phi_2 \,\mathrm{d}\phi_2 + \csc\phi_1 \csc\phi_2 \,\mathrm{d}\kappa \right)$$

as well as the components of the angular momentum operator

$$\begin{aligned} L_1 &= \frac{\partial}{\partial\phi_1} \\ L_2 &= \sin\phi_1 \tan\frac{\phi_2}{2} \frac{\partial}{\partial\phi_1} + \cos\phi_1 \frac{\partial}{\partial\phi_2} + \sin\phi_1 \csc\phi_2 \frac{\partial}{\partial\kappa} \\ L_3 &= \cos\phi_1 \tan\frac{\phi_2}{2} \frac{\partial}{\partial\phi_1} - \sin\phi_1 \frac{\partial}{\partial\phi_2} + \cos\phi_1 \csc\phi_2 \frac{\partial}{\partial\kappa}. \end{aligned}$$

The Laplacian

The associated Laplace operator (quantum Hamiltonian) has the form

$$\Delta = \sec^2\frac{\phi_2}{2}\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2} - \tan\frac{\phi_2}{2}\frac{\partial}{\partial\phi_2} + \csc^2\phi_2\frac{\partial^2}{\partial\kappa^2}$$

Using the notation $(\phi_1, \phi_2) \to (\alpha, \vartheta)$ one may rewrite the above as

$$\Delta = \sec^2\frac{\vartheta}{2}\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial}{\partial\vartheta}\left(\cos^2\frac{\vartheta}{2}\frac{\partial}{\partial\vartheta}\right)\right] + \csc^2\vartheta\frac{\partial^2}{\partial\kappa^2}.$$

For $\kappa = \mathrm{const.}$ we obtain the restriction on the quadric $r_{21} = g_{21}$

$$\Delta_0 = \sec^2\frac{\vartheta}{2}\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\vartheta^2} - \tan\frac{\vartheta}{2}\frac{\partial}{\partial\vartheta}.$$

The Hyperbolic Case

The factors $\epsilon_k = \hat{\mathbf{c}}_k^2$ distinguish between space-like ($\epsilon_k = 1$), time-like ($\epsilon_k = -1$) and light-like ($\epsilon_k = 0$) vectors in $\mathbb{R}^{2,1}$. We may denote

$$\tau = \tanh \frac{\phi}{2}\ (\epsilon{=}1), \qquad \tau = \tan \frac{\phi}{2}\ (\epsilon{=}{-1}), \qquad \tau = \frac{\phi}{2}\ (\epsilon{=}0)$$

thus unifying angles and rapidities. Note also the isotropic singularity

Theorem

If $\hat{\mathbf{c}}_{1,2} \in \mathbf{c}_\circ^\perp$ with $\mathbf{c}_\circ^2 = 0$, we may decompose a pseudo-rotation

$$\Lambda(\mathbf{n}, \tau) = \Lambda(\hat{\mathbf{c}}_2, \tau_2)\Lambda(\hat{\mathbf{c}}_1, \tau_1), \qquad \Lambda \in SO(2,1)$$

if and only if $\mathbf{n} \in \mathbf{c}_\circ^\perp$ lies the same tangent plane to the null cone.

The Monodromy Matrix

The quantum mechanical *monodromy matrix*

$$\mathcal{M} = \begin{pmatrix} 1/\bar{t} & -\bar{r}/\bar{t} \\ -r/t & 1/t \end{pmatrix} \in \mathrm{SU}(1,1)$$

relates left and right free particle asymptotic solutions

$$\begin{array}{lcll} \Psi(k,x) & \sim & \mathrm{e}^{\mathrm{i}kx} + r(k)\,\mathrm{e}^{-\mathrm{i}kx}, & x \to -\infty \\ \Psi(k,x) & \sim & t(k)\,\mathrm{e}^{\mathrm{i}kx}, & x \to \infty. \end{array}$$

In a standard split-quaternion basis $\mathcal{M}$ may be decomposed as

$$\mathcal{M} \;\to\; \zeta_{\mathcal{M}} = (\,\Re(t),\, -\Re(r\bar{t}),\, \Im(r\bar{t}),\, \Im(t)\,)^{t}$$

associating $t \in \mathbb{R}$ with pure boosts and $r = 0$ with pure rotations.

Decomposition of Scattering Potentials

Choosing $\hat{\mathbf{c}}_1$ to be aligned with the z-axis we find

$$\phi_1 = 2\theta, \qquad \theta = \arg(t)$$

as well as

$$\tau_2 = \sqrt{1 - |t|^2} \quad \Rightarrow \quad \phi_2 = 2 \operatorname{arccosh} |t|^{-1}$$

which finally yields

$$\mathcal{M} = \frac{1}{|t|} \begin{pmatrix} 1 & -\bar{r}\mathrm{e}^{2\mathrm{i}\theta} \\ -r\mathrm{e}^{-2\mathrm{i}\theta} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$$

and we decompose the monodromy into a product of a pure phase shift and a phase preserving scattering.

Extension to $SO(3,1)$

The local isomorphism

$$SO^+(3,1) \cong SO(3,\mathbb{C})$$

allows for extending via complexification e.g using biquaternions.

The existence of invariant axes is ensured by the Plücker relations

$$(\Re\,\tau\mathbf{n}, \Im\,\tau\mathbf{n}) = (\Re\,\tau_k\hat{\mathbf{c}}_k, \Im\,\tau_k\hat{\mathbf{c}}_k) = 0$$

and decomposability with real scalar parameters demands

$$(\Re\,\tau\mathbf{n}, \Im\,\tau_k\hat{\mathbf{c}}_k) + (\Re\,\tau_k\hat{\mathbf{c}}_k, \Im\,\tau\mathbf{n}) = 0$$

which projects the problem to a three-dimensional hyperplane.

Similar arguments (and Plücker relations) hold for the groups

$$SO(4), \quad SO(2,2), \quad SO^*(4).$$

Thomas Precession and Wigner Little Groups

Consider the generalized Rodrigues' vector for SO(3, 1)

$$\mathbf{c} = \tau\mathbf{n} = \boldsymbol{\alpha} + \mathrm{i}\beta \in \mathbb{CP}^3.$$

In the Plücker setting $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$ we may write

$$\Lambda(\boldsymbol{\alpha}+\mathrm{i}\boldsymbol{\beta})=\Lambda(\mathrm{i}\tilde{\boldsymbol{\beta}}_+)\Lambda(\boldsymbol{\alpha})=\Lambda(\boldsymbol{\alpha})\Lambda(\mathrm{i}\tilde{\boldsymbol{\beta}}_-), \qquad \tilde{\boldsymbol{\beta}}_\pm=\frac{\mathcal{I}\pm\boldsymbol{\alpha}^\times}{1+\boldsymbol{\alpha}^2}\,\boldsymbol{\beta}.$$

One example is the Thomas precession, in which

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}, \qquad \boldsymbol{\beta} = \frac{\boldsymbol{\beta}_1+\boldsymbol{\beta}_2}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}$$

where $\mathrm{i}\boldsymbol{\beta}_{1,2}$ are the Rodrigues' vectors of the two consecutive boosts.

A Numerical Example

Consider a proper Lorentz transformation

$$\Lambda = \frac{1}{5}\begin{pmatrix} -6 & 2 & -1 & -4 \\ -4 & -3 & -6 & -6 \\ -3 & -4 & 2 & -2 \\ 6 & 2 & 4 & 9 \end{pmatrix}$$

and let us choose $\hat{\mathbf{c}}_1 = (1+\mathrm{i},\ 1-\mathrm{i},\ 1)$, so that $\Lambda(\hat{\mathbf{c}}_1, \tau_1)$ preserves

$$\varsigma = (1,1,1,0)^t.$$

Then, we easily obtain $\Lambda = \Lambda_2\Lambda_1$ with

$$\Lambda_1 = \frac{1}{17}\begin{pmatrix} -15 & 8 & 24 & 24 \\ -8 & -15 & 40 & 40 \\ 40 & 24 & -47 & -64 \\ -40 & -24 & 64 & 81 \end{pmatrix}, \quad \Lambda_2 = \frac{1}{85}\begin{pmatrix} 178 & 138 & -401 & -452 \\ 36 & 77 & -334 & -334 \\ 109 & 224 & -438 & -506 \\ -194 & -278 & 676 & 761 \end{pmatrix}.$$

Rational Coordinates

Given a rotation matrix $\mathcal{R} \in \mathrm{SO}(3)$ with rational coefficients and a rational unit vector $\hat{\mathbf{c}}_1$ we may construct $\hat{\mathbf{c}}_2 = \hat{\mathbf{c}}_1 \times \mathcal{R}\hat{\mathbf{c}}$, which yields

$$\tau_1 = \rho_1, \qquad \tau_2 = \frac{1}{1 + r_{11}}$$

in the Euclidean case and

$$\tau_1 = \epsilon_1^{-1}\rho_1, \qquad \tau_2 = (\epsilon_1 + r_{11})^{-1}$$

for $\mathrm{SO}(3,1)$, so the two factors are rational as well.

Recommended Readings

- Brezov D., Mladenova C. and Mladenov I., *Two-Axes Decompositions of (Pseudo-) Rotations and Some of Their Applications*, AIP Conf. Proc. **1629** (2014) 226-234.
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- Tsiotras P. and Longuski J., *A New Parameterization of the Attitude Kinematics*, J. Austron. Sci. **43** (1995) 243-262.
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Thank You!

THANKS FOR YOUR PATIENCE!