Alternative Description of Rigid Body Kinematics and Quantum Mechanical Angular Momenta

Danail Brezov\textsuperscript{1}, Clementina Mladenova\textsuperscript{2} and Ivailo Mladenov\textsuperscript{3}

\textsuperscript{1}University of Architecture, Civil Engineering and Geodesy
\textsuperscript{2}Institute of Mechanics, Bulgarian Academy of Sciences
\textsuperscript{3}Institute of Biophysics, Bulgarian Academy of Sciences

Geometry, Integrability and Quantization
June 03-08, 2016, Varna, Bulgaria
Two-Axes Decompositions

Given two non-parallel unit vectors $\hat{c}_{1,2} \in S^2$ the NSC for decomposing

$$\mathcal{R}(n, \phi) = \mathcal{R}(\hat{c}_2, \phi_2)\mathcal{R}(\hat{c}_1, \phi_1)$$

is the coincidence of the matrix entries

$$r_{21} = g_{21}$$

and the solution has the form

$$\phi_1 = 2 \arctan \left( \frac{n, \hat{c}_1 \times \hat{c}_2}{(n, \hat{c}_1)g_{21}} \right), \quad \phi_2 = 2 \arctan \left( \frac{n, \hat{c}_1 \times \hat{c}_2}{(n, \hat{c}_2)g_{12}} \right)$$

where we denote

$$r_{ij} = (\hat{c}_i, \mathcal{R}(n, \phi)\hat{c}_j), \quad g_{ij} = (\hat{c}_i, \hat{c}_j), \quad a[i;b_j] = a_i b_j - a_j b_i.$$
Euler Angles and Conjugation

Consider the Euler decomposition of a rotation $\mathcal{R} \in SO(3)$

$$\mathcal{R}(n, \phi) = \mathcal{R}(\hat{c}_1, \psi)\mathcal{R}(\hat{c}_2, \vartheta)\mathcal{R}(\hat{c}_1, \varphi)$$

where $\phi, \varphi, \vartheta$ and $\psi$ denote the angles and $n, \hat{c}_{1,2}$ - the invariant axes (with the Davenport condition $\hat{c}_1 \perp \hat{c}_2$). A simple conjugation yields

$$\mathcal{R}_1(\psi)\mathcal{R}_2(\vartheta)\mathcal{R}_1(\phi) = \mathcal{R}_1(\psi)\mathcal{R}_2(\vartheta)\mathcal{R}_1^{-1}(\psi)\mathcal{R}_1(\varphi + \psi)$$

so we have an equivalent two-factor decomposition with respect to a (shifted) pair of orthogonal axes

$$\mathcal{R}(n, \phi) = \mathcal{R}(\mathcal{R}(\hat{c}_1, \psi) \hat{c}_2, \vartheta)\mathcal{R}(\hat{c}_1, \varphi + \psi).$$
One Particular Solution

It is not difficult to satisfy the condition $r_{21} = g_{21}$ choosing arbitrary $\hat{c}_1$

$$\hat{c}_2 = \lambda \hat{c}_1 \times \mathcal{R}(c) \hat{c}_1, \quad \lambda = (1 - r_{11}^2)^{-1/2}$$

and as for the solutions, we have (note that $\phi_1 = \psi$ and $\phi_2 = \varphi + \psi$)

$$\phi_1 = 2 \arctan \rho_1, \quad \phi_2 = \arccos r_{11}$$

with the notation $\rho_i = \tan \frac{\phi}{2}(n, \hat{c}_i)$. One exception is the setting

$$\mathcal{R}(n, \pi) = 2 n \otimes n^t - \mathcal{I}, \quad n \perp \hat{c}_1$$

that yields a one-parameter set of solutions. Choosing $\hat{c}_2 \perp \hat{c}_1$ one has

$$\mathcal{R}(n, \pi) = \mathcal{R}(\hat{c}_2, \pi) \mathcal{R}(\hat{c}_1, \phi_1), \quad \phi_1 = 2 \angle(c_2, n).$$
The Orthonormal Frame

One easily constructs a basis with a third vector

\[ \hat{c}_3 = \hat{c}_1 \times \hat{c}_2 = \lambda [r_{11} I - \mathcal{R}] \hat{c}_1. \]

In order to parameterize the SO(3) we choose the third coordinate \( \kappa \) as the normal component of the rate, at which \( \hat{c}_2 \) varies with \( \mathcal{R} \), i.e.,

\[ \hat{c}'_2 = \kappa' \hat{c}_1 \times \hat{c}_2. \]
Kinematics

The \( \{\phi_1, \phi_2, \kappa\} \) coordinates provide the kinematic equations in the form

\[
\begin{align*}
\dot{\phi}_1 &= \Omega_1 - \Omega_3 \tan \frac{\phi_2}{2} \\
\dot{\phi}_2 &= \Omega_2 \\
\dot{\kappa} &= \Omega_1 + \Omega_3 \cot \phi_2
\end{align*}
\]

where \( \Omega_k \) denote the components of the angular velocity in the so chosen basis. Inverting the matrix of the above system, one easily obtains

\[
\begin{align*}
\Omega_1 &= \dot{w} - \cos v \, \dot{u} \\
\Omega_2 &= \dot{v} \\
\Omega_3 &= \sin v \, \dot{u}
\end{align*}
\]

where we make use of the notation \( u = \kappa - \phi_1 \), \( v = \phi_2 \) and \( w = \kappa \).
Dynamics

Consider the free Euler equations for a rotational inertial ellipsoid

\[
\ddot{u} = - \cot v \dot{u} \dot{v} \\
\ddot{v} = \mu (\cos v \dot{u} - \dot{w}) \sin v \dot{u} \\
\ddot{w} = (\mu \sin v - \csc v) \dot{u} \dot{v}
\]

with \( I_1 = I_2 = l \) and \( \mu = 1 - l_3/l \). One has \( \Omega_3 = \text{const.} \) and

\[
v = a \cos(\omega t + \varphi_0) + b, \quad \omega = \mu \Omega_3.
\]

The kinematic equations then yield directly

\[
\Omega_1(t) = a\omega \cos(\omega t + \varphi_0), \quad \Omega_2(t) = -a\omega \sin(\omega t + \varphi_0)
\]

while for the \( u \) and \( v \) variables one ends up with

\[
u = \mp \frac{1}{\mu} \int \frac{\csc v \, dv}{\sqrt{a^2 - (v - b)^2}}, \quad w = \mp \frac{1}{\mu} \int \frac{\cot v + \mu(v - b)}{\sqrt{a^2 - (v - b)^2}} \, dv.
\]
Infinitesimal Variations

Infinitesimal left and right deck transformations yield the differential

\[ d\phi = \sin \phi \left( \csc \phi_1 \, d\phi_1 + \frac{1 - \cos \phi_2}{\cos \phi_1 - \cos \phi} \, \csc \phi_2 \, d\phi_2 + \csc \phi_1 \, \csc \phi_2 \, d\kappa \right) \]

as well as the components of the angular momentum operator

\[ L_1 = \frac{\partial}{\partial \phi_1} \]
\[ L_2 = \sin \phi_1 \tan \frac{\phi_2}{2} \frac{\partial}{\partial \phi_1} + \cos \phi_1 \frac{\partial}{\partial \phi_2} + \sin \phi_1 \csc \phi_2 \frac{\partial}{\partial \kappa} \]
\[ L_3 = \cos \phi_1 \tan \frac{\phi_2}{2} \frac{\partial}{\partial \phi_1} - \sin \phi_1 \frac{\partial}{\partial \phi_2} + \cos \phi_1 \csc \phi_2 \frac{\partial}{\partial \kappa} . \]
The associated Laplace operator (quantum Hamiltonian) has the form

\[ \Delta = \sec^2 \frac{\phi_2}{2} \frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \phi_2^2} - \tan \frac{\phi_2}{2} \frac{\partial}{\partial \phi_2} + \csc^2 \frac{\phi_2}{2} \frac{\partial^2}{\partial \kappa^2} \]

Using the notation \((\phi_1, \phi_2) \rightarrow (\alpha, \vartheta)\) one may rewrite the above as

\[ \Delta = \sec^2 \frac{\vartheta}{2} \left[ \frac{\partial^2}{\partial \alpha^2} + \frac{\partial}{\partial \vartheta} \left( \cos^2 \frac{\vartheta}{2} \frac{\partial}{\partial \vartheta} \right) \right] + \csc^2 \frac{\vartheta}{2} \frac{\partial^2}{\partial \kappa^2}. \]

For \(\kappa = \text{const.}\) we obtain the restriction on the quadric \(r_{21} = g_{21}\)

\[ \Delta_0 = \sec^2 \frac{\vartheta}{2} \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \vartheta^2} - \tan \frac{\vartheta}{2} \frac{\partial}{\partial \vartheta}. \]
The factors $\epsilon_k = \hat{c}_k^2$ distinguish between space-like ($\epsilon_k = 1$), time-like ($\epsilon_k = -1$) and light-like ($\epsilon_k = 0$) vectors in $\mathbb{R}^{2,1}$. We may denote

$$\tau = \tanh \frac{\phi}{2} \ (\epsilon = 1), \quad \tau = \tan \frac{\phi}{2} \ (\epsilon = -1), \quad \tau = \frac{\phi}{2} \ (\epsilon = 0)$$

thus unifying angles and rapidities. Note also the isotropic singularity.

**Theorem**

*If $\hat{c}_{1,2} \in c_\circ^\perp$ with $c_\circ^2 = 0$, we may decompose a pseudo-rotation*

$$\Lambda(n, \tau) = \Lambda(\hat{c}_2, \tau_2) \Lambda(\hat{c}_1, \tau_1), \quad \Lambda \in SO(2,1)$$

*if and only if $n \in c_\circ^\perp$ lies the same tangent plane to the null cone.*
The Monodromy Matrix

The quantum mechanical monodromy matrix

\[ \mathcal{M} = \begin{pmatrix} 1/\bar{t} & -\bar{r}/\bar{t} \\ -r/t & 1/t \end{pmatrix} \in SU(1, 1) \]

relates left and right free particle asymptotic solutions

\[ \Psi(k,x) \sim e^{ikx} + r(k)e^{-ikx}, \quad x \to -\infty \]
\[ \Psi(k,x) \sim t(k)e^{ikx}, \quad x \to \infty. \]

In a standard split-quaternion basis \( \mathcal{M} \) may be decomposed as

\[ \mathcal{M} \to \zeta_{\mathcal{M}} = (\mathcal{R}(t), -\mathcal{R}(rt), \mathcal{I}(rt), \mathcal{I}(t))^t \]

associating \( t \in \mathbb{R} \) with pure boosts and \( r = 0 \) with pure rotations.
Choosing \( \hat{c}_1 \) to be aligned with the \( z \)-axis we find

\[
\phi_1 = 2\theta, \quad \theta = \arg(t)
\]

as well as

\[
\tau_2 = \sqrt{1 - |t|^2} \quad \Rightarrow \quad \phi_2 = 2 \arccosh |t|^{-1}
\]

which finally yields

\[
\mathcal{M} = \frac{1}{|t|} \begin{pmatrix}
1 & -re^{-2i\theta} \\
-re^{2i\theta} & 1
\end{pmatrix}
\begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{pmatrix}
\]

and we decompose the monodromy into a product of a pure phase shift and a phase preserving scattering.
Extension to SO(3, 1)

The local isomorphism

\[ \text{SO}^+(3, 1) \cong \text{SO}(3, \mathbb{C}) \]

allows for extending via complexification e.g using biquaternions.

The existence of invariant axes is ensured by the Plücker relations

\[ (\Re \tau n, \Im \tau n) = (\Re \tau_k \hat{c}_k, \Im \tau_k \hat{c}_k) = 0 \]

and decomposability with real scalar parameters demands

\[ (\Re \tau n, \Im \tau_k \hat{c}_k) + (\Re \tau_k \hat{c}_k, \Im \tau n) = 0 \]

which projects the problem to a three-dimensional hyperplane.

Similar arguments (and Plücker relations) hold for the groups

\[ \text{SO}(4), \quad \text{SO}(2, 2), \quad \text{SO}^*(4). \]
Consider the generalized Rodrigues’ vector for $\text{SO}(3,1)$

$$\mathbf{c} = \tau \mathbf{n} = \alpha + i \beta \in \mathbb{CP}^3.$$ 

In the Plücker setting $\alpha \perp \beta$ we may write

$$\Lambda(\alpha + i\beta) = \Lambda(i\tilde{\beta}_+)\Lambda(\alpha) = \Lambda(\alpha)\Lambda(i\tilde{\beta}_-), \quad \tilde{\beta}_\pm = \frac{I \pm \alpha \times}{1 + \alpha^2} \beta.$$ 

One example is the Thomas precession, in which

$$\alpha = \frac{\beta_1 \times \beta_2}{1 + \beta_1 \cdot \beta_2}, \quad \beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \cdot \beta_2}$$

where $i\beta_{1,2}$ are the Rodrigues’ vectors of the two consecutive boosts.
A Numerical Example

Consider a proper Lorentz transformation

\[
\Lambda = \frac{1}{5} \begin{pmatrix}
-6 & 2 & -1 & -4 \\
-4 & -3 & -6 & -6 \\
-3 & -4 & 2 & -2 \\
6 & 2 & 4 & 9
\end{pmatrix}
\]

and let us choose \( \hat{c}_1 = (1+i, 1-i, 1) \), so that \( \Lambda(\hat{c}_1, \tau_1) \) preserves

\[
\varsigma = (1, 1, 1, 0)^t.
\]

Then, we easily obtain \( \Lambda = \Lambda_2 \Lambda_1 \) with

\[
\Lambda_1 = \frac{1}{17} \begin{pmatrix}
-15 & 8 & 24 & 24 \\
-8 & -15 & 40 & 40 \\
40 & 24 & -47 & -64 \\
-40 & -24 & 64 & 81
\end{pmatrix}, \quad \Lambda_2 = \frac{1}{85} \begin{pmatrix}
178 & 138 & -401 & -452 \\
36 & 77 & -334 & -334 \\
109 & 224 & -438 & -506 \\
-194 & -278 & 676 & 761
\end{pmatrix}.
\]
Given a rotation matrix $\mathcal{R} \in \text{SO}(3)$ with rational coefficients and a rational unit vector $\hat{\mathbf{c}}_1$ we may construct $\hat{\mathbf{c}}_2 = \hat{\mathbf{c}}_1 \times \mathcal{R} \hat{\mathbf{c}}$, which yields

$$
\tau_1 = \rho_1, \quad \tau_2 = \frac{1}{1 + r_{11}}
$$

in the Euclidean case and

$$
\tau_1 = \epsilon_1^{-1} \rho_1, \quad \tau_2 = (\epsilon_1 + r_{11})^{-1}
$$

for $\text{SO}(3, 1)$, so the two factors are rational as well.
Recommended Readings


Thank You!

THANKS FOR YOUR PATIENCE!