Local and non-local conservation laws for quadratic constrained Lagrangians and applications to cosmology

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Outline

Mini-superspace description

$$S = \int_{\Omega} d^4x \sqrt{-g} R + S_m \tag{1}$$

Einstein's equations

$$R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \tag{2}$$

For a spatially homogeneous space-time

$$\begin{aligned} ds^2 =& (N_\mu(t) N^\mu(t) - N(t)^2) dt^2 + N_\mu(t) \sigma^\mu_i(x) dt dx^i \\ &+ \gamma_{\alpha\beta}(t) \sigma^\alpha_i(x) \sigma^\beta(x) dx^i dx^j, \end{aligned} \tag{3}$$

with

$$\sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = C^\alpha_{\mu\nu} \sigma^\mu_i \sigma^\nu_i, \tag{4}$$

Equations (2) reduce to ODEs.

Without loss of generality $N_\alpha(t) = 0$.

$$ds^2 = -N(t)^2 dt^2 + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta(x)dx^i dx^j, \tag{5}$$

For some types of systems

$$L = \frac{1}{2N} G^{\kappa\lambda\mu\nu}\dot{\gamma}_{\kappa\lambda}\dot{\gamma}_{\mu\nu} - N\sqrt{\gamma}\mathcal{R} + L_m \tag{6}$$

is a valid Lagrangian, with

$$G^{\kappa\lambda\mu\nu} = \frac{1}{4}\sqrt{\gamma}\left(\gamma^{\kappa\mu}\gamma^{\lambda\nu} + \gamma^{\kappa\nu}\gamma^{\lambda\mu} - 2\gamma^{\kappa\lambda}\gamma^{\mu\nu}\right) \tag{7}$$

being the mini-superspace metric.

Constrained Systems

Prototype mini-superspace Lagrangian

$$L = \frac{1}{2N} G_{\alpha\beta}(q)\, \dot{q}^\alpha\, \dot{q}^\beta - N\, V(q) \tag{8}$$

$d+1$ degrees of freedom: $\mathrm{v}^i(t) := (N(t), q^\alpha(t)) \quad , \quad \alpha = 1, ..., d$

but the Hessian matrix is of rank d

$$\det\left(\frac{\partial^2 L}{\partial \dot{\mathrm{v}}^i \partial \dot{\mathrm{v}}^j}\right) = 0$$

$$p_\alpha := \frac{\partial L}{\partial \dot{q}^\alpha} = \frac{1}{N} G_{\alpha\beta}\, \dot{q}^\beta \qquad p_N \approx 0 \qquad \text{(Primary Constraint)}$$

$$\begin{aligned} H &= \dot{q}^{\gamma} p_{\gamma} - L + u_N p_N \\ &= N \left(\frac{1}{2} G^{\alpha\beta}(q)\, p_\alpha\, p_\beta + V(q) \right) + u_N p_N \\ &= N \mathcal{H} + u_N p_N \end{aligned} \tag{9}$$

$$\dot{p}_N = \{p_N, H\} \approx 0 \Rightarrow \qquad \mathcal{H} = \frac{1}{2} G^{\alpha\beta}(q)\, p_\alpha\, p_\beta + V(q) \approx 0$$

($\mathcal{H}$ Secondary Constraint)

$$\{p_N, \mathcal{H}\} \approx 0 \Rightarrow p_N, \mathcal{H}$$ First Class Constraints

Variational symmetries of the action

T. Christodoulakis, N. Dimakis and Petros A. Terzis *J. Phys. A: Math. Theor.* **47** (2014) 095202

Generator of transformations in $(t, q(t), N(t))$

$$X = \chi(t,q,N)\frac{\partial}{\partial t} + \xi^\alpha(t,q,N)\frac{\partial}{\partial q^\alpha} + \omega(t,q,N)\frac{\partial}{\partial N} \tag{10}$$

k-th prolongation

$$pr^{(k)}X = X + \Phi^\alpha_t \frac{\partial}{\partial \dot{q}^\alpha} + \Omega_t \frac{\partial}{\partial \dot{N}} + \ldots + \Phi^\alpha_{t^k}\frac{\partial}{\partial(\partial_{t^k} q^\alpha)} + \Omega_{t^k}\frac{\partial}{\partial(\partial_{t^k}N)} \tag{11}$$

$$\Phi^\alpha_{t^k} = \frac{d^k}{dt^k}\left(\xi^\alpha - \chi\dot{q}^\alpha\right) + \chi\frac{d^{k+1}q^\alpha}{dt^{k+1}}$$

$$\Omega_{t^k} = \frac{d^k}{dt^k}\left(\omega - \chi\dot{N}\right) + \chi\frac{d^{k+1}N}{dt^{k+1}}$$

Infinitesimal criterion of invariance

$$pr^{(1)}X(L) + L\frac{d\chi}{dt} = \frac{df}{dt}, \quad \text{where } f = f(t, q, N)$$

Final form for the generator

$$X = X_1 + X_2$$

$$X_1 = \xi^{\alpha}(q)\frac{\partial}{\partial q^{\alpha}} + N\tau(q)\frac{\partial}{\partial N} \tag{12}$$

$$X_2 = \chi(t)\frac{\partial}{\partial t} - N\dot{\chi}(t)\frac{\partial}{\partial N} \tag{13}$$

with $\pounds_{\xi} G_{\alpha\beta} = \tau(q) G_{\alpha\beta}$ and $\pounds_{\xi} V = -\tau(q) V$

$$\pounds_{\xi} G_{\alpha\beta} = -\frac{1}{V}\left(\pounds_{\xi} V\right) G_{\alpha\beta}$$

Lie-point symmetries of the equations of motion

$$E^0 := \frac{\partial L}{\partial N} = 0 \tag{14a}$$

$$E^\alpha := \frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) = 0 \tag{14b}$$

Infinitesimal criterion

$$pr^{(1)}X(E^0) = T(t,q,N)E^0$$

$$pr^{(2)}X(E^\alpha)\big|_{E^\alpha=0} = \left(P_{1\,\alpha}^{\kappa}(t,q,N)\dot{q}^\alpha + P_2^\kappa(t,q,N)\dot{N} + P_3^\kappa(t,q,N)\right)E^0$$

The Lie - point symmetries of the system are

$$X = \tilde{X}_1 + X_2$$

$$\tilde{X}_1 = X_1 + c\frac{\partial}{\partial N} = \xi^{\alpha}(q)\frac{\partial}{\partial q^{\alpha}} + N(\tau(q)+c)\frac{\partial}{\partial N} \quad (16)$$

$$X_2 = \chi(t)\frac{\partial}{\partial t} - N\dot{\chi}(t)\frac{\partial}{\partial N} \quad (17)$$

with $\pounds_{\xi} G_{\alpha\beta} = \tau(q) G_{\alpha\beta}$ and $\pounds_{\xi} V = -(\tau(q)+2c)V$

$$\pounds_{\xi} G_{\alpha\beta} = -\left(\frac{1}{V}\pounds_{\xi} V + \tilde{c}\right) G_{\alpha\beta}$$

The effective constant potential parametrization

Lapse function scaling: $N \to n = N\, V(q)$
Equivalent Lagrangian:

$$L = \frac{1}{2n}\,\overline{G}_{\alpha\beta}(q)\,\dot{q}^{\alpha}\,\dot{q}^{\beta} - n \qquad (18)$$

with $\overline{G}_{\alpha\beta} := V\, G_{\alpha\beta}$

- Variational symmetries

$$\pounds_{\xi}\overline{G}_{\alpha\beta} = 0$$

- Lie-point symmetries of the equations of motion

$$\pounds_{\xi}\overline{G}_{\alpha\beta} = (\text{const.})\;\overline{G}_{\alpha\beta}$$

Conditional symmetries of phase space

$$H_T = n\mathcal{H} + u_n p_n \approx 0$$

$$\mathcal{H} = \frac{1}{2}\overline{G}^{\alpha\beta} p_\alpha p_\beta + 1 \approx 0$$

Assume a quantity $Q = Q(t, q, p)$

$$\frac{dQ}{dt} \approx 0 \Rightarrow \frac{\partial Q}{\partial t} + \{Q, H_T\} = \omega\mathcal{H} \tag{19}$$

If Q is linear in the momenta p_α, then

$$Q = \xi^\alpha p_\alpha + \int n(t)\omega(q(t))dt \tag{20}$$

is a conditional symmetry whenever

$$\pounds_\xi \overline{G}_{\alpha\beta} = \omega(q)\,\overline{G}_{\alpha\beta}$$

Integrals of motion

- $\omega = 0$, ξ is a Killing vector of $\overline{G}_{\alpha\beta}$

$$Q = \xi^{\alpha} p_{\alpha} \qquad \text{(Variational/Lie-point symmetries)}$$

- $\omega \neq 0$
  - $\omega = 1$, ξ is a Homothetic vector of $\overline{G}_{\alpha\beta}$

$$Q = \xi^{\alpha} p_{\alpha} + \int n(t) dt \qquad \text{(Lie-point symmetry)}$$

  - $\omega \neq const.$

$$Q = \xi^{\alpha} p_{\alpha} + \int n(t) \omega(q) dt \qquad \text{(Conditional symmetries)}$$

Variational $\subset$ Lie-point symmetries $\subset$ Conditional symmetries

Higher order symmetries

Petros A. Terzis, N. Dimakis, T. Christodoulakis, A. Paliathanasis and M. Tsamparlis *J. Geom. and Phys.* **101** (2016) 52-64

Conditional symmetry of order $k+1$ in the momenta

$$Q = S^{\kappa\alpha_1\ldots\alpha_k} p_\kappa p_{\alpha_1} \ldots p_{\alpha_n} + \int n\,\omega^{\alpha_1\ldots\alpha_k} \frac{\partial L}{\partial \dot{q}^{\alpha_1}} \cdots \frac{\partial L}{\partial \dot{q}^{\alpha_n}} dt$$

whenever

$$S^{(\nu\alpha_1\ldots\alpha_n;\mu)} = \frac{1}{2}\omega^{(\alpha_1\ldots\alpha_n}\overline{G}^{\mu\nu)}$$

The case $S_{(\nu\alpha_1\ldots\alpha_n;\mu)} = 0$ corresponds to contact symmetries of the action

$$X = \Xi^\kappa(n, q, \dot{q}) \frac{\partial}{\partial q^\kappa} + \Omega(n, q, \dot{q}) \frac{\partial}{\partial n}$$

$$\Xi^\kappa = \xi^\kappa(q) + \frac{1}{n} S^\kappa{}_{\alpha_1}(q)\dot{q}^{\alpha_1} + \ldots + \frac{1}{n^k} S^\kappa{}_{\alpha_1\ldots\alpha_k}(q)\dot{q}^{\alpha_1}\ldots\dot{q}^{\alpha}_k$$

Canonical Quantization

$$p_\alpha \longmapsto \widehat{p}_\alpha = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial q^\alpha} \tag{21a}$$

$$p_n \longmapsto \widehat{p}_n = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial n} \tag{21b}$$

$$\{\ ,\ \} \longrightarrow -\frac{\mathrm{i}}{\hbar}[\ ,\ ]$$

$$\widehat{p}_n\Psi(q,n) = 0 \Rightarrow \Psi = \Psi(q) \tag{22a}$$

$$\widehat{\mathcal{H}}\Psi(q) = 0 \Rightarrow \left[-\frac{1}{2\mu}\partial_\alpha(\mu\,G^{\alpha\beta}\partial_\beta) + V(q) + \frac{d-2}{4\,(d-1)}\mathcal{R}\right]\Psi = 0 \tag{22b}$$

$$\widehat{Q}_I := -\frac{\mathrm{i}}{2\mu}\left(\mu\,\xi_I^\alpha\,\partial_\alpha + \partial_\alpha\,\mu\,\xi_I^\alpha\right) \tag{23}$$

Eigenvalue equations

$$\widehat{Q}_I\Psi = \kappa_I\Psi, \qquad 1 \leq I \leq \frac{d\,(d+1)}{2} \tag{24}$$

$$\{Q_I, Q_J\} = C^M{}_{IJ} Q_M \Rightarrow [\widehat{Q}_I, \widehat{Q}_J] = \mathrm{i} C^M{}_{IJ} \widehat{Q}_M$$

Integrability conditions of (24):

$$C^M{}_{IJ}\,\kappa_M = 0 \tag{25}$$

Mini-superspace reduction

N. Dimakis, A. Karagiorgos, A. Zampeli, A. Paliathanasis, T. Christodoulakis and Petros A. Terzis To appear in *Phys. Rev. D*

$$S = \int d^4x \sqrt{-g}\,(R + \epsilon\,\phi_{,\mu}\phi^{,\mu} + 2\,V(\phi))$$

Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

with

$$T_{\mu\nu} = \epsilon\,\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\,(\epsilon\,\phi^{,\kappa}\phi_{,\kappa} - 2\,V(\phi))\,g_{\mu\nu}$$

Klein-Gordon equation

$$\epsilon\,\Box\phi - V'(\phi) = 0$$

Ansatz for the metric

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left(\frac{1}{1 - k\, r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right)$$

Mini-superspace Lagrangian

$$L = \frac{2a^2}{n} \left(a^2 V(\phi) - 3k \right) \left(-6\dot{a}^2 + \epsilon\, a^2 \dot{\phi}^2 \right) - n \qquad (26)$$

with

$$n = N \left(2\, a \left(a^2 V(\phi) - 3k \right) \right) \qquad (27)$$

2d mini-superspace metric

$$G_{\mu\nu} = 4\, a^2 \left(a^2 V(\phi) - 3k \right) \begin{pmatrix} -6 & 0 \\ 0 & \epsilon\, a^2 \end{pmatrix} \qquad (28)$$

Conformal vector $\xi = \frac{\partial}{\partial \phi}$ with a corresponding factor $\frac{a^2 V'(\phi)}{a^2 V(\phi) - 3k}$.
Integral of motion

$$Q = p_\phi + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt = \frac{\partial L}{\partial \dot{\phi}} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt$$
$$= \frac{4\,\epsilon\, a^4 \dot{\phi}\,(a^2 V(\phi) - 3k)}{n} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt$$

Strategy:

- re-parametrize $n(t) \Rightarrow n(t) = \frac{2\dot{h}(a^2 V - 3k)}{a^2 \dot{V}}$
- Fix the gauge $\phi(t) = t \Rightarrow$ re-parametrize $V(t)$ in terms of a new function of t
- Solve $Q =$ const. together with $\frac{\partial L}{\partial n} = 0$

$$ds^2 = \frac{-e^{\omega}\dot{\omega}^2}{36\left(2\,e^{\omega-6\int(\epsilon/\dot{\omega})dt}\left(c_2 + 3\,k\int \frac{\exp\left(6\int(\epsilon/\dot{\omega})dt - \frac{\omega}{3}\right)}{\dot{\omega}}dt\right) - ke^{\frac{2\omega}{3}}\right)}dt^2$$
$$+ e^{\omega/3}\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right)$$

$$V(t) = \frac{6\,e^{-\omega}}{\dot{\omega}^2}\Bigg[\left(\dot{\omega}^2 - 6\,\epsilon\right)\times$$
$$e^{\omega-6\int(\epsilon/\dot{\omega})dt}\left(c_2 + 3\,k\int \frac{\exp\left(6\int(\epsilon/\dot{\omega}) - \frac{\omega}{3}dt\right)}{\dot{\omega}}dt\right) + 3\,k\,e^{\frac{2\omega}{3}}\Bigg]$$

Under the time change

$$\phi = t = \pm \int \left[\frac{1}{6\,\epsilon} \left(\frac{S''(\omega)}{S'(\omega)} + \frac{1}{3} \right) \right]^{1/2} d\omega$$

where $S(\omega) = \exp\left(12\,k \int e^{F(\omega)-\omega/3} d\omega\right) - \frac{6\,c_2}{k}$

$$ds^2 = -e^{F(\omega)} d\omega^2 + e^{\omega/3} \left(\frac{1}{1-kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\varphi^2 \right)$$

$$V(\omega) = \frac{1}{12} e^{-F(\omega)} \left(1 - F'(\omega)\right) + 2\,k\,e^{-\omega/3} \tag{29}$$

Equivalent Perfect Fluid formalism

$$\rho_\phi(t) = T^{\mu\nu} u_\mu u_\nu, \qquad u_\mu = \frac{\phi_{,\mu}}{\sqrt{-g^{\kappa\lambda}\phi_{,\kappa}\phi_{,\lambda}}}$$

$$P_\phi(t) = \frac{1}{3} T^{\mu\nu} h_{\mu\nu}, \qquad h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

- $k = 0$

$$P_\phi = (2F'(\omega) - 1)\rho_\phi$$

- $k \neq 0$

$$P_\phi = \left(\frac{2\, e^{\omega/3}\,(3F'(\omega) - 1)}{3\,\left(36\, k\, e^{F(\omega)} + e^{\omega/3}\right)} - \frac{1}{3} \right) \rho_\phi$$

Final Remarks

Quadratic constrained Lagrangians

- All conformal Killing tensors generate integrals of motion.
- In the constant potential parametrization:
  - Killing tensors $\longrightarrow$ Noether symmetries
  - CKTs $\longrightarrow$ non-local conditional symmetries
- Use of these symmetries at both the classical and quantum level to achieve integrability