On Special Coordinate Systems

Irena Hinterleitner, Josef Mikeš

Brno University of Technology, Palacky University Olomouc

Varna 2016

Topics

1. Geodesics
2. Geodesic coordinates
3. Semigeodesic coordinates
4. Pre-semigeodesic coordinates
5. On the existence of pre-semigeodesic coordinates

A geodesic

- is an object analogous to straight lines in Euclidean space,
- is a curve whose tangent vectors in all of its points are parallel

Definition

A curve ℓ in A_n is a geodesic when its tangent vector field remains in the tangent distribution of ℓ during parallel transport along the curve.

A curve $\ell(t) \subset A_n$ is a geodesic iff the covariant derivative of its tangent vector $\lambda(t) = \dot{\ell}(t)$ is proportional to the tangent vector

$$\nabla_\lambda \lambda = \rho(t)\lambda,$$

where ρ is some function of the parameter t of the curve ℓ.

When the parameter t of the geodesic is chosen so that $\rho(t) \equiv 0$, then this parameter is called natural or affine.
A natural parameter is usually denoted by s.

$$\frac{d^2x^h(s)}{ds^2} + \Gamma^h_{ij}(x(s))\frac{dx^i(s)}{ds}\frac{dx^j(s)}{ds} = 0,$$

where in (pseudo-) Riemannian spaces

$$\Gamma^h_{ij} \equiv \frac{1}{2}\left(\frac{\partial g_{ik}(x)}{\partial x^j} + \frac{\partial g_{jk}(x)}{\partial x^i} - \frac{\partial g_{ij}(x)}{\partial x^k}\right) g^{kh}(x), \quad h, i, j, k = 1, 2, \ldots, n.$$

$\ell: \quad x^h = x^h(t)$ for any parameter t:

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ij}(x(t))\frac{dx^i}{dt}\frac{dx^j}{dt} = \rho(t)\frac{dx^h}{dt},$$

where $\rho(t)$ is some function of the parameter t.

Examples: some properties of geodesic lines in mechanics

- a point mass on a plane without external influences moves on a geodesic line,
- an ideal elastic ribbon without friction between two points on a curved surface lies along a geodesic.

A coordinate system will be called *geodesic in the point x_0* if

$$\Gamma^h_{ij}|_{x_0} = 0.$$

A coordinate system will be called *geodesic along a curve ℓ*

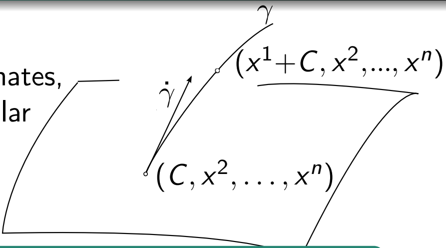
(*Fermi coordinates*)

if for the prescribed curve ℓ

$$\Gamma^h_{ij}|_\ell = 0$$

is satisfied.

For any Riemannian manifold V_n let us introduce semigeodesic coordinates, which can be considered as a particular case of orthogonal coordinates based on a system of hypersurfaces.



Definition

Let us consider a non-isotropic coordinate hypersurface Σ*:* $x^1 = C$*. Let us fix some point* $(C, x^2, \dots, x^n)$ *on* Σ *and construct the geodesic* γ *passing through the point and tangent to the unit normal of* Σ*;* γ *is an* x^1*-curve, it is parametrized by*

$$\gamma(x^1) = (x^1 + C, x^2, \dots, x^n)$$

and x^1 *is the arc length on the geodesic. Coordinates introduced in this way are called* *semigeodesic coordinates* *in* V_n*.*

It is well known that the metric form of V_n in semigeodesic coordinate has the following form:

$$(1) \qquad ds^2 = e\,(dx^1)^2 + g_{ab}(x)\,dx^a\,dx^b, \quad a, b > 1,\; e = \pm 1.$$

In this case for the Christoffel symbols of the second type follows

$$(2) \qquad \Gamma^h_{11} = 0, \quad h = 1, \ldots, n.$$

On the other hand the coordinate form (1) of the metric is a sufficient condition for the coordinate system to be semigeodesic.

Advantages of such coordinates are known since C.F. Gauss:
Geodätische Parallelkoordinaten, [Kreyzig, Diff. Geom. p. 201].

Geodesic polar coordinates:
can be also interpreted as a "limit case"
of semigeodesic coordinates:

all geodesic coordinate lines $\varphi = x^2 = \mathrm{const}$ pass through one point called the pole, corresponding to $r = x^1 = 0$,
and lines $r = x^1 = \mathrm{const}$ are geodesic circles.
Geodätische Polarkoordinaten, [Kreyzig, Diff. Geom. pp. 197-204].

Let $A_n = (M, \nabla)$ be an n-dimensional manifold M with the affine connection ∇, dimension $n \geq 2$, and let $U \subset M$ be a coordinate neighbourhood at the point $x_0 \in U$. A couple (U, x) is a coordinate map on A_n.

It is well known that semigeodesic coordinate systems on surfaces and (pseudo-) Riemannian manifolds are generalized in the following way (Mikeš, Vanžurová, Hinterleitner, *Geodesic mappings and some generalizations*, Olomouc Univ. Press 2009, see p. 43):

Definition

Coordinates in A_n are called

pre-semigeodesic coordinates

if one system of coordinate lines is consists of geodesics and their natural parameter is just the first coordinate.

Theorem (1)

The conditions

$$(3) \qquad \Gamma^h_{11}(x) = 0, \quad h = 1, \dots, n,$$

are satisfied in (U, x) *if and only if the parametrized curves*

$\ell \colon I \to U$, $\ell(s) = (s, a_2, \dots, a_n)$, $s \in I$, $a_i \in R$, $i = 2, \dots, n$,

are canonically parametrized geodesics of $\nabla_{|U}$,

I *is some interval,* a_k *are suitable constants chosen so that* $\ell(I) \subset U$,

Γ^h_{ij} *are components of the connection* ∇.

Theorem (2)

The conditions (3) *are satisfied in* (U, x) *if and only if the coordinate system* (U, x) *is pre-semigeodesic.*

The existence of Pre-semigeodesic coordinates

We thought that the existence of this chart is trivial. This problem is obviously more difficult than we supposed.

This was observed by Z. Dušek and O. Kowalski [3] who precisely proved the existence of pre-semigeodesic charts in the case when the components of the affine connection are real analytic functions. We proved the following

Theorem (3)

For any affine connection determined by $\Gamma_{ij}^h(x) \in C^r$, $r \geq 2$, there exists a local transformation of coordinates determined by $x' = f(x) \in C^r$ such that the connection in the new coordinates satisfies ${\Gamma'}_{11}^h(x') = 0$, for $h = 1, \ldots, n$,
i.e. x' is pre-semigeodesic.

The existence of this chart is not excluded in the case when the components are only continuous.

Because the transformation of the components of affine connection has the following form

$$\Gamma^h_{ij}(x) = \left(\Gamma'^{\gamma}_{\alpha\beta}(x'(x))\partial_i x'^{\alpha}\partial_j x'^{\beta} + \partial_{ij} x'^{\gamma}\right)\frac{\partial x^h}{\partial x'^{\gamma}}.$$

If $\Gamma'^h_{ij}(x') \in C^r$ then in the pre-semigeodesic coordinates x in general $\Gamma^h_{ij}(x') \in C^{r-2}$ hold.

If $\Gamma'^h_{ij}(x') \in C^\infty$ then $\Gamma^h_{ij}(x') \in C^\infty$, too.

References

[1] J. Mikeš, A. Vanžurová, I. Hinterleitner, Geodesic mappings and some generalizations. Palacky University Press, Olomouc, 2009.
[2] J. Mikeš, et al., Differential geometry of special mappings. Palacky University Press, Olomouc, 2015.
[3] Z. Dušek, O. Kowalski, How many are affine connections with torsion. Arch. Math. 50:5, 257-264, 2014.

Thank you for your attention!