On F_2^ε-planar mappings of (pseudo-) Riemannian manifolds

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1. Introduction

- T. Levi-Civita used geodesic mappings for modeling mechanical processes, and A.Z. Petrov used quasigeodesic mappings for modeling in theoretical physics. More general mappings were studied by Hrdina, Slovák and Vašík.
- In 2003 Topalov introduced PQ^ε-projectivity of Riemannian metrics, $\varepsilon \in \mathbb{R}(\neq 1, 1+n)$. In 2013 these mappings were studied by Matveev and Rosemann. They found that for $\varepsilon = 0$ they are projective.
- We show that PQ^ε-projective equivalence corresponds to a special case of F-planar mapping studied by Mikeš and Sinyukov (1983) and F_2-planar mappings (Mikeš, 1994), with $F = Q$. Moreover, the tensor P is derived from the tensor Q and the non-zero number ε.

2. On F-planar mappings

Let $A_n = (M, \nabla, F)$ be an n-dimensional manifold M with affine connection ∇, and affinor structure F, i.e. a tensor field of type $(1,1)$.

Definition 1. [Mikeš, Sinyukov]

A curve ℓ, which is given by the equations $\ell = \ell(t)$, $\lambda(t) = d\ell(t)/dt$ $(\neq 0)$, $t \in I$, where t is a parameter, is called *F-planar*, if its tangent vector $\lambda(t_0)$, for any initial value t_0 of the parameter t, remains under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.

A curve ℓ is F-planar if and only if the following condition holds:

$$\nabla_{\lambda(t)}\lambda(t) = \varrho_1(t)\lambda(t) + \varrho_2(t)F\lambda(t),$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

We suppose two spaces $A_n = (M, \nabla, F)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla}, \bar{F})$ with torsion-free affine connections ∇ and $\bar{\nabla}$, respectively. Affine structures F and $\bar{F}$ are defined on A_n, resp. $\bar{A}_n$.

Definition 2. [Mikeš, Sinyukov]

A diffeomorphism f between manifolds with affine connection A_n and $\bar{A}_n$ is called an *F-planar mapping* if any F-planar curve in A_n is mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$.

Assume an F-planar mapping $f\colon A_n \to \bar{A}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M and $\bar{M}$, respectively, such that locally, $f\colon A_n \to \bar{A}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$. We always suppose that ∇, $\bar{\nabla}$ and the affinors F, $\bar{F}$ are defined on M $(\equiv \bar{M})$.

The following theorem holds.

Theorem 1

An F-planar mapping f from A_n onto $\bar{A}_n$ preserves F-structures (i.e. $\bar{F} = a\,F + b\,Id$, a,b are some functions on M), and is characterized by the following condition

$$P(X,Y) = \psi(X)\cdot Y + \psi(Y)\cdot X + \varphi(X)\cdot FY + \varphi(Y)\cdot FX \quad (1)$$

for any vector fields X, Y, where $P = f^*\bar{\nabla} - \nabla$ is the deformation tensor field of f, ψ and φ are some linear forms on M.

This Theorem was proved by Mikeš and Sinyukov for finite dimension $n > 3$, a more concise proof of this Theorem for $n > 3$ and also a proof for $n = 3$ was given by J. Mikeš and I. Hinterleitner.

Definition 3

1. An F-planar mapping of a manifold $A_n = (M, \nabla)$ with affine connection onto a (pseudo-) Riemannian manifold $\bar{V}_n = (M, \bar{g})$ is called an *F_1-planar mapping* if the metric tensor $\bar{g}$ satisfies the condition

$$\bar{g}(X, FX) = 0, \quad \text{for all} \quad X. \tag{2}$$

2. An F_1-planar mapping $A_n \to \bar{V}_n$ is called an *F_2-planar mapping* if the one-form ψ is gradient-like, i.e.

$$\psi(X) = \nabla_X \Psi,$$

where Ψ is a function on A_n.

If a manifold A_n admits F_2-planar mapping onto $\bar{V}_n$, then the following equations are satisfied

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \xi^i F^j_k + \xi^j F^i_k, \quad (3)$$

where

$$a^{ij} = e^{2\psi} \bar{g}^{ij}, \quad \lambda^i = -a^{i\alpha}\psi_\alpha, \quad \xi^i = -a^{i\alpha}\varphi_\alpha, \quad (4)$$

where $\psi_j,\ \varphi_i,\ F^h_i$ are components of $\psi,\ \varphi,\ F$ and $\bar{g}^{ij}$ are components of the inverse matrix to the metric $\bar{g}$.
From (2) and (4) follows that $a^{i\alpha}F^j_\alpha + a^{j\alpha}F^i_\alpha = 0$.
If A_n is a (pseudo-) Riemannian manifold $V_n = (M, g)$ with metric tensor g, after lowering indices in (3), we obtain

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \xi_i F_{jk} + \xi_j F_{ik}, \quad (5)$$

where $a_{ij} = a^{\alpha\beta} g_{i\alpha} g_{j\beta},\ \lambda_i = g_{i\alpha}\lambda^\alpha,\ \xi_i = g_{i\alpha}\xi^\alpha,\ F_{ik} = g_{i\alpha}F^\alpha_k$.
Evidently $a_{i\alpha}F^\alpha_j + a_{j\alpha}F^\alpha_i = 0$.

3. PQ^{ε}-projective Riemannian manifolds

3.1 Definition of PQ^{ε}-projective Riemannian manifolds

Let g and $\bar{g}$ be two Riemannian metrics on an n-dimensional manifold M. Consider the $(1,1)$-tensors P, Q which are satisfying the following conditions:

$$PQ = \varepsilon\, Id,\ g(X, PX) = 0,\ \bar{g}(X, PX) = 0, \quad (6)$$

$$g(X, QX) = 0,\ \bar{g}(X, QX) = 0,$$

for all X and where $\varepsilon \neq 1,\ n+1$ is a real number.

Definition 4. [Topalov]

The metrics $g, \bar{g}$ are called *PQ^{ε}- projective* ($\varepsilon \in \mathbb{R}, \varepsilon \neq 1, n+1$) if for the 1-form Φ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$(\bar{\nabla} - \nabla)_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \quad (7)$$

for all X, Y.

Two metrics g and $\bar{g}$ are denoted by the synonym PQ^{ε}-projective if they are *PQ^{ε}-projective equivalent*.
On the other hand this notation can be seen from the point of view of mappings. The study of these mappings lead us to implement F_2^{ε}-planar mapping.

Assume two Riemannian manifolds (M, g) and $(\bar{M}, \bar{g})$.

A diffeomorphism $f : M \to \bar{M}$ allows to identify the manifolds M and $\bar{M}$. For this reason we can speak about PQ^{ε}-projective mappings (or more precisely diffeomorphisms) between (M, g) and $(\bar{M}, \bar{g})$, when equations (6) and (7) hold.

In these formulas $\bar{g}$ and $\bar{\nabla}$ mean in fact the pullbacks $f^*\bar{g}$ and $f^*\bar{\nabla}$.

F-planar mapping

$$P(X,Y) = \psi(X)\cdot Y + \psi(Y)\cdot X + \varphi(X)\cdot FY + \varphi(Y)\cdot FX \quad (1)$$

PQ^{ε}-projective mappings

$$P(X,Y) = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \quad (7)$$

Comparing formulas (1) and (7) we make sure that PQ^{ε}-projective equivalence is a special case of the F-planar mapping between Riemannian manifolds (M,g) and $(M,\bar{g})$. Evidently, this is if $\psi \equiv \Phi$, $F \equiv Q$ and $\varphi(\cdot) = -\Phi(P(\cdot))$. Moreover, it follows elementary from (7) that ψ is a gradient-like form, thus a PQ^{ε}-projective equivalence is a special case of an F_2-planar mapping.

Therefore the PQ^ε-projective equivalence formula 3:

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \xi^i F^j_k + \xi^j F^i_k, \quad (3)$$

after lowering the indices i and j by the metric g, has the following form:

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} - \lambda_\alpha P^\alpha_i g_{j\beta} Q^\beta_k - \lambda_\alpha P^\alpha_j g_{i\beta} Q^\beta_k. \quad (8)$$

From conditions (4):

$$a^{ij} = e^{2\psi} \bar{g}^{ij}, \quad \lambda^i = -a^{i\alpha}\psi_\alpha, \quad \xi^i = -a^{i\alpha}\varphi_\alpha, \quad (4)$$

and (6):

$$PQ = \varepsilon\, Id,\ g(X, PX) = 0,\ \bar{g}(X, PX) = 0,$$
$$(6)$$
$$g(X, QX) = 0,\ \bar{g}(X, QX) = 0,$$

we obtain $a(X, PX) = 0$ and $a(X, QX) = 0$ for all X, and equivalently in local form

$$a_{i\alpha} P^\alpha_j + a_{j\alpha} P^\alpha_i = 0 \text{ and } a_{i\alpha} Q^\alpha_j + a_{j\alpha} Q^\alpha_i = 0. \quad (9)$$

3.2. New results about PQ^ε-projective Riemannian manifolds for $\varepsilon \neq 0$

We will study PQ^ε-projective mappings for f $\varepsilon \neq 0$. From the condition $PQ = \varepsilon\, Id$ follows

$$P = \varepsilon\, Q^{-1}. \quad (10)$$

This implies that P depends on Q and ε. Moreover two conditions in (6) depend on the other ones, i.e. in the definition of PQ^ε-projective mappings we can restrict on the conditions $PQ = \varepsilon\, Id$, $g(X, QX) = 0$, $\bar{g}(X, QX) = 0$. This fact implies the following lemma:

Lemma 1.

If Q satisfies the conditions $g(X, QX) = 0$ and $\bar{g}(X, QX) = 0$ for $\varepsilon \neq 0$, then we obtain $g(X, PX) = 0$ and $\bar{g}(X, PX) = 0$.

4. F_2^{ε}-projective mapping with $\varepsilon \neq 0$

Due to the above properties, from formula (7) and Lemma 1, we can simplify the Definition 4.

Let g and $\bar{g}$ be two (pseudo-) Riemannian metrics on an n-dimensional manifold M. Consider the regular $(1,1)$-tensors F which are satisfying the following conditions

$$g(X, FX) = 0 \quad \text{and} \quad \bar{g}(X, FX) = 0 \quad \text{for all } X. \tag{11}$$

Definition 5.

The metrics g and $\bar{g}$ are called *F_2^{ε}-projective* if for a certain gradient-like form ψ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$(f^*\bar{\nabla} - \nabla)_X Y = \psi(X)Y + \psi(Y)X - \varepsilon\,\psi(F^{-1}X)FY - \varepsilon\,\psi(F^{-1}Y)FX,$$

for all vector fields X, Y and for all $x \in M$, ε is non-zero constant.

From the discussion in section 3 we obtain the following proposition:

Proposition 1.

A PQ^{ε}-projective mapping can be understood as an F_2^{ε} with

$$P = \varepsilon F^{-1} \quad \text{and} \quad Q = F. \qquad (13)$$

We also prooved following theorem:

Theorem 2.

If a (pseudo-) Riemannian manifold (M, g, F) with regular structure F, for which $F^2 \neq \kappa Id$ and $g(X, FX) = 0$ for all X, admits an F_2^ε-projective mapping onto a (pseudo-) Riemannian manifold $(\bar{M}, \bar{g})$, then the linear system of differential equations

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} - \lambda_\alpha P_i^\alpha g_{j\beta} F_k^\beta - \lambda_\alpha P_j^\alpha g_{i\beta} F_k^\beta \qquad (14)$$

and

$$a_{i\alpha} F_j^\alpha + a_{j\alpha} F_i^\alpha = 0 \qquad (15)$$

hold, where $P = \varepsilon F^{-1}$, $\lambda_i = a_{\alpha\beta} T_i^{\alpha\beta}$ and $T_i^{\alpha\beta}$ is a certain tensor obtained from g_{ij} and F_i^h.

Proof.

We covariantly differentiate (15) and obtain

$$\nabla_k a_{i\alpha} F_j^\alpha + \nabla_k a_{j\alpha} F_i^\alpha = \overset{1}{T}_{ijk},$$

where $\overset{1}{T}_{ijk} = -a_{i\alpha} \nabla_k F_j^\alpha - a_{j\alpha} \nabla_k F_i^\alpha$.

Using formula (14), we obtain

$$\lambda_i g_{\alpha k} F_j^\alpha + \lambda_\alpha F_j^\alpha g_{ik} - \lambda_\beta P_i^\beta g_{\alpha\gamma} F_j^\alpha F_k^\gamma - \varepsilon \lambda_j g_{i\alpha} F_k^\alpha + \lambda_j g_{\alpha k} F_i^\alpha$$
$$+\lambda_\alpha F_i^\alpha g_{jk} - \lambda_\beta P_j^\beta g_{\alpha\gamma} F_i^\alpha F_k^\gamma - \varepsilon \lambda_i g_{j\alpha} F_k^\alpha = \overset{1}{T}_{ijk}.$$

After some calculation we get

$$(\varepsilon + 1)(g_{\alpha k} F_j^\alpha \lambda_i + g_{\alpha k} F_i^\alpha \lambda_j) + \lambda_\alpha F_j^\alpha g_{ik} + \lambda_\alpha F_i^\alpha g_{jk} -$$
$$-\lambda_\alpha P_i^\alpha g_{\beta\gamma} F_j^\beta F_k^\gamma - \lambda_\alpha P_j^\alpha g_{\beta\gamma} F_i^\beta F_k^\gamma = \overset{1}{T}_{ijk}. \qquad (16)$$

By cyclic permutation of the indices i, j, k we obtain

$$\begin{aligned}
&\lambda_\alpha F^\alpha_j g_{ik} + \lambda_\alpha F^\alpha_i g_{jk} + \lambda_\alpha F^\alpha_k g_{ij} - \lambda_\alpha P^\alpha_i g_{\beta\gamma} F^\beta_j F^\gamma_k - \\
&-\lambda_\alpha P^\alpha_j g_{\beta\gamma} F^\beta_i F^\gamma_k - \lambda_\alpha P^\alpha_k g_{\beta\gamma} F^\beta_i F^\gamma_j = \overset{1}{T}_{ijk} + \overset{1}{T}_{jki} + \overset{1}{T}_{kij}.
\end{aligned} \tag{17}$$

Next, we will subtract equations (16) and (17) :

$$(\varepsilon+1)(g_{\alpha k} F^\alpha_j \lambda_i + g_{\alpha k} F^\alpha_i \lambda_j) - \lambda_\alpha F^\alpha_k g_{ij} + \lambda_\alpha P^\alpha_k g_{\beta\gamma} F^\beta_i F^\gamma_j = \overset{2}{T}_{ijk}, \tag{18}$$

where $\overset{2}{T}_{ijk} = -\overset{1}{T}_{jki} - \overset{1}{T}_{kij}$.

We write the homogeneous linear equation to equation (18)

$$g_{\alpha k} F_j^\alpha A_i + g_{\alpha k} F_i^\alpha A_j - B_k g_{ij} + C_k g_{\beta\gamma} F_i^\beta F_j^\gamma = 0, \quad (19)$$

where $A_i = (\varepsilon + 1)\lambda_i,\ B_k = \lambda_\alpha F_k^\alpha,\ C_k = \lambda_\alpha P_k^\alpha.$

Now we prove that (19) has only trivial solution. From that follows that $\lambda_i = \overset{3}{T}$, i.e. is a linear combination of the tensor components a_{ij} with coefficients generated by g and F on V_n.

If $A_i \neq 0$, from (19) follows rank $\left\| g_{\alpha k} F_j^\alpha \right\| \leq 3$, in the other case $g_{\alpha k} F_j^\alpha$ we can decompose into 3 bivectors.

And because the tensors g and F are regular, it follows that rank $\left\| g_{\alpha k} F_j^\alpha \right\| = n$.

We suppose that $n \geq 4$.

$$-B_k g_{ij} + C_k g_{\beta\gamma} F_i^\beta F_j^\gamma = 0. \qquad (20)$$

If B_k or $C_k \neq 0$:

$$g_{\beta\gamma} F_i^\beta F_j^\gamma = \rho g_{ij}, \qquad (21)$$

where ρ is a function.
We multiply formula (21) by P_k^i. From that follows $F^2 = \kappa Id$, where κ is a function, which is in contradiction with our assumption. For this reason in the formula (19) we suppose that $A_i = B_i = C_i = 0$. Therefore $\lambda_\alpha F_k^\alpha = \overset{3}{T}_k$, where $\overset{3}{T}_k$ is a tensor which is a linear combination of a_{ij} with coefficients generated by g and F. Let be $G = F^{-1}$, then $\lambda_i = \overset{3}{T}_k G_i^k$. This means $\lambda_i = a_{\alpha\beta} T_i^{\alpha\beta}$. $\square$

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Thank you for your attention!