Deformation Quantization of Kähler Manifods and Their Twisted Fock Representation

30min Version

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1. Intro I

- Deformation quantization is a way to construct noncommutative geometry, which is first introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheime
- Several ways of deformation quantization were established by De Wilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, Kontsevich
- Deformation quantizations of Kähler manifolds were provided in Moreno, Cahen.
- Deformation quantization with separation of variables to construct noncommutative Kähler manifolds is introduced by Karabegov

1. Intro II

- The deformation quantization is an associative algebra on a set of formal power series of C^∞ functions with a star product between formal power series.
- One of the advantages of deformation quantization is that usual analytical techniques are available on them.
- On the other hand, when we consider physics on N.C. manifolds given by deformation quantization, physical quantities are given as formal power series, and there are difficulties to understand them from a viewpoint of physics.

A typical way to solve the difficulties is to make a representation of the noncommutative algebra.

1. Intro III

The purpose is to construct the Fock representation of noncommutative Kähler manifolds.

- A Fock space is spanned by a vacuum and states obtained by acting creation operators on this vacuum.
- The algebras on noncommutative Kähler mfs are represented as those of linear operators acting on the Fock sp.
- In my talk, creation operators and annihilation operators are not Hermitian conjugate with each other, in general.
- The bases of the Fock space are not the Hermitian conjugates of those of the dual vector space. In this case, we call the representation the twisted Fock representation.

1. Intro IV

Main result: Functions - Fock operators Dictionary

Table: $z^i, \bar{z}^i$ $(i = 1, \cdots N)$ are local complex coordinates. Φ is a Kähler potential and H is defined by $e^{\Phi/\hbar} = \sum H_{\vec{m},\vec{n}} z^{\vec{m}} \bar{z}^{\vec{n}}$, where $z^{\vec{m}} = z_1^{m_1} z_2^{m_2} \cdots z_N^{m_N}$ for $\vec{m} = (m_1, m_2, \cdots, m_N)$

Functions	Fock operators
$e^{-\Phi/\hbar}$	$\lvert\vec{0}\rangle\langle\vec{0}\rvert$
z_i	$a_i^\dagger$
$\frac{1}{\hbar}\partial_i\Phi$	$\underline{a_i}$
$\bar{z}^i$	$a_i = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k}+\vec{e}_i,\vec{n}} \lvert\vec{m}\rangle\underline{\langle\vec{n}\rvert}$
$\frac{1}{\hbar}\partial_{\bar{i}}\Phi$	$\underline{a_i^\dagger} = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_i + 1) H_{\vec{m},\vec{k}+\vec{e}_i} H^{-1}_{\vec{k},\vec{n}} \lvert\vec{m}\rangle\underline{\langle\vec{n}\rvert}$

Note that $\underline{a_i^\dagger}$ is NOT a Hermitian conjugate of $\underline{a_i}$, in general.

talk plan

Section 2: we review several facts of deformation quantization with separation of variables which are used in this talk.
Section 3: a twisted Fock representation is constructed on a chart of a general Kähler manifold.
Section 4: transition maps between the twisted Fock representations on two local coordinate charts are constructed.
Section 5: the Fock representations of $\mathbb{C}^N$, $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are given as examples.
Section 6: Summary

2-1. What's "Deform. Quantiz."? I

Definition (Deformation quantization (weak sense))

Let M be a Poisson manifold and $\mathcal{F}$ be a set of formal power series: $\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k, \ f_k \in C^\infty(M) \right\}$. Products are defined by star products;

$$f * g = \sum_k C_k(f, g)\hbar^k \tag{1}$$

such that the products satisfy the following conditions.

1. $*$ is associative. $f * 1 = 1 * f = f$, C_k is a bidifferential op.
2. C_0 and C_1 are defined as

$$C_0(f, g) = fg, \ \ C_1(f, g) - C_1(g, f) = i\{f, g\}. \tag{2}$$

2-2. Defo. Quant. with S of V I

As a special case of deformation quantizations of Kähler manifold M, Deformation Quantization with Separation of Variables is introduced by Karabegov

Definition (A star product with separation of variables)

$*$ is called a star product with separation of variables when

$$a * f = af \tag{3}$$

for a holomorphic function a and

$$f * b = fb \tag{4}$$

for an anti-holomorphic function b.

2-3. Notation of Kähler Manifolds I

Let M be an N-dimensional complex Kähler manifold,
Φ be its Kähler potential and ω be its Kähler 2-form:

$$\omega := i g_{k\bar{l}} dz^k \wedge d\bar{z}^l, \quad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \tag{5}$$

Here g is the Kähler metric and $z^i, \bar{z}^j$ $(i, j = 1, \cdots, N)$ are local coordinates on an open set $U \subset M$.
The $g^{\bar{k}l}$ is the inverse of the metric $g_{k\bar{l}}$:

$$g^{\bar{k}l} g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}. \tag{6}$$

In the following, we use the following abridged notations

$$\partial_k = \frac{\partial}{\partial z^k}, \qquad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \tag{7}$$

2-4. Karabegov's Deform. Quant. I

Karabegov constructed a $*$ product with S of V for Kähler mfds. There exists a differential operator L_f such that

$$L_f g = f * g. \tag{8}$$

L_f is given as a formal power series in $\hbar$

$$L_f = \sum_{n=0}^{\infty} \hbar^n A^{(n)}, \tag{9}$$

where $A^{(n)}$ is a differential operator which contains only partial derivatives by z^i $(i = 1, \cdots, N)$ and has the following form

$$A^{(n)} = \sum_{k \geq 0} a^{(n;k)}_{\bar{i}_1 \cdots \bar{i}_k} D^{\bar{i}_1} \cdots D^{\bar{i}_k}, \tag{10}$$

where

$$D^{\bar{i}} = g^{\bar{i}j}\partial_j, \tag{11}$$

and each $a^{(n;k)}_{\bar{i}_1\cdots\bar{i}_k}$ is a C^∞ function on M. Note that the differential operators $D^{\bar{i}}$ satisfy the following relations,

$$[D^{\bar{i}}, D^{\bar{j}}] = 0,\ [D^{\bar{i}}, \partial_{\bar{j}}\Phi] = \delta_{ij}. \tag{12}$$

Karabegov showed the following theorem.

Theorem (Karabegov)

L_f is uniquely determined by requiring the following conditions,

$$L_f 1 = f * 1 = f, \tag{13}$$
$$[L_f, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0, \tag{14}$$

2-5.Commutation Relations

The left star product by $\partial_i\Phi$ and the right star product by $\partial_{\bar{i}}\Phi$ are

$$L_{\partial_i\Phi} = \hbar\partial_i + \partial_i\Phi = \hbar e^{-\Phi/\hbar}\partial_i e^{\Phi/\hbar}, \tag{15}$$

$$R_{\partial_{\bar{i}}\Phi} = \hbar\partial_{\bar{i}} + \partial_{\bar{i}}\Phi = \hbar e^{-\Phi/\hbar}\partial_{\bar{i}} e^{\Phi/\hbar}. \tag{16}$$

From the definition of the star product, we easily find

$$[\frac{1}{\hbar}\partial_i\Phi,\ z^j]_* = \delta_{ij}, \qquad [z^i,\ z^j]_* = 0, \qquad [\partial_i\Phi,\ \partial_j\Phi]_* = 0, \tag{17}$$

$$[\bar{z}^i,\ \frac{1}{\hbar}\partial_{\bar{j}}\Phi]_* = \delta_{ij}, \qquad [\bar{z}^i,\ \bar{z}^j]_* = 0, \qquad [\partial_{\bar{i}}\Phi,\ \partial_{\bar{j}}\Phi]_* = 0, \tag{18}$$

where $[A,\ B]_* = A * B - B * A$.
Hence, $\{z^i, \partial_j\Phi\}$ and $\{\bar{z}^i, \partial_{\bar{j}}\Phi\}$ constitute $2N$ sets of the creation and annihilation operators ! But, Note that operators in $\{z^i, \partial_j\Phi\}$ does not commute with ones in $\{\bar{z}^i, \partial_{\bar{j}}\Phi\}$, e.g., $z^i * \bar{z}^j - \bar{z}^j * z^i \neq 0$.

2-6. Examples

Deformation quantization of $\mathbb{C}P^N$

inhomogeneous coordinates z^i $(i = 1, 2, \cdots, N)$
Kähler potential of $\mathbb{C}P^N$: $\Phi = \ln\left(1 + |z|^2\right)$,
metric :

$$
\begin{aligned}
ds^2 &= 2g_{i\bar{j}} dz^i d\bar{z}^j , \\
g_{i\bar{j}} &= \partial_i \partial_{\bar{j}} \Phi = \frac{(1 + |z|^2)\delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2} , \\
g^{\bar{i}j} &= (1 + |z|^2)\left(\delta_{ij} + z^j \bar{z}^i\right) .
\end{aligned}
$$

Deformation Quantization of $\mathbb{C}P^N$

$*$-products between the coordinates

$$
\begin{aligned}
z^i * z^j &= z^i z^j, \\
z^i * \bar{z}^j &= z^i \bar{z}^j, \\
\bar{z}^i * \bar{z}^j &= \bar{z}^i \bar{z}^j, \\
\bar{z}^i * z^j &= \bar{z}^i z^j + \hbar \delta_{ij} (1 + |z|^2) {}_2F_1\left(1, 1; 1 - 1/\hbar; -|z|^2\right) \\
&\quad + \frac{\hbar}{1 - \hbar} \bar{z}^i z^j (1 + |z|^2) {}_2F_1\left(1, 2; 2 - 1/\hbar; -|z|^2\right),
\end{aligned}
$$

${}_2F_1$: Gauss hypergeometric function.

Deformation Quantization of $\mathbb{C}H^N$ I

$\mathbb{C}H^N$ is similar to $\mathbb{C}P^N$
Kähler potential and the metric :

$$\Phi = -\ln\left(1 - |z|^2\right), \tag{19}$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1-|z|^2)\delta_{ij} + \bar{z}^i z^j}{(1-|z|^2)^2}, \tag{20}$$

$$g^{\bar{i}j} = (1-|z|^2)\left(\delta_{ij} - \bar{z}^i z^j\right). \tag{21}$$

Deformation Quantization of $\mathbb{C}H^N$ II

$*$-products between the coordinates

$$
\begin{aligned}
z^i * z^j =& z^i z^j, \\
z^i * \bar{z}^j =& z^i \bar{z}^j, \\
\bar{z}^i * \bar{z}^j =& \bar{z}^i \bar{z}^j, \\
\bar{z}^i * z^j =& \bar{z}^i z^j + \hbar \delta_{ij} (1 - |z|^2) {}_2F_1 \left(1, 1; 1 + 1/\hbar; |z|^2 \right) \\
& - \frac{\hbar}{1+\hbar} \bar{z}^i z^j (1 - |z|^2) {}_2F_1 \left(1, 2; 2 + 1/\hbar; |z|^2 \right).
\end{aligned}
$$

Twisted Fock rep. of N.C. Kähler mfds

In this section we introduce
the Twisted Fock Representation
on an open set $U \subset M$.

3-1. Heisenberg-like algebra

As mentioned in Section 2, $\{z^i, \partial_j\Phi \mid i,j = 1, 2, \cdots, N\}$ and $\{\bar{z}^i, \partial_{\bar{j}}\Phi \mid i,j = 1, 2, \cdots, N\}$ are candidates for the creation and annihilation operators

$$a_i^\dagger = z^i, \quad \underline{a}_i = \frac{1}{\hbar}\partial_i\Phi, \quad a_i = \bar{z}^i, \quad \underline{a}_i^\dagger = \frac{1}{\hbar}\partial_{\bar{i}}\Phi. \tag{22}$$

They satisfy the following commutation relations

$$[\underline{a}_i,\ a_j^\dagger]_* = \delta_{ij}, \qquad [a_i^\dagger,\ a_j^\dagger]_* = 0, \qquad [\underline{a}_i,\ \underline{a}_j]_* = 0, \tag{23}$$

$$[a_i,\ \underline{a}_j^\dagger]_* = \delta_{ij}, \qquad [\underline{a}_i^\dagger,\ \underline{a}_j^\dagger]_* = 0, \qquad [a_i,\ a_j]_* = 0. \tag{24}$$

but slightly different !

$$[a_i, a_i^\dagger]_* \ \text{ and } \ [\underline{a}_i,\ \underline{a}_j^\dagger]_* \tag{25}$$

do not vanish in general.

3-2. Fock Space I

The Fock space is defined by a vector space spanned by the bases which is generated by acting $a_i^\dagger$ on $|\vec{0}\rangle$,

$$|\vec{n}\rangle = |n_1, \cdots, n_N\rangle = c_1(\vec{n})(a_1^\dagger)_*^{n_1} * \cdots * (a_N^\dagger)_*^{n_N} * |\vec{0}\rangle, \qquad (26)$$

where $|\vec{0}\rangle = |0, \cdots, 0\rangle$ satisfies $\underline{a}_i * |\vec{0}\rangle = 0 \quad (i = 1, \cdots, N)$ and $(A)_*^n$ stands for $\overbrace{A * \cdots * A}^{n}$. $c_1(\vec{n})$ is a normalization coefficient. Here, we define the basis of a dual vector space by

$$\underline{\langle \vec{m}|} = \underline{\langle m_1, \cdots, m_N|} = \langle \vec{0}| * (\underline{a}_1)_*^{m_1} * \cdots * (\underline{a}_N)_*^{m_N} c_2(\vec{m}), \qquad (27)$$

and $\langle \vec{0}| * a_i^\dagger = 0 \qquad (i = 1, \cdots, N)$, where $c_2(\vec{m})$ is also a normalization constant.

3-2. Fock Space II

The underlines are attached to the bra vectors in order to emphasize that $\underline{\langle \vec{m}|}$ is not Hermitian conjugate to $|\vec{m}\rangle$. In this article, we set the normalization constants as

$$c_1(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \quad c_2(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \tag{28}$$

where $\vec{n}! = n_1!n_2!\cdots n_N!$.

3-3. Twisted Fock representation I

Definition

The local twisted Fock algebra (representation) F_U is defined as a algebra given by a set of linear operators acting on the Fock space defined on U:

$$F_U := \{\sum_{\vec{n},\vec{m}} A_{\vec{n}\vec{m}} |\vec{n}\rangle \underline{\langle\vec{m}|} \mid A_{\vec{n}\vec{m}} \in \mathbb{C}\}. \tag{29}$$

and products between its elements are given by the star product $*$.

In the remaining part of this section, we construct concrete expressions of functions which are elements of this local twisted Fock algebra.

3-3.Vacuum states I

Lemma

For arbitrary Kähler manifolds (M,ω), there exists a Kähler potential $\Phi(z^1,\ldots,z^N,\bar{z}^1,\ldots,\bar{z}^N)$ such that

$$\Phi(0,\ldots,0,\bar{z}^1,\ldots,\bar{z}^N)=0,\ \ \Phi(z^1,\ldots,z^N,0,\ldots,0)=0. \qquad (30)$$

3-3.Vacuum states II

Proposition

Let (M, ω) be a Kähler manifold, Φ be its Kähler potential with the property (30), and $*$ be a star product with separation of variables given in the previous section. Then the following function

$$|\vec{0}\rangle\langle\vec{0}| := e^{-\Phi/\hbar}, \tag{31}$$

satisfies

$$\underline{a}_i * |\vec{0}\rangle\langle\vec{0}| = 0, \qquad |\vec{0}\rangle\langle\vec{0}| * a_i^\dagger = 0, \tag{32}$$

$$\left(|\vec{0}\rangle\langle\vec{0}|\right) * \left(|\vec{0}\rangle\langle\vec{0}|\right) = e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar} = |\vec{0}\rangle\langle\vec{0}|. \tag{33}$$

3-3.Vacuum states III

Overview of Proof

We define the following normal ordered quantity,

$$: e^{-\sum_i a_i^\dagger \underline{a}_i} : \quad := \prod_{i=1_*}^{N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n. \tag{34}$$

Here $\prod_{i=1_*}^{N}$ is defined by $\prod_{i=1_*}^{N} f_i = f_1 * f_2 * \cdots * f_N$.
It is easy to show that $\underline{a}_i * \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n = 0$, in the same way as in the case of the ordinary harmonic oscillator.
Therefore, all we have to do is to show

$$: e^{-\sum_i a_i^\dagger \underline{a}_i} := e^{-\Phi/\hbar}. \tag{35}$$

3-3.Vacuum states IV

This can be done as follows:

$$
\begin{aligned}
: e^{-\sum_i a_i^\dagger \underline{a}_i} : &= \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}!} (a^\dagger)_*^{\vec{n}} * (\underline{a})_*^{\vec{n}} \\
&= \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}! \hbar^{|n|}} (z)_*^{\vec{n}} * (\partial\Phi)_*^{\vec{n}}. \qquad (36)
\end{aligned}
$$

We use the following notation: for an N-tuple A_i $(i = 1, 2, \cdots, N)$ and an N-vector $\vec{n} = (n_1, n_2, \cdots, n_N)$,

$$
(A)_*^{\vec{n}} = (A_1)_*^{n_1} * (A_2)_*^{n_2} * \cdots * (A_N)_*^{n_N}, \qquad (37)
$$

$$
\vec{n}! = n_1! n_2! \cdots n_N!, \qquad |n| = \sum_{i=1}^{N} n_i. \qquad (38)
$$

3-3.Vacuum states V

By using $(z)^{\vec{n}}_* = (z)^{\vec{n}} = (z^1)^{n_1} \cdots (z^N)^{n_N}$ and (16), (36) is recast as

$$\sum_{n_1,n_2,\ldots,n_N=0}^{\infty} \frac{1}{n_1! n_2! \cdots n_N!} (-z^1)^{n_1} \cdots (-z^N)^{n_N} e^{-\frac{\Phi(z,\bar{z})}{\hbar}} \partial_1^{n_1} \cdots \partial_N^{n_N} e^{\frac{\Phi(z,\bar{z})}{\hbar}}$$
$$= e^{-\frac{\Phi(z,\bar{z})}{\hbar}} e^{\frac{\Phi(0,\bar{z})}{\hbar}}$$
$$= e^{-\frac{\Phi(z,\bar{z})}{\hbar}}. \tag{39}$$

Here, the final equality follows from the condition (30). $\square$

3-3.Dictionary I

Lemma

$e^{-\Phi/\hbar} = |0\rangle\langle 0|$ satisfies the relation

$$|0\rangle\langle 0| * f(z,\bar{z}) = e^{-\Phi/\hbar} * f(z,\bar{z}) = e^{-\Phi/\hbar} f(0,\bar{z}) = |0\rangle\langle 0| f(0,\bar{z}), \tag{40}$$

$$f(z,\bar{z}) * |0\rangle\langle 0| = f(z,\bar{z}) * e^{-\Phi/\hbar} = f(z,0) e^{-\Phi/\hbar} = f(z,0)|0\rangle\langle 0|. \tag{41}$$

for a function $f(z,\bar{z})$ such that $f(z,\bar{w})$ can be expanded as Taylor series with respect to z^i and $\bar{w}^j$, respectively.

3-3.Dictionary II

We expand a function $\exp \Phi(z, \bar{z})/\hbar$ as a power series,

$$e^{\Phi(z,\bar{z})/\hbar} = \sum_{\vec{m},\vec{n}} H_{\vec{m},\vec{n}}(z)^{\vec{m}}(\bar{z})^{\vec{n}} \tag{42}$$

where $(z)^{\vec{n}} = (z^1)^{n_1} \cdots (z^N)^{n_N}$ and $(\bar{z})^{\vec{n}} = (\bar{z}^1)^{n_1} \cdots (\bar{z}^N)^{n_N}$. Since $\exp \Phi/\hbar$ is real and satisfies (30), the expansion coefficients $H_{\vec{m},\vec{n}}$ obey

$$\bar{H}_{\vec{m},\vec{n}} = H_{\vec{n},\vec{m}}, \tag{43}$$

$$H_{\vec{0},\vec{n}} = H_{\vec{n},\vec{0}} = \delta_{\vec{n},\vec{0}}. \tag{44}$$

Using this expansion, the following relations are obtained.

3-3.Dictionary III

Proposition

The right $*$-multiplication of $(\underline{a})^{\vec{n}}_* = (\partial\Phi/\hbar)^{\vec{n}}_*$ on $|\vec{0}\rangle\langle\vec{0}|$ is related to the right $*$-multiplication of $(a)^{\vec{n}}_* = (\bar{z})^{\vec{n}}_*$ on $|\vec{0}\rangle\langle\vec{0}|$ as follows,

$$|\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{n}}_* = \vec{n}! \sum_{\vec{m}} H_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (a)^{\vec{m}}_*. \tag{45}$$

Similarly, the following relation holds,

$$(\underline{a}^\dagger)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| = \vec{n}! \sum_{\vec{m}} H_{\vec{m},\vec{n}} (a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\langle\vec{0}|. \tag{46}$$

3-3.Dictionary IV

Corollary

$$|\vec{0}\rangle\langle\vec{0}| * (a)^{\vec{n}}_{*} = \sum_{\vec{m}} \frac{1}{\vec{m}!} H^{-1}_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{m}}_{*}, \tag{47}$$

$$(a^{\dagger})^{\vec{n}}_{*} * |\vec{0}\rangle\langle\vec{0}| = \sum_{\vec{m}} \frac{1}{\vec{m}!} H^{-1}_{\vec{m},\vec{n}} (\underline{a}^{\dagger})^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}|, \tag{48}$$

where $H^{-1}_{\vec{n},\vec{m}}$ *is the inverse matrix of the matrix* $H_{\vec{n},\vec{m}}$,
$\sum_{\vec{k}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k},\vec{n}} = \delta_{\vec{m},\vec{n}}$.

3-3.Dictionary V

We introduce bases of the Fock representation as follows,

$$|\vec{m}\rangle\underline{\langle\vec{n}|} := \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(a^\dagger)_*^{\vec{m}}*|\vec{0}\rangle\langle\vec{0}|*(\underline{a})_*^{\vec{n}} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(z)_*^{\vec{m}}*e^{-\Phi/\hbar}*\left(\frac{1}{\hbar}\partial\Phi\right)_*^{\vec{n}}. \tag{49}$$

By using (45), the bases are also written as

$$\begin{aligned}|\vec{m}\rangle\underline{\langle\vec{n}|} &= \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H_{\vec{n},\vec{k}}(z)_*^{\vec{m}} * e^{-\Phi/\hbar} * (\bar{z})_*^{\vec{k}} \\ &= \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H_{\vec{n},\vec{k}}(z)^{\vec{m}}(\bar{z})^{\vec{k}} e^{-\Phi/\hbar}.\end{aligned} \tag{50}$$

3-3.Dictionary VI

The completeness of the bases are formally shown as

$$\begin{aligned}\sum_{\vec{n}} |\vec{n}\rangle\underline{\langle\vec{n}|} &= \sum_{\vec{m},\vec{n}} H_{\vec{n},\vec{m}}(z)^{\vec{n}}(\bar{z})^{\vec{m}} e^{-\Phi/\hbar} \\ &= e^{\Phi/\hbar}e^{-\Phi/\hbar} \\ &= 1. \end{aligned} \tag{51}$$

The $*$-products between the bases are calculated as

$$\begin{aligned}|\vec{m}\rangle\underline{\langle\vec{n}|} * |\vec{k}\rangle\underline{\langle\vec{l}|} &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\vec{k}!\vec{l}!}}(a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{n}} * (a^\dagger)_*^{\vec{k}} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})_*^{\vec{l}} \\ &= \delta_{\vec{n},\vec{k}}|\vec{m}\rangle\underline{\langle\vec{l}|}. \end{aligned} \tag{52}$$

3-3.Dictionary VII

The creation and annihilation operators $a_i^\dagger, \underline{a_i}$ act on the bases as follows,

$$a_i^\dagger * |\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{m_i+1}|\vec{m}+\vec{e}_i\rangle\underline{\langle\vec{n}|}, \tag{53}$$

$$\underline{a_i} * |\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{m_i}|\vec{m}-\vec{e}_i\rangle\underline{\langle\vec{n}|}, \tag{54}$$

$$|\vec{m}\rangle\underline{\langle\vec{n}|} * a_i^\dagger = \sqrt{n_i}|\vec{m}\rangle\underline{\langle\vec{n}-\vec{e}_i|}, \tag{55}$$

$$|\vec{m}\rangle\underline{\langle\vec{n}|} * \underline{a_i} = \sqrt{n_i+1}|\vec{m}\rangle\underline{\langle\vec{n}+\vec{e}_i|}, \tag{56}$$

where $\vec{e}_i$ is a unit vector, $(\vec{e}_i)_j = \delta_{ij}$. The action of a_i and $\underline{a_i^\dagger}$ is derived by the Hermitian conjugation of the above equations.

3-3.Dictionary VIII

The creation and annihilation operators can be expanded with respect to the bases,

$$a_i^{\dagger} = \sum_{\vec{n}} \sqrt{n_i + 1} |\vec{n} + \vec{e}_i\rangle \underline{\langle \vec{n}|}, \tag{57}$$

$$\underline{a}_i = \sum_{\vec{n}} \sqrt{n_i + 1} |\vec{n}\rangle \underline{\langle \vec{n} + \vec{e}_i|}, \tag{58}$$

$$a_i = \sum_{\vec{m},\vec{n},\vec{k}} \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k}+\vec{e}_i,\vec{n}} |\vec{m}\rangle \underline{\langle \vec{n}|}, \tag{59}$$

$$\underline{a}_i^{\dagger} = \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_i + 1) H_{\vec{m},\vec{k}+\vec{e}_i} H^{-1}_{\vec{k},\vec{n}} |\vec{m}\rangle \underline{\langle \vec{n}|}. \tag{60}$$

4. Transition maps I

- Let $M = \cup_a U_a$ be a locally finite open covering and $\phi_a : U_a \to \mathbb{C}^N$. Consider the case $U_a \cap U_b \neq \emptyset$.
- Denote by $\phi_{a,b}$ the transition map from $\phi_a(U_a)$ to $\phi_b(U_b)$. The local coordinates $(z, \bar{z}) = (z^1, \cdots, z^N, \bar{z}^1, \cdots, \bar{z}^N)$ on U_a are transformed into the coordinates $(w, \bar{w}) = (w^1, \cdots, w^N, \bar{w}^1, \cdots, \bar{w}^N)$ on U_b by $(w, \bar{w}) = (w(z), \bar{w}(\bar{z}))$, where $w(z) = (w^1(z), \cdots, w^N(z))$ is a holomorphic function and $\bar{w}(\bar{z}) = (\bar{w}^1(\bar{z}), \cdots, \bar{w}^N(\bar{z}))$ is an anti-holomorphic function.
- $f *_a g$ and $f *_b g$: the star products defined on U_a and U_b, respectively.

In general, there is a nontrivial transition maps T between two star products i.e. $f *_b g = T(f) *_a T(g)$. But the transition maps are trivial in our case.

4. Transition maps II

Proposition

For an overlap $U_a \cap U_b \neq \emptyset$,

$$f *_b g(w, \bar{w}) = \phi^*_{a,b}\, f *_a g(w, \bar{w}) \tag{61}$$

Here $\phi^*_{a,b}$ is the pull back of $\phi_{a,b}$.

Proof

The Kähler potentials $\Phi_a(z, \bar{z})$ on U_a and $\Phi_b(w, \bar{w})$ on U_b satisfy, in general,

$$\Phi_b(w, \bar{w}) = \Phi_a(z, \bar{z}) + \phi(z) + \bar{\phi}(\bar{z}),$$

where ϕ is a holomorphic function and $\bar{\phi}$ is an anti-holomorphic function. We define a differential operator $L_{b,f}$ by $L_{b,f}\, g := f *_b g$

4. Transition maps III

on U_b. Similarly, we use $g_b^{\bar{i}j}$, $D_b^{\bar{i}}$ etc. as the metric on U_b, differential operator $D^{\bar{i}}$ on U_b, etc. As mentioned in Section 2,

$$L_{b,f} = \sum_{n=0}^{\infty} \hbar^n a^b_{n,\vec{\bar{i}}}(f) D_b^{\vec{\bar{i}}} = \sum_{n=0}^{\infty} \hbar^n \sum_{k \geq 0} a^{b(n;k)}_{\bar{i}_1 \cdots \bar{i}_k} D_b^{\bar{i}_1} \cdots D_b^{\bar{i}_k}, \tag{62}$$

is determined by

$$[L_{b,f} \ , \ R_{b, \partial_l \Phi_b}] = [L_{b,f} \ , \ \frac{\partial \Phi_b}{\partial \bar{w}^l} + \hbar \frac{\partial}{\partial \bar{w}^l}] = 0 \tag{63}$$

On the overlap $U_a \cap U_b$,

$$D_b^{\bar{i}} = \frac{\partial \bar{w}^i}{\partial \bar{z}^l} D_a^{\bar{l}}, \tag{64}$$

4. Transition maps IV

because $g_b^{\bar{i}j} = \frac{\partial \bar{w}^i}{\partial \bar{z}^k} \frac{\partial w^j}{\partial z^l} g_a^{\bar{k}l}$. From the fact that differential operators $D_b^{\vec{\bar{i}}}$ contain only differentiation with respect to holomorphic coordinates w^i, $D_b^{\vec{\bar{i}}}$ commutes with anti-holomorphic functions, then we obtain

$$L_{b,f} = \sum_{n=0}^{\infty} \hbar^n a^b_{n,\vec{\bar{i}}}(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{i}}}_{\vec{\bar{j}}} D_b^{\vec{\bar{j}}}, \tag{65}$$

where $\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{i}}}_{\vec{\bar{j}}}$ is an anti-holomorphic function

$$\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{i}}}_{\vec{\bar{j}}} = \frac{\partial \bar{w}^{i_1}}{\partial \bar{z}^{j_1}} \cdots \frac{\partial \bar{w}^{i_k}}{\partial \bar{z}^{j_k}}. \tag{66}$$

$$\left[L_{b,f}\ ,\ \frac{\partial \Phi_b}{\partial \bar{w}^l} + \hbar \frac{\partial}{\partial \bar{w}^l}\right]$$

$$= \left[\sum_{n=0}^{\infty} \hbar^n a^b_{n,\vec{\bar{i}}}(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{i}}}_{\vec{\bar{j}}} D^{\vec{\bar{j}}}_a\ ,\ \frac{\partial \bar{z}^k}{\partial \bar{w}^l}\left(\frac{\partial \Phi_a}{\partial \bar{z}^k} + \frac{\partial \bar{\phi}}{\partial \bar{z}^k} + \hbar \frac{\partial}{\partial \bar{z}^k}\right)\right]$$

$$= \frac{\partial \bar{z}^k}{\partial \bar{w}^l}\left[\sum_{n=0}^{\infty} \hbar^n a^b_{n,\vec{\bar{i}}}(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{i}}}_{\vec{\bar{j}}} D^{\vec{\bar{j}}}_a\ ,\ \frac{\partial \Phi_a}{\partial \bar{z}^k} + \hbar \frac{\partial}{\partial \bar{z}^k}\right] = 0, \tag{67}$$

and thus we obtain

$$L_{a,f} = \sum_{n=0}^{\infty} \hbar^n a^b_{n,\vec{\bar{i}}}(f) \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\vec{\bar{i}}}_{\vec{\bar{j}}} D^{\vec{\bar{j}}}_a = L_{b,f} \tag{68}$$

which satisfies the condition $[L_{a,f}\,, R_{a,\partial_{\bar{i}}\Phi_a}] = 0$. $\square$

4-2. Trans. maps for Twisted Fock rep. I

As a next step, we consider the transition function between twisted Fock reps. we can choose $\Phi_a(z,\bar{z})$ and $\Phi_b(w,\bar{w})$ s. t.

$$\Phi_a(0,\bar{z}) = \Phi_a(z,0) = 0, \quad \Phi_b(0,\bar{w}) = \Phi_b(w,0) = 0. \tag{69}$$

Using these Kähler potentials, $|\vec{0}\rangle_{pp}\langle\vec{0}|$ is defined as

$$|\vec{0}\rangle_{pp}\langle\vec{0}| = e^{-\Phi_p/\hbar}, \quad (p = a, b),$$

and $|\vec{m}\rangle_{p}\underline{{}_{p}\langle\vec{n}|}$ are defined by

$$|\vec{m}\rangle_{a}\underline{{}_{a}\langle\vec{n}|} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(z)_*^{\vec{m}} * e^{-\Phi_a/\hbar} * \left(\frac{1}{\hbar}\partial\Phi_a\right)_*^{\vec{n}},$$

$$|\vec{m}\rangle_{b}\underline{{}_{b}\langle\vec{n}|} = \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(w)_*^{\vec{m}} * e^{-\Phi_b/\hbar} * \left(\frac{1}{\hbar}\partial\Phi_b\right)_*^{\vec{n}}.$$

4-2. Trans. maps for Twisted Fock rep. II

Let us consider the case that on the overlap $U_a \cap U_b$ the coordinate transition function $w(z)$, $\bar{w}(\bar{z})$, and the functions $\exp(\phi(w)/\hbar)$ and $\exp(\bar{\phi}(\bar{w})/\hbar)$ are given by analytic functions. Then the products $(w(z))^{\vec{\alpha}} \exp -(\phi(w)/\hbar)$ and $(\bar{w}(\bar{z}))^{\vec{\alpha}} \exp -(\bar{\phi}(\bar{w})/\hbar)$ are also analytic functions;

$$
\begin{aligned}
(w(z))^{\vec{\alpha}} e^{-\phi(w)/\hbar} &= \sum_{\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} z^{\vec{\beta}}, \\
(\bar{w}(\bar{z}))^{\vec{\alpha}} e^{-\bar{\phi}(\bar{w})/\hbar} &= \sum_{\vec{\beta}} \bar{C}^{\vec{\alpha}}_{\vec{\beta}} \bar{z}^{\vec{\beta}}.
\end{aligned} \tag{70}
$$

4-2. Trans. maps for Twisted Fock rep. III

After some calculations, we obtain trans. between the bases,

$$T^{ab} : F_{U_a} \to F_{U_b}, \tag{71}$$

as

$$|\vec{m}\rangle_{b\,\underline{b}}\underline{\langle\vec{n}|} = \sum_{\vec{i},\vec{j}} T^{ba,\vec{i}\vec{j}}_{\vec{m}\vec{n}} |\vec{i}\rangle_{a\,\underline{a}}\underline{\langle\vec{j}|}, \tag{72}$$

where

$$T^{ba,\vec{i}\vec{j}}_{\vec{m}\vec{n}} = \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sqrt{\frac{\vec{i}!}{\vec{j}!}} \sum_{\vec{k}} H^{b}_{\vec{n},\vec{k}}(C^{\vec{m}}_{\vec{i}})(\sum_{\vec{\beta}} \bar{C}^{\vec{k}}_{\vec{\beta}} H^{a-1}_{\vec{\beta}\vec{j}}). \tag{73}$$

Using this transformation, the twisted Fock representation is extended to M. We call it the twisted Fock representation of M.

5. Example-1 I

Some examples of the Twisted Fock reps. are given.
Example 1: Fock representation of $\mathbb{C}^N$
The first example is $\mathbb{C}^N$. The Kähler potential is given by

$$\Phi_{\mathbb{C}^N} = \sum_{i=1}^{N} |z^i|^2. \tag{74}$$

The star product is easily obtained as

$$f * g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \delta^{k_1 l_1} \cdots \delta^{k_n l_n} (\partial_{\bar{k}_1} \cdots \partial_{\bar{k}_n} f)(\partial_{l_1} \cdots \partial_{l_n} g).$$

We put

$$a_i^\dagger = z^i, \quad \underline{a}_i = \frac{1}{\hbar}\bar{z}^i, \quad a_i = \bar{z}^i, \quad \underline{a}_i^\dagger = \frac{1}{\hbar} z^i. \tag{75}$$

5. Example-1 II

Then they satisfy the commutation relations:

$$[\underline{a}_i,\ a_j^\dagger]_* = \delta_{ij}, \quad [a_i,\ \underline{a}_j^\dagger]_* = \hbar\delta_{ij} \tag{76}$$

and the others are zero. The basis of the twisted Fock algebra is given by

$$|\vec{m}\rangle\langle\vec{n}| = \frac{1}{\hbar^{|\vec{n}|}\sqrt{\vec{m}!\vec{n}!}}(z)^{\vec{m}}e^{-\Phi/\hbar}(\bar{z})^{\vec{n}}. \tag{77}$$

These are defined globally, so the trace operations for the twisted Fock algebras given by

$$\mathrm{Tr}_{\mathbb{C}^N}|\vec{m}\rangle\langle\vec{n}| := \frac{\hbar^N}{\pi^N}\int_{\mathbb{C}^N} dx^{2D}|\vec{m}\rangle\langle\vec{n}| = \delta_{\vec{m}\vec{n}}. \tag{78}$$

These results coincide with well known facts for N.C. $\mathbb{R}^{2N}$.

Example 2 : Fock representation of N.C. $\mathbb{C}P^N$
Let denote ζ^a $(a = 0, 1, \dots, N)$ homogeneous coordinates and $\bigcup U_a$ $(U_a = \{[\zeta^0 : \zeta^1 : \cdots : \zeta^N]\} | \zeta^a \neq 0)$ an open covering of $\mathbb{C}P^N$. We define inhomogeneous coordinates on U_a as

$$z_a^0 = \frac{\zeta^0}{\zeta^a}, \ \cdots, \ z_a^{a-1} = \frac{\zeta^{a-1}}{\zeta^a}, \ z_a^{a+1} = \frac{\zeta^{a+1}}{\zeta^a}, \ \cdots, \ z_a^N = \frac{\zeta^N}{\zeta^a}. \quad (79)$$

We choose a Kähler potential on U_a

$$\Phi_a = \ln(1 + |z_a|^2), \quad (80)$$

where $|z_a|^2 = \sum_i |z_a^i|^2$. A star product on U_a is given as

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f \right) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g, \quad (81)$$

5. Example-2 II

where

$$c_n(\hbar) = \frac{\Gamma(1-n+1/\hbar)}{n!\Gamma(1+1/\hbar)}, \qquad D^{\bar{i}} = g^{\bar{i}j}\partial_j, \qquad D^i = g^{i\bar{j}}\partial_{\bar{j}}. \tag{82}$$

On U_a, creation and annihilation operators are given as

$$a^\dagger_{a,i} = z^i_a, \quad \underline{a_{a,i}} = \frac{\bar{z}^i_a}{\hbar(1+|z_a|^2)}, \quad a_{a,i} = \bar{z}^i_a, \quad \underline{a_{a,i}}^\dagger = \frac{z^i_a}{\hbar(1+|z_a|^2)}.$$

and a vacuum is

$$|\vec{0}\rangle_{a\,a}\underline{\langle\vec{0}|} = e^{-\Phi_a/\hbar} = (1+|z_a|^2)^{-1/\hbar}. \tag{83}$$

The bases can be explicitly written as

$$|\vec{m}\rangle_{a\,a}\underline{\langle\vec{n}|} = \frac{\Gamma(1/\hbar+1)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar-|n|+1)}(z_a)^{\vec{m}}(\bar{z}_a)^{\vec{n}}e^{-\Phi/\hbar}. \tag{84}$$

5. Example-2 III

$H_{\vec{m},\vec{n}}$ s.t. $e^{\Phi_a/\hbar} = \sum H_{\vec{m},\vec{n}}(z_a)^{\vec{m}}(\bar{z})^{\vec{n}}$ is obtained as

$$H_{\vec{m},\vec{n}} = \delta_{\vec{m},\vec{n}} \frac{\Gamma(1/\hbar + 1)}{\vec{m}!\Gamma(1/\hbar - |m| + 1)}, \tag{85}$$

The transformations for the coordinates and the Kähler potential on $U_a \bigcap U_b$ are

$$z_a^i = \frac{z_b^i}{z_b^a}, \quad (i \neq a), \qquad z_a^b = \frac{1}{z_b^a}, \tag{86}$$

$$\Phi_a = \Phi_b - \ln z_b^a - \ln \bar{z}_b^a. \tag{87}$$

5. Example-2 IV

Thus, $|\vec{m}\rangle_{a\,a}\langle\vec{n}|$ is written on $U_a \bigcap U_b$ as

$$\frac{\Gamma(1/\hbar+1)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar-|n|+1)}e^{-\Phi_b/\hbar}$$
$$\times (z_b^0)^{m_0}\cdots(z_b^{a-1})^{m_{a-1}}(z_b^a)^{1/\hbar-|m|}(z_b^{a+1})^{m_{a+1}}\cdots(z_b^{b-1})^{m_{b-1}}(z_b^{b+1})^{m_{b+1}}\cdots(z_b^N)^{m_N}$$
$$\times (\bar{z}_b^0)^{n_0}\cdots(\bar{z}_b^{a-1})^{n_{a-1}}(\bar{z}_b^a)^{1/\hbar-|n|}(\bar{z}_b^{a+1})^{n_{a+1}}\cdots(\bar{z}_b^{b-1})^{n_{b-1}}(\bar{z}_b^{b+1})^{n_{b+1}}\cdots(\bar{z}_b^N)^{n_N},$$

where

$$\vec{m} = (m_0, \ldots, m_{a-1}, m_{a+1}, \ldots, m_N), \tag{88}$$
$$\vec{n} = (n_0, \ldots, n_{a-1}, n_{a+1}, \ldots, n_N). \tag{89}$$

We should treat $(z_b^a)^{1/\hbar-|m|}$ and $(\bar{z}_b^a)^{1/\hbar-|n|}$ carefully, because if they are not monomials some trick is needed to express them as the twisted Fock representation.

5. Example-2 V

A function $f(z,\bar{z})e^{-\Phi/\hbar}$ is expressed as the Twisted Fock algebra when $f(z,\bar{z})$ is given as a Taylor expansion in z and $\bar{z}$. For simplicity, we consider the one dimensional case. When a non-monomial function z^q of some complex coordinate z with a nonpositive integer q is given, z^q should be Taylor expanded around the some non-zero point $p \in \mathbb{C}$ to express it as a twisted Fock algebra:

$$z^q = p^q + qp^{q-1}(z-p) + \frac{q(q-1)}{2}p^{q-2}(z-p)^2 + \cdots \ . \quad (90)$$

Then they are written by using the Twisted Fock rep.

5. Example-2 VI

To avoid the complications concerning $(z^a_b)^{1/\hbar-|m|}$ and $(\bar{z}^a_b)^{1/\hbar-|n|}$, we can introduce a slightly different representation.
Let us consider the case

$$1/\hbar = L \in \mathbb{Z}, \quad L \geq 0, \tag{91}$$

Then, we define F^L_a on U_a as a subspace of F_{U_a},

$$F^L_a = \{\sum_{\vec{m},\vec{n}} A_{\vec{m}\vec{n}} |\vec{m}\rangle_{a} \underline{{}_a\langle\vec{n}|} \mid A_{\vec{m}\vec{n}} \in \mathbb{C},\ |m| \leq L,\ |n| \leq L\}. \tag{92}$$

The bases on U_a are related to those on U_b as,

$$\sqrt{\frac{(L-|n|)!}{(L-|m|)!}} |\vec{m}\rangle_{a} \underline{{}_a\langle\vec{n}|} = \sqrt{\frac{(L-|n'|)!}{(L-|m'|)!}} |\vec{m'}\rangle_{b} \underline{{}_b\langle\vec{n'}|}, \tag{93}$$

where

$$\vec{m}' = (m_0, \cdots, m_{a-1}, L - |m|, m_{a+1}, \cdots, m_{b-1}, m_{b+1}, \cdots, m_N), \tag{94}$$

$$\vec{n}' = (n_0, \cdots, n_{a-1}, L - |n|, n_{a+1}, \cdots, n_{b-1}, n_{b+1}, \cdots, n_N). \tag{95}$$

Using the expression of (93), we can define $|\vec{m}\rangle_{a\,a}\underline{\langle\vec{n}|}$ on the whole of U_b. Therefore, the operators in F_a^L can be extended to the whole of $\mathbb{C}P^N$.

Example 3 : Fock representation of noncommutative $\mathbb{C}H^N$
We choose a Kähler potential

$$\Phi = -\ln(1 - |z|^2), \tag{96}$$

where $|z|^2 = \sum_i^N |z^i|^2$. A star product is given as

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f \right) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g, \tag{97}$$

where

$$c_n(\hbar) = \frac{\Gamma(1/\hbar)}{n!\Gamma(n + 1/\hbar)}, \qquad D^{\bar{i}} = g^{\bar{i}j} \partial_j, \qquad D^{i} = g^{i\bar{j}} \partial_{\bar{j}}. \tag{98}$$

6. Example-3 II

The creation and annihilation operators are given as

$$a_i^\dagger = z^i, \quad \underline{a_i} = \frac{\bar{z}^i}{\hbar(1-|z|^2)}, \quad a_i = \bar{z}^i, \quad \underline{a_i}^\dagger = \frac{z^i}{\hbar(1-|z|^2)}. \tag{99}$$

and a vacuum is

$$|\vec{0}\rangle\langle\vec{0}| = e^{-\Phi/\hbar} = (1-|z|^2)^{1/\hbar}. \tag{100}$$

Bases of the Fock representation on $\mathbb{C}H^N$ are constructed as

$$|\vec{m}\rangle\underline{\langle\vec{n}|} = \frac{(-1)^{|n|}\Gamma(1/\hbar+|n|)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar)}(z)^{\vec{m}}(\bar{z})^{\vec{n}}(1-|z|^2)^{1/\hbar}. \tag{101}$$

These are defined globally.

6. Example-3 III

For $\mathbb{C}H^N$, trace density is given by the usual Riemannian vol.

$$\mu_g = \frac{1}{(1-|z|^2)^{N+1}}. \tag{102}$$

The trace is given by the integration

$$\mathrm{Tr}_{\mathbb{C}H^N} |\vec{m}\rangle \underline{\langle \vec{n}|} = \frac{\Gamma(1/\hbar)}{\pi^N \Gamma(1/\hbar - N)} \int_{\mathbb{C}H^N} dz^{2D} \mu_g |\vec{m}\rangle \underline{\langle \vec{n}|} = \delta_{\vec{m}\vec{n}}. \tag{103}$$

6. Summary I

- Twisted Fock reps. of N.C. Kähler manifolds are constructed. The N.C. Kähler manifolds are given by deformation quantization with separation of variables. Using this, the twisted Fock representation which constructed based on two sets of creation and annihilation operators was introduced with the concrete expressions of them on a local coordinate chart.
- The dictionary to translate bases of the twisted Fock reps into functions is given.
- They are defined on a local coordinate chart, and they are extended by the transition functions. This extension is achieved by essentially the result that the star products with separation of variables have a trivial transition function.
- Examples of the twisted Fock representation of Kähler manifolds: $\mathbb{C}^N$, $\mathbb{C}P^N$ and $\mathbb{C}H^N$.

6. Summary II

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