Deformation Quantization of Kähler Manifolds and Their Twisted Fock Representation

30min Version

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Deformation quantization is a way to construct noncommutative geometry, which is first introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheime.

Several ways of deformation quantization were established by De Wilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, Kontsevich.

Deformation quantizations of Kähler manifolds were provided in Moreno, Cahen.

Deformation quantization with separation of variables to construct noncommutative Kähler manifolds is introduced by Karabegov.
The deformation quantization is an associative algebra on a set of formal power series of $C^\infty$ functions with a star product between formal power series.

One of the advantages of deformation quantization is that usual analytical techniques are available on them.

On the other hand, when we consider physics on N.C. manifolds given by deformation quantization, physical quantities are given as formal power series, and there are difficulties to understand them from a viewpoint of physics.

A typical way to solve the difficulties is to make a representation of the noncommutative algebra.
The purpose is to construct the Fock representation of noncommutative Kähler manifolds.

- A Fock space is spanned by a vacuum and states obtained by acting creation operators on this vacuum.
- The algebras on noncommutative Kähler manifolds are represented as those of linear operators acting on the Fock space.
- In my talk, creation operators and annihilation operators are not Hermitian conjugates of each other, in general.
- The bases of the Fock space are not the Hermitian conjugates of those of the dual vector space. In this case, we call the representation the twisted Fock representation.
Main result: **Functions - Fock operators Dictionary**

Table: $z^i, \bar{z}^i \ (i = 1, \cdots N)$ are local complex coordinates. $\Phi$ is a Kähler potential and $H$ is defined by $e^{\Phi/\hbar} = \sum H_{\vec{m}, \vec{n}} z^{\vec{m}} \bar{z}^\vec{n}$, where $z^{\vec{m}} = z_1^{m_1} z_2^{m_2} \cdots z_N^{m_N}$ for $\vec{m} = (m_1, m_2, \cdots, m_N)$

<table>
<thead>
<tr>
<th>Functions</th>
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<tr>
<td>$e^{-\Phi/\hbar}$</td>
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<tr>
<td>$z_i$</td>
<td>$a_i^\dagger$</td>
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<tr>
<td>$\frac{1}{\hbar} \partial_i \Phi$</td>
<td>$a_i$</td>
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<tr>
<td>$\bar{z}^i$</td>
<td>$a_i = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m}, \vec{k}} H^{-1}_{\vec{k} + \vec{e}_i, \vec{n}}</td>
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<tr>
<td>$\frac{1}{\hbar} \partial_i \Phi$</td>
<td>$a_i^\dagger = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_i + 1) H_{\vec{m}, \vec{k} + \vec{e}<em>i} H^{-1}</em>{\vec{k}, \vec{n}}</td>
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Note that $a_i^\dagger$ is **NOT** a Hermitian conjugate of $a_i$, in general.
Section 2: we review several facts of deformation quantization with separation of variables which are used in this talk.
Section 3: a twisted Fock representation is constructed on a chart of a general Kähler manifold.
Section 4: transition maps between the twisted Fock representations on two local coordinate charts are constructed.
Section 5: the Fock representations of $\mathbb{C}^N$, $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are given as examples.
Section 6: Summary
2-1. What’s “Deform. Quantiz.”? I

Definition (Deformation quantization (weak sense))

Let $M$ be a Poisson manifold and $\mathcal{F}$ be a set of formal power series: $\mathcal{F} := \left\{ f \mid f = \sum_k f_k \hbar^k, \; f_k \in C^\infty(M) \right\}$. Products are defined by star products;

$$f \ast g = \sum_k C_k(f, g) \hbar^k \quad (1)$$

such that the products satisfy the following conditions.

1. * is associative. $f \ast 1 = 1 \ast f = f$, $C_k$ is a bidifferential op.
2. $C_0$ and $C_1$ are defined as

$$C_0(f, g) = fg, \quad C_1(f, g) - C_1(g, f) = i\{f, g\}. \quad (2)$$
As a special case of deformation quantizations of Kähler manifold $M$, *Deformation Quantization with Separation of Variables* is introduced by Karabegov.

**Definition (A star product with separation of variables)**

* is called a star product with separation of variables when

$$a * f = af$$

(3)

for a holomorphic function $a$ and

$$f * b = fb$$

(4)

for an anti-holomorphic function $b$. 
Let $M$ be an $N$-dimensional complex Kähler manifold, $\Phi$ be its Kähler potential and $\omega$ be its Kähler 2-form:

$$
\omega := ig_{k\bar{l}}dz^k \wedge d\bar{z}^l, \quad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}.
$$

Here $g$ is the Kähler metric and $z^i, \bar{z}^j \ (i, j = 1, \cdots, N)$ are local coordinates on an open set $U \subset M$. The $g^{\bar{k}l}$ is the inverse of the metric $g_{k\bar{l}}$:

$$
g^{\bar{k}l}g_{l\bar{m}} = \delta_{k\bar{m}}.
$$

In the following, we use the following abridged notations

$$
\partial_k = \frac{\partial}{\partial z^k}, \quad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}.
$$
Karabegov constructed a $\ast$ product with $S$ of $V$ for Kähler mfds. There exists a differential operator $L_f$ such that

$$L_f g = f \ast g.$$  \hspace{1cm} (8)

$L_f$ is given as a formal power series in $\hbar$

$$L_f = \sum_{n=0}^{\infty} \hbar^n A^{(n)},$$  \hspace{1cm} (9)

where $A^{(n)}$ is a differential operator which contains only partial derivatives by $z^i$ ($i = 1, \cdots, N$) and has the following form

$$A^{(n)} = \sum_{k \geq 0} \sum_a a^{(n;k)}_{i_1 \cdots i_k} D^{\tilde{i}_1} \cdots D^{\tilde{i}_k},$$  \hspace{1cm} (10)
where

\[ D^i = g^i{}_j \partial_j, \]  \hspace{1cm} (11)\]

and each \( a^{(n;k)}_{i_1 \cdots i_k} \) is a \( C^\infty \) function on \( M \). Note that the differential operators \( D^i \) satisfy the following relations,

\[ [D^i, D^j] = 0, \quad [D^i, \partial_j \Phi] = \delta_{ij}. \]  \hspace{1cm} (12)\]

Karabegov showed the following theorem.

**Theorem (Karabegov)**

\( L_f \) is uniquely determined by requiring the following conditions,

\[ L_f 1 = f \ast 1 = f, \]  \hspace{1cm} (13)\]

\[ [L_f, \partial_i \Phi + \hbar \partial_i] = 0, \]  \hspace{1cm} (14)\]
2-5. Commutation Relations

The left star product by $\partial_i \Phi$ and the right star product by $\partial^*_j \Phi$ are

$$L_{\partial_i \Phi} = \hbar \partial_i + \partial_i \Phi = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar},$$

$$R_{\partial_j \Phi} = \hbar \partial_j + \partial_j \Phi = \hbar e^{-\Phi/\hbar} \partial_j e^{\Phi/\hbar}. \tag{16}$$

From the definition of the star product, we easily find

$$\left[ \frac{1}{\hbar} \partial_i \Phi, \ z^j \right]_* = \delta_{ij}, \quad \left[ z^i, \ z^j \right]_* = 0, \quad \left[ \partial_i \Phi, \ \partial_j \Phi \right]_* = 0, \tag{17}$$

$$\left[ \bar{z}^i, \ \frac{1}{\hbar} \partial_j \Phi \right]_* = \delta_{ij}, \quad \left[ \bar{z}^i, \ \bar{z}^j \right]_* = 0, \quad \left[ \partial^*_i \Phi, \ \partial^*_j \Phi \right]_* = 0, \tag{18}$$

where $[A, B]_* = A \ast B - B \ast A$.

Hence, $\{z^i, \partial_j \Phi\}$ and $\{\bar{z}^i, \partial_j \Phi\}$ constitute $2N$ sets of the creation and annihilation operators! But, Note that operators in $\{z^i, \partial_j \Phi\}$ does not commute with ones in $\{\bar{z}^i, \partial_j \Phi\}$, e.g., $z^i \ast \bar{z}^j - \bar{z}^j \ast z^i \neq 0$. 
2-6. Examples

Deformation quantization of $\mathbb{C}P^N$

inhomogeneous coordinates $z^i$ ($i = 1, 2, \cdots, N$)

Kähler potential of $\mathbb{C}P^N$ : $\Phi = \ln (1 + |z|^2) ,$

metric :

$$ds^2 = 2g_{ij} dz^i d\bar{z}^j ,$$

$$g_{ij} = \partial_i \partial_j \Phi = \frac{(1 + |z|^2)\delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2} ,$$

$$g^{ij} = (1 + |z|^2) (\delta^{ij} + z^j \bar{z}^i) .$$
Deformation Quantization of $\mathbb{C}P^N$

$\ast$-products between the coordinates

\[
\begin{align*}
    z^i \ast z^j &= z^i z^j, \\
    z^i \ast \bar{z}^j &= z^i \bar{z}^j, \\
    \bar{z}^i \ast \bar{z}^j &= \bar{z}^i \bar{z}^j, \\
    \bar{z}^i \ast z^j &= \bar{z}^i z^j + \hbar \delta_{ij} (1 + |z|^2) _2 F_1 (1, 1; 1 - 1/\hbar; -|z|^2) \\
    &\quad + \frac{\hbar}{1 - \hbar} \bar{z}^i z^j (1 + |z|^2) _2 F_1 (1, 2; 2 - 1/\hbar; -|z|^2), \\
\end{align*}
\]

$_2F_1$ : Gauss hypergeometric function.
\( \mathbb{C}H^N \) is similar to \( \mathbb{C}P^N \)

Kähler potential and the metric:

\[
\Phi = - \ln \left( 1 - |z|^2 \right), \quad (19)
\]

\[
g_{\bar{i} \bar{j}} = \partial_{\bar{i}} \partial_{\bar{j}} \Phi = \frac{(1 - |z|^2) \delta_{\bar{i} \bar{j}} + \bar{z}^i z^j}{(1 - |z|^2)^2}, \quad (20)
\]

\[
g^{\bar{i} \bar{j}} = (1 - |z|^2) \left( \delta_{\bar{i} \bar{j}} - \bar{z}^i z^j \right). \quad (21)
\]
Deformation Quantization of $\mathbb{C}H^N$ II

*-products between the coordinates

\[ z^i * z^j = z^i z^j, \]
\[ z^i * \bar{z}^j = z^i \bar{z}^j, \]
\[ \bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \]
\[ \bar{z}^i * z^j = \bar{z}^i z^j + \hbar \delta_{ij} (1 - |z|^2) {}_2F_1(1, 1; 1 + 1/\hbar; |z|^2) \]
\[ - \frac{\hbar}{1 + \hbar} \bar{z}^i z^j (1 - |z|^2) {}_2F_1(1, 2; 2 + 1/\hbar; |z|^2). \]
In this section we introduce the Twisted Fock Representation on an open set $U \subset M$. 
3-1. Heisenberg-like algebra

As mentioned in Section 2, \( \{ z^i, \partial_j \Phi \mid i, j = 1, 2, \cdots, N \} \) and \( \{ \bar{z}^i, \partial_j \Phi \mid i, j = 1, 2, \cdots, N \} \) are candidates for the creation and annihilation operators

\[
\begin{align*}
    a_i^\dagger &= z^i, & a_i &= \frac{1}{\hbar} \partial_i \Phi, & a_i &= \bar{z}^i, & a_i^\dagger &= \frac{1}{\hbar} \partial_i \Phi. & (22)
\end{align*}
\]

They satisfy the following commutation relations

\[
\begin{align*}
    [a_i, a_j^\dagger]_* &= \delta_{ij}, & [a_i^\dagger, a_j^\dagger]_* &= 0, & [a_i, a_j]_* &= 0, & (23) \\
    [a_i, a_j^\dagger]_* &= \delta_{ij}, & [a_i^\dagger, a_j^\dagger]_* &= 0, & [a_i, a_j]_* &= 0. & (24)
\end{align*}
\]

but slightly different!

\[
\begin{align*}
    [a_i, a_j^\dagger]_* \quad \text{and} \quad [a_i, a_j^\dagger]_*
\end{align*}
\]

do not vanish in general.
The Fock space is defined by a vector space spanned by the bases which is generated by acting $a_i^\dagger$ on $|\vec{0}\rangle$,

$$|\vec{n}\rangle = |n_1, \cdots, n_N\rangle = c_1(\vec{n})(a_1^\dagger)^{n_1} \cdots (a_N^\dagger)^{n_N} * |\vec{0}\rangle,$$

where $|\vec{0}\rangle = |0, \cdots, 0\rangle$ satisfies $a_i^\dagger * |\vec{0}\rangle = 0$ $(i = 1, \cdots, N)$ and $(A)^n$ stands for $\underbrace{A \ast \cdots \ast A}$. $c_1(\vec{n})$ is a normalization coefficient. Here, we define the basis of a dual vector space by

$$\langle \vec{m}| = \langle m_1, \cdots, m_N| = \langle \vec{0}| * (a_1^*)^{m_1} \cdots (a_N^*)^{m_N} c_2(\vec{m}),$$

and $\langle 0| * a_i^\dagger = 0$ $(i = 1, \cdots, N)$, where $c_2(\vec{m})$ is also a normalization constant.
3-2. Fock Space II

The underlines are attached to the bra vectors in order to emphasize that $\langle \vec{m} \mid$ is not Hermitian conjugate to $| \vec{m} \rangle$. In this article, we set the normalization constants as

$$c_1(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \quad c_2(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}},$$

(28)

where $\vec{n}! = n_1!n_2! \cdots n_N!$. 

The local twisted Fock algebra (representation) $F_U$ is defined as an algebra given by a set of linear operators acting on the Fock space defined on $U$:

$$F_U := \{ \sum_{\vec{n},\vec{m}} A_{\vec{n}\vec{m}} |\vec{n}\rangle \langle \vec{m}| \mid A_{\vec{n}\vec{m}} \in \mathbb{C} \}. \quad (29)$$

and products between its elements are given by the star product $\ast$.

In the remaining part of this section, we construct concrete expressions of functions which are elements of this local twisted Fock algebra.
Lemma

For arbitrary Kähler manifolds \((M, \omega)\), there exists a Kähler potential \(\Phi(z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N)\) such that

\[
\Phi(0, \ldots, 0, \bar{z}^1, \ldots, \bar{z}^N) = 0, \quad \Phi(z^1, \ldots, z^N, 0, \ldots, 0) = 0. \quad (30)
\]
Let \((M, \omega)\) be a Kähler manifold, \(\Phi\) be its Kähler potential with the property (30), and \(*\) be a star product with separation of variables given in the previous section. Then the following function

\[
|0\rangle \langle 0| := e^{-\Phi/\hbar},
\]

satisfies

\[
a_i * |0\rangle \langle 0| = 0, \quad |0\rangle \langle 0| * a_i^\dagger = 0, \quad (|0\rangle \langle 0|) * (|0\rangle \langle 0|) = e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar} = |0\rangle \langle 0|.
\]
Overview of Proof
We define the following normal ordered quantity,

\[ : e^{-\sum_i a_i^\dagger a_i} : = \prod_{i=1}^{N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)^n_\ast * (a_i)^n_\ast. \]  

(34)

Here \( \prod_{i=1}^{N} \) is defined by \( \prod_{i=1}^{N} f_i = f_1 \ast f_2 \ast \cdots \ast f_N \).

It is easy to show that \( a_i^\dagger \ast \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)^n_\ast * (a_i)^n_\ast = 0 \), in the same way as in the case of the ordinary harmonic oscillator.

Therefore, all we have to do is to show

\[ : e^{-\sum_i a_i^\dagger a_i} : = e^{-\Phi/\hbar}. \]  

(35)
This can be done as follows:

\[ : e^{-\sum_i a_i^\dagger a_i} : = \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}!} (a^\dagger)^*_* (a)^*_* \]

\[ = \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}! \hbar |n|} (z)^*_* (\partial \Phi)^*_*. \quad (36) \]

We use the following notation: for an \( N \)-tuple \( A_i \) \( (i = 1, 2, \cdots, N) \) and an \( N \)-vector \( \vec{n} = (n_1, n_2, \cdots, n_N) \),

\[ (A)^{\vec{n}} = (A_1)^{n_1}_* (A_2)^{n_2}_* \cdots (A_N)^{n_N}*, \quad \text{ (37)} \]

\[ \vec{n}! = n_1! n_2! \cdots n_N!, \quad |n| = \sum_{i=1}^{N} n_i. \quad \text{ (38)} \]
By using \((z)\vec{n} = (z)^{n_1} \cdots (z^N)^{n_N}\) and (16), (36) is recast as

\[
\sum_{n_1, n_2, \ldots, n_N = 0}^{\infty} \frac{1}{n_1! n_2! \cdots n_N!} (-z^1)^{n_1} \cdots (-z^N)^{n_N} e^{-\frac{\Phi(z, \bar{z})}{\hbar}} \partial_1^{n_1} \cdots \partial_N^{n_N} e^{\frac{\Phi(z, \bar{z})}{\hbar}}
\]

\[
= e^{-\frac{\Phi(z, \bar{z})}{\hbar}} \Phi(0, \bar{z}) e^{\frac{\Phi(0, \bar{z})}{\hbar}}
\]

\[
= e^{-\frac{\Phi(z, \bar{z})}{\hbar}}. \tag{39}
\]

Here, the final equality follows from the condition (30).

\(\square\)
Lemma

\[ e^{-\Phi/\hbar} = |0\rangle\langle 0| \text{ satisfies the relation} \]

\[ |0\rangle\langle 0| * f(z, \bar{z}) = e^{-\Phi/\hbar} * f(z, \bar{z}) = e^{-\Phi/\hbar} f(0, \bar{z}) = |0\rangle\langle 0| f(0, \bar{z}), \tag{40} \]

\[ f(z, \bar{z}) * |0\rangle\langle 0| = f(z, \bar{z}) * e^{-\Phi/\hbar} = f(z, 0) e^{-\Phi/\hbar} = f(z, 0) |0\rangle\langle 0|. \tag{41} \]

for a function \( f(z, \bar{z}) \) such that \( f(z, \bar{w}) \) can be expanded as Taylor series with respect to \( z^i \) and \( \bar{w}^j \), respectively.
We expand a function \( \exp \Phi(z, \bar{z})/\hbar \) as a power series,

\[
e^{\Phi(z, \bar{z})/\hbar} = \sum_{\vec{m}, \vec{n}} H_{\vec{m}, \vec{n}}(z) \vec{m}(\bar{z}) \vec{n}
\]  

(42)

where \((z) \vec{n} = (z^1)^{n_1} \cdots (z^N)^{n_N}\) and \((\bar{z}) \vec{n} = (\bar{z}^1)^{n_1} \cdots (\bar{z}^N)^{n_N}\). Since \(\exp \Phi/\hbar\) is real and satisfies (30), the expansion coefficients \(H_{\vec{m}, \vec{n}}\) obey

\[
\vec{H}_{\vec{m}, \vec{n}} = H_{\vec{n}, \vec{m}}, \\
H_{\vec{0}, \vec{n}} = H_{\vec{n}, \vec{0}} = \delta_{\vec{n}, \vec{0}}.
\]  

(43)  

(44)

Using this expansion, the following relations are obtained.
Proposition

The right $\ast$-multiplication of $(a)^{\vec{n}}_\ast = (\frac{\partial \Phi}{\partial \vec{n}})^{\vec{n}}_\ast$ on $\langle 0 | \langle 0 |$ is related to the right $\ast$-multiplication of $(a)^{\vec{n}}_\ast = (\vec{z})^{\vec{n}}_\ast$ on $\langle 0 | \langle 0 |$ as follows,

$$| 0 \rangle \langle 0 | \ast (a)^{\vec{n}}_\ast = \vec{n}! \sum_{\vec{m}} H_{\vec{n},\vec{m}} | 0 \rangle \langle 0 | \ast (a)^{\vec{m}}_\ast. \quad (45)$$

Similarly, the following relation holds,

$$(a^\dag)^{\vec{n}}_\ast \ast | 0 \rangle \langle 0 | = \vec{n}! \sum_{\vec{m}} H_{\vec{m},\vec{n}} (a^\dag)^{\vec{m}}_\ast \ast | 0 \rangle \langle 0 |. \quad (46)$$
Corollary

\[
\langle 0 \rangle \langle 0 \rangle \ast (a)^\tilde{n} = \sum_{\tilde{m}} \frac{1}{\tilde{m}!} H_{\tilde{n},\tilde{m}}^{-1} \langle 0 \rangle \langle 0 \rangle \ast (a)^{\tilde{m}}, \quad (47)
\]

\[
(a^\dagger)^\tilde{n} \ast \langle 0 \rangle \langle 0 \rangle = \sum_{\tilde{m}} \frac{1}{\tilde{m}!} H_{\tilde{m},\tilde{n}}^{-1} (a^\dagger)^{\tilde{m}} \ast \langle 0 \rangle \langle 0 \rangle, \quad (48)
\]

where \( H_{\tilde{n},\tilde{m}}^{-1} \) is the inverse matrix of the matrix \( H_{\tilde{n},\tilde{m}} \),

\[
\sum_{\tilde{k}} H_{\tilde{m},\tilde{k}} H_{\tilde{k},\tilde{n}}^{-1} = \delta_{\tilde{m},\tilde{n}}.
\]
We introduce bases of the Fock representation as follows,

\[ |\vec{m}\rangle\langle\vec{n}| := \frac{1}{\sqrt{m!n!}} (a^\dagger)_{\vec{*}}^\dagger |\vec{0}\rangle\langle\vec{0}| (a)_{\vec{*}}^* \]

\[ = \frac{1}{\sqrt{m!n!}} (z)_{\vec{*}}^\dagger e^{-\Phi/\hbar}^* \left( \frac{1}{\hbar} \partial \Phi \right)^{\vec{n}}_{\vec{*}}. \]

By using (45), the bases are also written as

\[ |\vec{m}\rangle\langle\vec{n}| = \sqrt{\frac{n!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}}(z)_{\vec{*}}^\dagger e^{-\Phi/\hbar}^* (\bar{z})_{\vec{k}}^* \]

\[ = \sqrt{\frac{n!}{\vec{m}!}} \sum_{\vec{k}} H_{\vec{n},\vec{k}}(z) (\bar{z})_{\vec{k}}^* e^{-\Phi/\hbar}. \]
The completeness of the bases are formally shown as

$$\sum_{\vec{n}} |\vec{n}\rangle\langle\vec{n}| = \sum_{\vec{m},\vec{n}} H_{\vec{n},\vec{m}}(z)^{\vec{n}}(\vec{z})^{\vec{m}} e^{-\Phi/\hbar}$$

$$= e^{\Phi/\hbar} e^{-\Phi/\hbar}$$

$$= 1. \quad (51)$$

The *-products between the bases are calculated as

$$|\vec{m}\rangle\langle\vec{n}| * |\vec{k}\rangle\langle\vec{l}| = \frac{1}{\sqrt{\vec{m}!\vec{n}!\vec{k}!\vec{l}!}} (a^\dagger)_{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| * (a)_{\vec{n}} * (a^\dagger)_{\vec{k}} * |\vec{0}\rangle\langle\vec{0}| * (a)_{\vec{l}}$$

$$= \delta_{\vec{n},\vec{k}}|\vec{m}\rangle\langle\vec{l}|. \quad (52)$$
The creation and annihilation operators $a_i^\dagger$, $a_i$ act on the bases as follows,

$$a_i^\dagger \star |\bar{m}\rangle \langle \bar{n}| = \sqrt{m_i + 1} |\bar{m} + \vec{e}_i\rangle \langle \bar{n}|, \quad (53)$$

$$a_i \star |\bar{m}\rangle \langle \bar{n}| = \sqrt{m_i} |\bar{m} - \vec{e}_i\rangle \langle \bar{n}|, \quad (54)$$

$$|\bar{m}\rangle \langle \bar{n}| \star a_i^\dagger = \sqrt{n_i} |\bar{m}\rangle \langle \bar{n} - \vec{e}_i|, \quad (55)$$

$$|\bar{m}\rangle \langle \bar{n}| \star a_i = \sqrt{n_i + 1} |\bar{m}\rangle \langle \bar{n} + \vec{e}_i|, \quad (56)$$

where $\vec{e}_i$ is a unit vector, $(\vec{e}_i)_j = \delta_{ij}$. The action of $a_i$ and $a_i^\dagger$ is derived by the Hermitian conjugation of the above equations.
The creation and annihilation operators can be expanded with respect to the bases,

\[ a_i^\dagger = \sum_{\vec{n}} \sqrt{n_i + 1} |\vec{n} + \vec{e}_i\rangle \langle \vec{n}|, \]  
(57)

\[ a_i = \sum_{\vec{n}} \sqrt{n_i + 1} |\vec{n}\rangle \langle \vec{n} + \vec{e}_i|, \]  
(58)

\[ a_i = \sum_{\vec{m},\vec{n},\vec{k}} \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H_{\vec{k+\vec{e}_i},\vec{n}}^{-1} |\vec{m}\rangle \langle \vec{n}|, \]  
(59)

\[ a_i^\dagger = \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_i + 1) H_{\vec{m},\vec{k}+\vec{e}_i} H_{\vec{k},\vec{n}}^{-1} |\vec{m}\rangle \langle \vec{n}|. \]  
(60)
4. Transition maps I

- Let $M = \bigcup_a U_a$ be a locally finite open covering and $\phi_a : U_a \to \mathbb{C}^N$. Consider the case $U_a \cap U_b \neq \emptyset$.

- Denote by $\phi_{a,b}$ the transition map from $\phi_a(U_a)$ to $\phi_b(U_b)$. The local coordinates $(z, \bar{z}) = (z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N)$ on $U_a$ are transformed into the coordinates $(w, \bar{w}) = (w^1, \ldots, w^N, \bar{w}^1, \ldots, \bar{w}^N)$ on $U_b$ by $(w, \bar{w}) = (w(z), \bar{w}(\bar{z}))$, where $w(z) = (w^1(z), \ldots, w^N(z))$ is a holomorphic function and $\bar{w}(\bar{z}) = (\bar{w}^1(\bar{z}), \ldots, \bar{w}^N(\bar{z}))$ is an anti-holomorphic function.

- $f \ast_a g$ and $f \ast_b g$ : the star products defined on $U_a$ and $U_b$, respectively.

In general, there is a nontrivial transition maps $T$ between two star products i.e. $f \ast_b g = T(f) \ast_a T(g)$. But the transition maps are trivial in our case.
4. Transition maps II

Proposition

For an overlap $U_a \cap U_b \neq \emptyset$,

$$f *_b g(w, \bar{w}) = \phi_{a,b}^* f *_a g(w, \bar{w}) \quad (61)$$

Here $\phi_{a,b}^*$ is the pull back of $\phi_{a,b}$.

Proof

The Kähler potentials $\Phi_a(z, \bar{z})$ on $U_a$ and $\Phi_b(w, \bar{w})$ on $U_b$ satisfy, in general,

$$\Phi_b(w, \bar{w}) = \Phi_a(z, \bar{z}) + \phi(z) + \bar{\phi}(\bar{z}),$$

where $\phi$ is a holomorphic function and $\bar{\phi}$ is an anti-holomorphic function. We define a differential operator $L_{b,f}$ by $L_{b,f} g := f *_b g$. 
4. Transition maps III

on $U_b$. Similarly, we use $g^{ij}_b$, $D^i_b$ etc. as the metric on $U_b$, differential operator $D^i$ on $U_b$, etc. As mentioned in Section 2,

$$L_{b,f} = \sum_{n=0}^{\infty} \hbar^n a^b_{n,i}(f) \overleftarrow{D}^i_b = \sum_{n=0}^{\infty} \hbar^n \sum_{k \geq 0} a^{b(n;k)}_{i_1 \ldots i_k} D^i_{b_1} \cdots D^i_{b_k}, \quad (62)$$

is determined by

$$[L_{b,f} , R_{b,\partial_l \Phi_b}] = [L_{b,f} , \frac{\partial \Phi_b}{\partial \overleftarrow{w}^l} + \hbar \frac{\partial}{\partial \overleftarrow{w}^l}] = 0 \quad (63)$$

On the overlap $U_a \cap U_b$,

$$D^i_b = \frac{\partial \overleftarrow{w}^i}{\partial \overrightarrow{z}^l} D^i_a, \quad (64)$$
4. Transition maps IV

because $g_{ib}^{ij} = \frac{\partial \bar{w}^i}{\partial \bar{z}^k} \frac{\partial w^j}{\partial z^l} g_a^{kl}$. From the fact that differential operators $\bar{D}^i_b$ contain only differentiation with respect to holomorphic coordinates $w^i$, $\bar{D}^i_b$ commutes with anti-holomorphic functions, then we obtain

$$L_{b,f} = \sum_{n=0}^{\infty} \hbar^n a_{n,i}^b (f) \left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_j D^j_b,$$

(65)

where $\left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_j$ is an anti-holomorphic function

$$\left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\bar{i}}_j = \frac{\partial \bar{w}^{i_1}}{\partial \bar{z}^{j_1}} \cdots \frac{\partial \bar{w}^{i_k}}{\partial \bar{z}^{j_k}}.$$

(66)
4. Transition maps V

\[
\left[ L_{b,f} , \frac{\partial \Phi_b}{\partial \bar{w}^l} + \hbar \frac{\partial}{\partial \bar{w}^l} \right]
= \left[ \sum_{n=0}^{\infty} \hbar^n a^{b}_{n,i}(f) \left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^i_j \overset{\bar{v}^i_j}{D^j_a} , \frac{\partial \bar{z}^k}{\partial \bar{w}^l} \left( \frac{\partial \Phi_a}{\partial \bar{z}^k} + \frac{\partial \bar{\phi}}{\partial \bar{z}^k} + \hbar \frac{\partial}{\partial \bar{z}^k} \right) \right]
= \frac{\partial \bar{z}^k}{\partial \bar{w}^l} \left[ \sum_{n=0}^{\infty} \hbar^n a^{b}_{n,i}(f) \left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^i_j \overset{\bar{v}^i_j}{D^j_a} , \frac{\partial \Phi_a}{\partial \bar{z}^k} + \hbar \frac{\partial}{\partial \bar{z}^k} \right] = 0, \quad (67)
\]

and thus we obtain

\[
L_{a,f} = \sum_{n=0}^{\infty} \hbar^n a^{b}_{n,i}(f) \left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^i_j \overset{\bar{v}^i_j}{D^j_a} = L_{b,f} \quad (68)
\]

which satisfies the condition \([L_{a,f} , R_{a,\partial_i \Phi_a}] = 0. \quad \square\)
4-2. Trans. maps for Twisted Fock rep. 1

As a next step, we consider the transition function between twisted Fock reps. we can choose $\Phi_a(z, \bar{z})$ and $\Phi_b(w, \bar{w})$ s. t.

$$\Phi_a(0, \bar{z}) = \Phi_a(z, 0) = 0, \quad \Phi_b(0, \bar{w}) = \Phi_b(w, 0) = 0. \quad (69)$$

Using these Kähler potentials, $|\vec{0}\rangle_{pp} \langle \vec{0}|$ is defined as

$$|\vec{0}\rangle_{pp} \langle \vec{0}| = e^{-\Phi_p/\hbar}, \quad (p = a, b),$$

and $|\vec{m}\rangle_{pp} \langle \vec{n}|$ are defined by

$$|\vec{m}\rangle_{aa} \langle \vec{n}| = \frac{1}{\sqrt{\vec{m}! \vec{n}!}} (z)^{\vec{m}}_* e^{-\Phi_a/\hbar} * \left( \frac{1}{\hbar} \partial \Phi_a \right)^{\vec{n}}_*,$$

$$|\vec{m}\rangle_{bb} \langle \vec{n}| = \frac{1}{\sqrt{\vec{m}! \vec{n}!}} (w)^{\vec{m}}_* e^{-\Phi_b/\hbar} * \left( \frac{1}{\hbar} \partial \Phi_b \right)^{\vec{n}}_*. $$
Let us consider the case that on the overlap $U_a \cap U_b$ the coordinate transition function $w(z)$, $\tilde{w}(\tilde{z})$, and the functions $\exp(\phi(w)/\hbar)$ and $\exp(\phi(\tilde{w})/\hbar)$ are given by analytic functions. Then the products $(w(z))^{\tilde{\alpha}} \exp - (\phi(w)/\hbar)$ and $(\tilde{w}(\tilde{z}))^{\alpha} \exp - (\phi(\tilde{w})/\hbar)$ are also analytic functions;

\[
(w(z))^{\tilde{\alpha}} e^{-\phi(w)/\hbar} = \sum_{\tilde{\beta}} C_{\tilde{\alpha} \tilde{\beta}} z^{\tilde{\beta}},
\]

\[
(\tilde{w}(\tilde{z}))^{\alpha} e^{-\phi(\tilde{w})/\hbar} = \sum_{\beta} \tilde{C}_{\alpha \beta} \tilde{z}^{\beta}. \quad (70)
\]
After some calculations, we obtain trans. between the bases,

\[ T^{ab} : F_{U_a} \rightarrow F_{U_b}, \]  

as

\[ |\tilde{m}\rangle_{bb}|\tilde{n}\rangle = \sum_{i,j} T^{ba,ij}_{mbn} |i\rangle_{aa} |j\rangle, \]  

where

\[ T^{ba,ij}_{mbn} = \sqrt{\frac{n!}{m!}} \sqrt{\frac{i!}{j!}} \sum_{k} H^{b}_{n,k}(C^{-m}_{i})(\sum_{\beta} C^{k}_{\beta} H^{a-1}_{\beta,j}). \]  

Using this transformation, the twisted Fock representation is extended to \( M \). We call it the twisted Fock representation of \( M \).
5. Example-1

Some examples of the Twisted Fock reps. are given.

**Example 1**: Fock representation of \( \mathbb{C}^N \)

The first example is \( \mathbb{C}^N \). The Kähler potential is given by

\[
\Phi_{\mathbb{C}^N} = \sum_{i=1}^{N} |z^i|^2.
\]

The star product is easily obtained as

\[
f \ast g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \delta^{k_1 l_1} \cdots \delta^{k_n l_n} (\partial_{\bar{k}_1} \cdots \partial_{\bar{k}_n} f)(\partial_{l_1} \cdots \partial_{l_n} g).
\]

We put

\[
a_{i}^\dagger = z^i, \quad a_i = \frac{1}{\hbar} \bar{z}^i, \quad a_i = \bar{z}^i, \quad a_{i}^\dagger = \frac{1}{\hbar} z^i.
\]

(74)

(75)
Then they satisfy the commutation relations:

\[
[a_i, a_j^\dagger]_* = \delta_{ij}, \quad [a_i, a_j^\dagger]_* = \hbar \delta_{ij}
\]  

(76)

and the others are zero. The basis of the twisted Fock algebra is given by

\[
|\vec{m}\rangle\langle\vec{n}| = \frac{1}{\hbar |\vec{n}| \sqrt{\vec{m}! \vec{n}!}} (z)^{\vec{m}} e^{-\Phi/\hbar} (\vec{z})^{\vec{n}}.
\]  

(77)

These are defined globally, so the trace operations for the twisted Fock algebras given by

\[
\text{Tr}_{CN}|\vec{m}\rangle\langle\vec{n}| := \frac{\hbar^N}{\pi^N} \int_{CN} d^{2D} \vec{m} |\vec{m}\rangle\langle\vec{n}| = \delta_{\vec{m}\vec{n}}.
\]  

(78)

These results coincide with well known facts for N.C. $\mathbb{R}^{2N}$.
Example 2: Fock representation of N.C. $\mathbb{C}P^N$

Let denote $\zeta^a$ ($a = 0, 1, \ldots, N$) homogeneous coordinates and $\bigcup U_a$ ($U_a = \{[\zeta^0 : \zeta^1 : \cdots : \zeta^N]|\zeta^a \neq 0\}$) an open covering of $\mathbb{C}P^N$. We define inhomogeneous coordinates on $U_a$ as

$$z_a^0 = \frac{\zeta^0}{\zeta^a}, \cdots, z_a^{a-1} = \frac{\zeta^{a-1}}{\zeta^a}, z_a^{a+1} = \frac{\zeta^{a+1}}{\zeta^a}, \cdots, z_a^N = \frac{\zeta^N}{\zeta^a}. \quad (79)$$

We choose a Kähler potential on $U_a$

$$\Phi_a = \ln(1 + |z_a|^2), \quad (80)$$

where $|z_a|^2 = \sum i |z_a^i|^2$. A star product on $U_a$ is given as

$$f \star g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} (D^{j_1} \cdots D^{j_n} f) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g, \quad (81)$$
5. Example-2 II

where

\[ c_n(\hbar) = \frac{\Gamma(1 - n + 1/\hbar)}{n! \Gamma(1 + 1/\hbar)}, \quad D^i = g^{ij} \partial_j, \quad D^i = g^{ij} \partial_j. \] (82)

On \( U_a \), creation and annihilation operators are given as

\[ a^\dagger_{a,i} = z^i_a, \quad a_{a,i} = \frac{\bar{z}^i_a}{\hbar(1 + |z_a|^2)}, \quad a_{a,i} = \bar{z}^i_a, \quad a_{a,i}^\dagger = \frac{z^i_a}{\hbar(1 + |z_a|^2)}. \]

and a vacuum is

\[ |\bar{0}\rangle_{aa} \langle 0| = e^{-\Phi_a/\hbar} = (1 + |z_a|^2)^{-1/\hbar}. \] (83)

The bases can be explicitly written as

\[ |\bar{m}\rangle_{aa} \langle \bar{n}| = \frac{\Gamma(1/\hbar + 1)}{\sqrt{m! \bar{n}! \Gamma(1/\hbar - |n| + 1)}} (z_a)^m (\bar{z}_a)^n e^{-\Phi/\hbar}. \] (84)
$H_{\vec{m}, \vec{n}}$ s.t. $e^{\Phi_a/\hbar} = \sum H_{\vec{m}, \vec{n}}(z_a)_{\vec{m}}(\bar{z})_{\vec{n}}$ is obtained as

$$H_{\vec{m}, \vec{n}} = \delta_{\vec{m}, \vec{n}} \frac{\Gamma(1/\hbar + 1)}{\vec{m}! \Gamma(1/\hbar - |m| + 1)}, \tag{85}$$

The transformations for the coordinates and the Kähler potential on $U_a \cap U_b$ are

$$z_a^i = \frac{z_b^i}{z_b^a}, \quad (i \neq a), \quad z_a^b = \frac{1}{z_b^a}, \tag{86}$$

$$\Phi_a = \Phi_b - \ln z_b^a - \ln \bar{z}_b^a. \tag{87}$$
Thus, $|\tilde{m}\rangle_{a a} \langle \tilde{n}|$ is written on $U_a \cap U_b$ as

$$\frac{\Gamma(1/\hbar + 1)}{\sqrt{\tilde{m}! \tilde{n}! \Gamma(1/\hbar - |n| + 1)}} e^{-\Phi_b/\hbar}$$

$$\times (z_b^0)^{m_0} \ldots (z_b^{a-1})^{m_{a-1}} (z_b^a)^{1/\hbar - |n|} (z_b^{a+1})^{m_{a+1}} \ldots (z_b^{b-1})^{m_{b-1}} (z_b^{b+1})^{m_{b+1}} \ldots (z_b^N)^{m_N}$$

$$\times (\tilde{z}_b^0)^{n_0} \ldots (\tilde{z}_b^{a-1})^{n_{a-1}} (\tilde{z}_b^a)^{1/\hbar - |n|} (\tilde{z}_b^{a+1})^{n_{a+1}} \ldots (\tilde{z}_b^{b-1})^{n_{b-1}} (\tilde{z}_b^{b+1})^{n_{b+1}} \ldots (\tilde{z}_b^N)^{n_N},$$

where

$$\tilde{m} = (m_0, \ldots, m_{a-1}, m_{a+1}, \ldots, m_N), \quad (88)$$

$$\tilde{n} = (n_0, \ldots, n_{a-1}, n_{a+1}, \ldots, n_N). \quad (89)$$

We should treat $(z_b^a)^{1/\hbar - |m|}$ and $(\tilde{z}_b^a)^{1/\hbar - |n|}$ carefully, because if they are not monomials some trick is needed to express them as the twisted Fock representation.
5. Example-2 V

A function $f(z, \bar{z})e^{-\Phi/\hbar}$ is expressed as the Twisted Fock algebra when $f(z, \bar{z})$ is given as a Taylor expansion in $z$ and $\bar{z}$. For simplicity, we consider the one dimensional case. When a non-monomial function $z^q$ of some complex coordinate $z$ with a nonpositive integer $q$ is given, $z^q$ should be Taylor expanded around the some non-zero point $p \in \mathbb{C}$ to express it as a twisted Fock algebra:

$$z^q = p^q + qp^{q-1}(z - p) + \frac{q(q - 1)}{2}p^{q-2}(z - p)^2 + \cdots . \quad (90)$$

Then they are written by using the Twisted Fock rep.
To avoid the complications concerning \( (z_b^a)^{1/\hbar - |m|} \) and \( (\bar{z}_b^a)^{1/\hbar - |n|} \), we can introduce a slightly different representation. Let us consider the case

\[
1/\hbar = L \in \mathbb{Z}, \quad L \geq 0,
\]

(91)

Then, we define \( F_a^L \) on \( U_a \) as a subspace of \( F_{U_a} \),

\[
F_a^L = \{ \sum_{\vec{m}, \vec{n}} A_{\vec{m}\vec{n}} |\vec{m}\rangle_{aa} \langle \vec{n}| \mid A_{\vec{m}\vec{n}} \in \mathbb{C}, \quad |m| \leq L, \quad |n| \leq L \}.
\]

(92)

The bases on \( U_a \) are related to those on \( U_b \) as,

\[
\sqrt{\frac{(L - |n|)!}{(L - |m|)!}} |\vec{m}\rangle_{aa} \langle \vec{n}| = \sqrt{\frac{(L - |n'|)!}{(L - |m'|)!}} |\vec{m}'\rangle_{bb} \langle \vec{n}'|,
\]

(93)
where

\[ \mathbf{\vec{m}}' = (m_0, \cdots, m_{a-1}, L - |m|, m_{a+1}, \cdots, m_{b-1}, m_{b+1}, \cdots, m_N), \]  

(94)

\[ \mathbf{\vec{n}}' = (n_0, \cdots, n_{a-1}, L - |n|, n_{a+1}, \cdots, n_{b-1}, n_{b+1}, \cdots, n_N). \]  

(95)

Using the expression of (93), we can define \( |\mathbf{\vec{m}}\rangle_{a a} \langle \mathbf{\vec{n}}| \) on the whole of \( U_b \). Therefore, the operators in \( F_a^L \) can be extended to the whole of \( \mathbb{C}P^N \).
Example 3: Fock representation of noncommutative $\mathbb{C}H^N$

We choose a Kähler potential

$$\Phi = -\ln(1 - |z|^2),$$  \hspace{1cm} (96)

where $|z|^2 = \sum_i^N |z^i|^2$. A star product is given as

$$f \star g = \sum_{n=0}^\infty c_n(\hbar) g_{j_1\bar{k}_1} \cdots g_{j_n\bar{k}_n} (D^{j_1} \cdots D^{j_n} f) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g,$$  \hspace{1cm} (97)

where

$$c_n(\hbar) = \frac{\Gamma(1/\hbar)}{n!\Gamma(n + 1/\hbar)}, \quad D^{\bar{i}} = g^{\bar{i}j} \partial_j, \quad D^i = g^{ij} \partial_j.$$  \hspace{1cm} (98)
The creation and annihilation operators are given as

\[ a_i^\dagger = z^i, \quad a_i = \frac{\bar{z}^i}{\hbar(1 - |z|^2)}, \quad a_i = \bar{z}^i, \quad a_i^\dagger = \frac{z^i}{\hbar(1 - |z|^2)}. \] (99)

and a vacuum is

\[ |\vec{0}\rangle\langle \vec{0}| = e^{-\Phi/\hbar} = (1 - |z|^2)^{1/\hbar}. \] (100)

Bases of the Fock representation on \( \mathbb{C}H^N \) are constructed as

\[ |\vec{m}\rangle\langle \vec{n}| = \frac{(-1)^{|n|\Gamma(1/\hbar + |n|)}}{\sqrt{\vec{m}!\vec{n}!\Gamma(1/\hbar)}}(z)^{\vec{m}}(\bar{z})^{\vec{n}}(1 - |z|^2)^{1/\hbar}. \] (101)

These are defined globally.
For $\mathbb{C}H^N$, trace density is given by the usual Riemannian vol.

$$\mu_g = \frac{1}{(1 - |z|^2)^{N+1}}.$$ \hfill (102)

The trace is given by the integration

$$\text{Tr}_{\mathbb{C}H^N} |\vec{m}\rangle \langle \vec{n}| = \frac{\Gamma(1/\hbar)}{\pi^N\Gamma(1/\hbar - N)} \int_{\mathbb{C}H^N} dz^{2D} \mu_g |\vec{m}\rangle \langle \vec{n}| = \delta_{\vec{m}\vec{n}}.$$ \hfill (103)
6. Summary

- Twisted Fock reps. of N.C. Kähler manifolds are constructed. The N.C. Kähler manifolds are given by deformation quantization with separation of variables. Using this, the twisted Fock representation which constructed based on two sets of creation and annihilation operators was introduced with the concrete expressions of them on a local coordinate chart.

- The dictionary to translate bases of the twisted Fock reps into functions is given.

- They are defined on a local coordinate chart, and they are extended by the transition functions. This extension is achieved by essentially the result that the star products with separation of variables have a trivial transition function.

- Examples of the twisted Fock representation of Kähler manifolds: $\mathbb{C}^N$, $\mathbb{C}P^N$ and $\mathbb{C}H^N$. 
Thank you for your attention.

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